PROJECTIVE DIFFERENTIAL GEOMETRY AND ASYMPTOTIC ANALYSIS IN GENERAL RELATIVITY

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I - Introduction

Overall goal: Relate the asymptotic behavior of solutions of PDEs to that of the geometry: "Geometric compactification"

Ⓐ "Geometric" compactification of pseudo-Riemannian manifolds

① "Weaken" the geometric structure

Levi-Civita connection

$$\begin{tikzcd} & (M, g) \arrow[dl, bend right, "\nabla_g"'] \arrow[dr, bend left, "g"] & \\ (M, [\nabla]) & & (M, c) \end{tikzcd}$$

projective class

conformal class

② Construct the boundary ∂M.

③ Extend the new structure to boundary points.

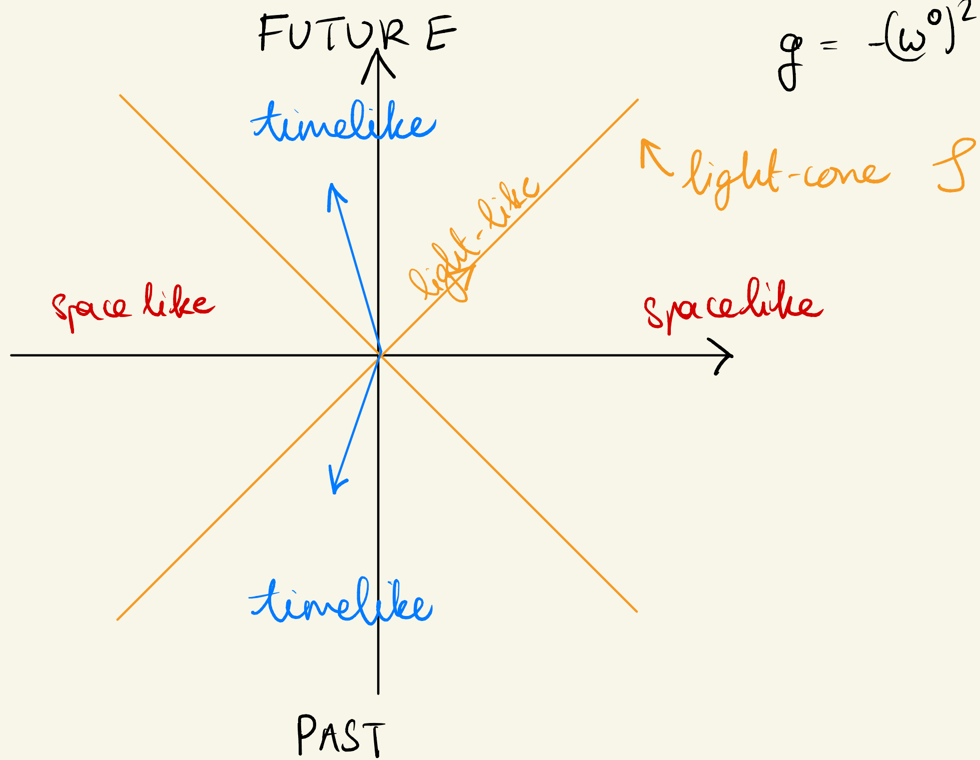
In a second time . . .

4. Construct invariant differential operators (that act on appropriate objects) in order to generalise equations on M.
   eg. conformal / projective Laplacian . . .

5. study their extension to boundary points.

Vocabulary of Lorentzian geometry

$$g = -(\omega^0)^2 + \sum_{i=1}^{n} (\omega^i)^2$$



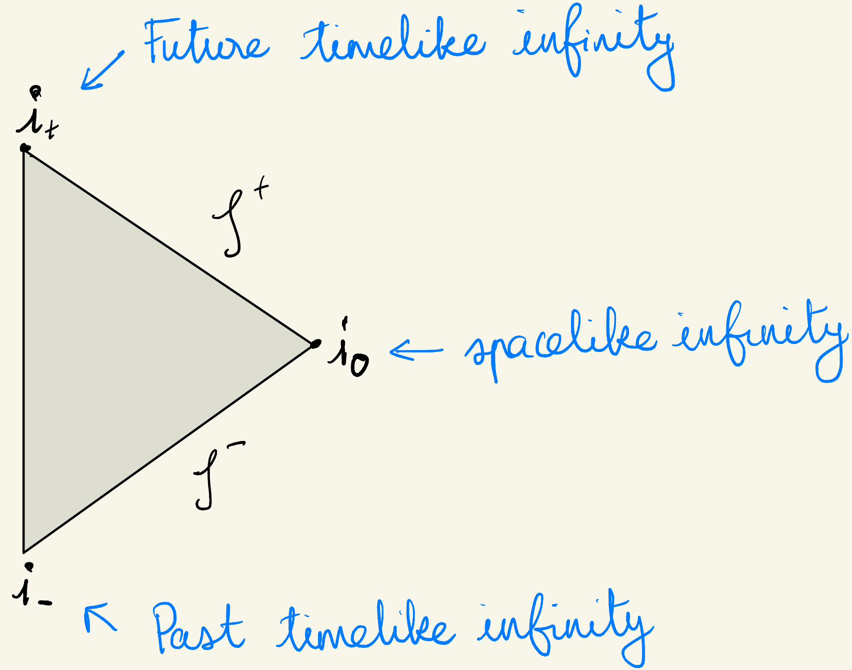
Time orientation: choice of connected component of $S \setminus \{0\}$ $\Leftrightarrow$ choice of $v \neq 0$, $g(v,v) \leq 0$.

(B) Motivations

- Massive fields

$$\tilde{g} = \Omega^2 g$$

$$\Omega^2 = \frac{4}{(1+u^2)(1+v^2)}$$



Penrose diagram of conformally compactified Minkowski spacetime

(B) Motivations

In $\mathbb{R}^{d+1}$: Klein-Gordon equation: $(\Box + m^2)\phi = 0$

$$\partial_t \underline{\phi} = \begin{pmatrix} 0 & \Delta - m^2 \\ 1 & 0 \end{pmatrix} \underline{\phi}, \quad \underline{\phi} = \begin{pmatrix} \partial_t \phi \\ \phi \end{pmatrix} \xrightarrow[(m=1)]{} \begin{cases} \partial_t u_\pm = \pm i (-\Delta + 1)^{\frac{1}{2}} u_\pm \\ u(0,x) = \varphi(x), \qquad \varphi \in \mathcal{S}(\mathbb{R}^d) \end{cases}$$

Theorem (Hörmander)

$$u_+(t,x) = U_0(t,x) + U_+(t,x)\, e^{i/\rho}, \quad \underline{\rho = (t^2 - |x|^2)^{-\frac{1}{2}}}, \quad U_0 \in \mathcal{S}(\mathbb{R}^{d+1})$$

$$\boxed{U_+(t,x) \sim (+0 + i\rho)^{\frac{d}{2}} \sum_j^{\infty} \rho^j w_j(t,x) .}$$

$$w_0(t,x) = \begin{cases} (2\pi)^{-\frac{d}{2}} \sqrt{1+|\tilde{x}|^2}\, \hat{\varphi}(-\tilde{x}) & \text{si } t^2 > |x|^2 \\ 0 & \text{sinon} \end{cases} \quad ; \quad w_j(t,x) = \underline{\underline{w_j\left(1, \frac{x}{t}\right)}} \longrightarrow \text{projective parameter}$$

$\tilde{x} = \rho x$

C - Projective differential geometry

M smooth n-dimensional manifold.

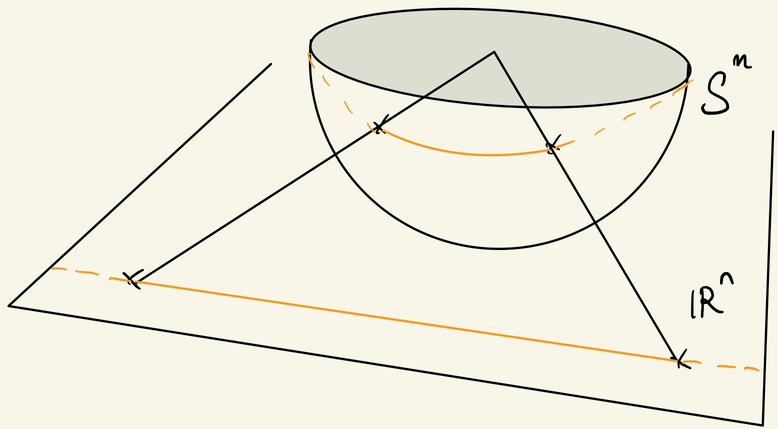
Definition: Two affine connections ∇ et $\hat{\nabla}$ on TM are *projectively equivalent* if and only if they have the same unparametrised geodesics.

Theorem (Weyl): ∇ et $\hat{\nabla}$ are projectively equivalent iff there is $\Upsilon \in \Gamma(T^*M)$ such that:

$$\boxed{\hat{\nabla}_a \xi^b = \nabla_a \xi^b + \Upsilon_a \xi^b + \Upsilon_c \xi^c \delta_a{}^b} \quad , \xi \in \Gamma(TM).$$

We will abbreviate this: $\underline{\hat{\nabla} = \nabla + \Upsilon}$.

Example. ∇_{g^n} is projectively equivalent to $\nabla_{\mathbb{R}^n}$



In the local chart:

$$\left\{ y_i = \frac{x_i}{\sqrt{1+|x|^2}} \right\}$$

$$\begin{cases} \nabla_{\mathbb{R}^n} = \nabla_{\varphi^* g_{S^n}} + \Upsilon \\ \Upsilon = -\dfrac{\nabla\left(\sqrt{1-|y|^2}\right)}{\sqrt{1-|y|^2}} = -\nabla\left(\dfrac{1}{\sqrt{1+|x|^2}}\right)\sqrt{1+|x|^2} \end{cases}$$

$$= -\frac{\nabla \rho}{\rho}$$

II - Projective compactification

Let $\overline{M} = M \cup \partial M$ be a manifold with boundary ∂M, and interior M. Let ∇ be an affine connection on M.

Def: ∇ is said projectively compact of order α if and only if at each point $x_0 \in \partial M$ there is a neighbourhood $\mathcal{U}$ and a boundary defining function (BDF) ρ on $\mathcal{U}$ such that the connection:

$$\boxed{\hat{\nabla} = \nabla + \frac{d\rho}{\alpha \rho}}$$

extends smoothly to $\mathcal{U} \cap \partial M$.

ρ BDF

$$\Updownarrow$$

$$\begin{cases} \rho > 0 & \text{on } M \cap \mathcal{U} \\ \rho = 0 & \text{on } \mathcal{U} \cap \partial M \\ d\rho \neq 0 & \text{on } \mathcal{U} \cap \partial M. \end{cases}$$

Def: Let $\overline{M} = M \cup \partial M$ and g a metric on M. g is said to be projectively compact of order α if its Levi-Civita connection is projectively compact in the preceding sense.

Examples:
- Minkowski spacetime is projectively compact of order 1.
- De-Sitter spacetime is projectively compact of order 2.

Example : Minkowski spacetime $= \dfrac{\mathbb{R}^{d+1} \rtimes SO(d,1)}{SO(d,1)}$

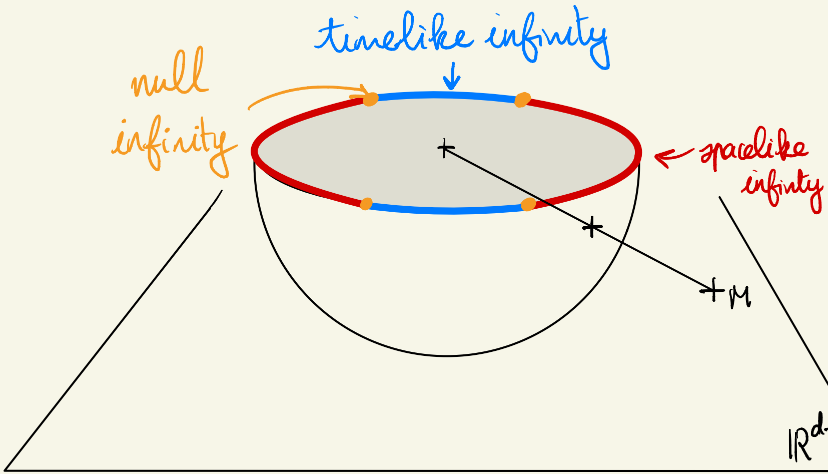
$\boxed{d=1}$

$$SL_{d+2}(\mathbb{R}) \longrightarrow \underbrace{G_1 = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}, A \in SL_{d+1}(\mathbb{R}), b \in \mathbb{R}^{d+1} \right\}}_{\text{Subgroup that preserves:}} \longrightarrow \underbrace{G_2 = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}, A \in SO(d,1), b \in \mathbb{R}^{d+1} \right\}}_{\text{subgroup that preserves:}}$$

$\uparrow$

Oriented projective group.

$$I_A = (0, \dots, 0, 1) \qquad H^{AB} = \mathrm{diag}\,(-1, 1, \dots, 1, 0)$$



$\simeq \{x_{d+2} = -1\}$

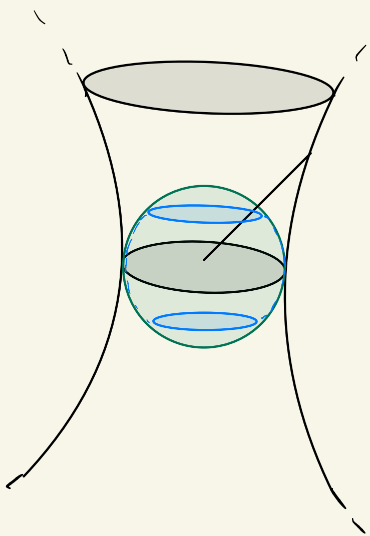
"Density" of weight 1. $\sigma = X^A I_A$

Fig : Description of the orbits of the restriction to G_2 of the action of $SL_{d+2}(\mathbb{R})$ on the projective sphere.

Examples : de-Sitter $= \dfrac{SO(4,1)}{SO(3,1)}$

$\boxed{\alpha = 2}$

$$\mathcal{DS} = \{ x \in \mathbb{R}^5 , \quad -x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \}$$



- H^{AB} on $(\mathbb{R}^5)^*$ given by $\mathrm{diag}(-1,1,\dots,1)$,
- The image under central projection on the projective sphere is defined by: $\sigma = H_{AB} X^A X^B > 0$
  density of weight $+2$
- $\sigma = 0 \Leftrightarrow$ boundary "boundary defining density"
- In order to retrieve the metric structure on $\mathcal{DS}$, one should restrict the action of $SL_{n+1}(\mathbb{R})$ to those elements that preserve H^{AB}.

Projective densities and projectively invariant equations

Def: A projective density of weight $w \in \mathbb{C}$ is a section of

$$\mathcal{E}(w) = L(TM) \times_\rho \mathbb{R}$$

where ρ is the action $A \mapsto |\det A|^{\frac{w}{n+1}} \in GL(\mathbb{R}) \simeq \mathbb{R}^*$

How can one construct projectively invariant differential operators.

- No preferred connection
- Some equations can be made projectively invariant by giving tensors a weight.

Example: the projective Killing equation.

If $\quad \hat{\nabla} = \nabla + \Upsilon$:

- $\sigma \in \Gamma(\mathcal{E}(\omega))$ then : $\hat{\nabla}_a \sigma = \nabla_a \sigma + \omega \Upsilon_a \sigma$
- $\mu_b \in \Gamma(T^*M)$ then : $\hat{\nabla}_a \mu_b = \nabla_a \sigma - 2\Upsilon_{(a}\mu_{b)}$

Hence if $\quad \underline{\mu_b} \in \Gamma(T^*M \otimes \mathcal{E}(+2))$:

$$\hat{\nabla}_{(a}\underline{\mu}_{b)} = \nabla_{(a}\mu_{b)} + \cancel{2\Upsilon_{(a}\mu_{b)}} - \cancel{2\Upsilon_{(a}\mu_{b)}}$$

$\Rightarrow$ projective invariance

N.B. One can also do this with the geodesic equation.

$\xi^a \hat{\nabla}_a \xi^b = \xi^a \nabla_a \xi^b + 2\Upsilon_c \xi^c \xi^b \Rightarrow$ if $\underline{\xi} \in \Gamma(TM \otimes \mathcal{E}(-2))$, $\underline{\xi}^a \hat{\nabla}_a \underline{\xi}^b = \underline{\xi}^a \nabla_a \underline{\xi}^b$.

III - Tractors (theory)

Theorem - E. Cartan (1924)

Every projective class induces a unique torsion free normal Cartan projective geometry (P,ω).

Conversely, each such geometry determines a class of projectively equivalent affine connections.

Def: The standard tractor bundle is the associated bundle:

$$\mathcal{T} = (P \times_H G) \times_G \mathbb{R}^{n+1}$$

where $G = Sl_{n+1}(\mathbb{R})$ acts on $\mathbb{R}^{n+1}$ canonically.

Ⓐ Tractors in practice

- The bundle $\mathcal{T}$ has the following decomposition structure:

$$TM \otimes \mathcal{E}(-1) \equiv$$

$$\boxed{0 \longrightarrow \mathcal{E}(-1) \xrightarrow{X} \mathcal{T} \xrightarrow{Z} TM(-1) \longrightarrow 0}$$

W: after a choice of connection.

- A tractor can be thought of as an equivalence class of $\Gamma(\mathcal{E}(-1) \oplus TM(-1)) \times [\hat{\nabla}]$ for the relation:

$$\left(\begin{pmatrix} v^a \\ e \end{pmatrix}, \nabla \right) \sim \left(\begin{pmatrix} \hat{v}^a \\ \hat{e} \end{pmatrix}, \hat{\nabla} \right) \iff \begin{cases} \hat{\nabla} = \nabla + \Upsilon \\ \hat{e} = e - v^a \Upsilon_a \, ; \; \hat{v}^a = v^a \end{cases}$$

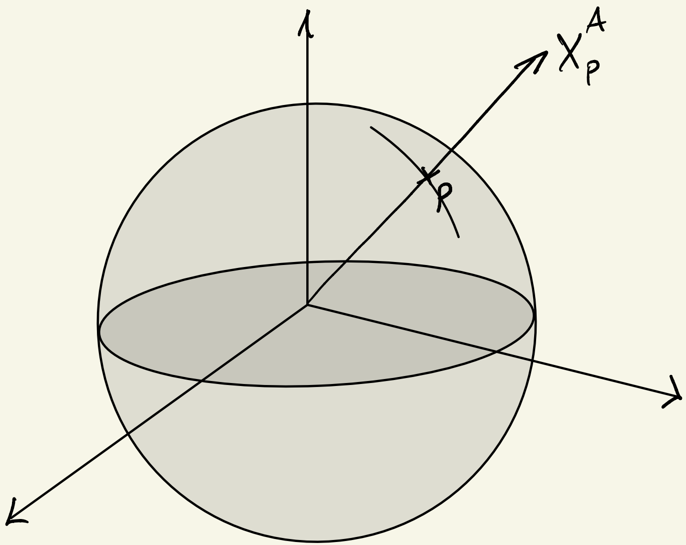
We write: $\boxed{T \overset{\nabla}{=} \begin{pmatrix} v^a \\ e \end{pmatrix}}$

Ⓑ The tractor connection

- The projective Cartan connection induces an affine connection $\nabla^{\mathcal{T}}$ on $\mathcal{T}$.

$$\boxed{\text{If} \quad T \stackrel{\nabla}{=} \begin{pmatrix} v^a \\ \rho \end{pmatrix} \qquad \nabla^{\mathcal{T}}_b T \stackrel{\nabla}{=} \begin{pmatrix} \nabla_b v^a + \delta^a_b \rho \\ \nabla_b \rho - \underbrace{P_{ba}}_{\text{Projective Schouten tensor.}} v^a \end{pmatrix}}$$

In the case of the projective sphere $\mathcal{S}^n$:



- $\mathcal{T} \simeq \mathcal{S}^n \times \mathbb{R}^{n+1}$
- $\nabla_{\mathcal{T}}$ = trivial connection
- A density of weight ω on $\mathcal{S}^n$.

$\Updownarrow$

$f : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}$

such that $f(tx) = t^{\omega} f(x)$, $t \in \mathbb{R}_+^*$, $x \in \mathbb{R}^{n+1} \setminus \{0\}$

- $X \rightarrow$ homogeneous coordinates; ρ homogeneous of weight -1 on $\mathbb{R}^{n+1} \setminus \{0\}$.

$T^A = \rho X^A$ corresponds to the map: $x \in \mathbb{R}^{n+1} \setminus \{0\} \longmapsto \rho(x) x \in \mathbb{R}^{n+1}$

(C) Tractor metric

- In GR we are mainly interested in the projective class of the Levi-Civita connection of a metric g.

- This equates to a metric H^{AB} on $\mathcal{T}^*$ that is a solution of the metrisability equation.

$$\boxed{\nabla_c H^{AB} + \frac{2}{n} X^{(A} W_{cE}{}^{B)}{}_F H^{EF} = 0}$$

$$W_{cE}{}^{B}{}_{F} = X^{(A} \underbrace{\Omega_{ce}{}^{B)}{}_{F}}_{\text{tractor curvature}} Z^{e}{}_{E}$$

- In some cases this reduces to : $\nabla_c H^{AB} = 0 \rightarrow$ normal solution.
  (Einstein manifolds)

D Natural differential operators on tractors

- The Thomas D-operator (a basic building block.)

$T \in \Gamma(\otimes^k \mathcal{T}^* \otimes^l \mathcal{T} \otimes \mathcal{E}(w))$

$$\boxed{D_A T \overset{\nabla}{=} \begin{pmatrix} w\,T \\ \nabla_a T \end{pmatrix}}$$

$n = d+1$

- Projective Laplacian : $\Delta^{\mathcal{T}} = H^{AB} D_A D_B \overset{\nabla}{=} \underline{g}^{ab} \nabla_a \nabla_b + \frac{P_{ab}\,\underline{g}^{ab}}{d+1} w(w+d)$

- If $F \in \Gamma(\Lambda^k \mathcal{T}^*)$, we set : $(\mathcal{D} F)_{A_1 \cdots A_{k+1}} = D_{[A_1} F_{A_2 \cdots A_{k+1}]}$

Proposition (B '21): $\boxed{\mathcal{D}^2 F = 0}$

IV - Exterior tractor calculus

We restrict now to (M, g) oriented and projectively compact of order 2 (de-Sitter).

In this case:

- H^{AB} is everywhere non-degenerate
- $\sigma = |\omega_g|^{-\frac{2}{n+1}}$ is a boundary defining density.

Prop: (B'21): One can equip $\mathcal{T}$ with an orientation induced by that of (M, g) and develop Hodge theory for tractors.

A few details ...

- $Z_A^a Z_B^b H^{AB} = \sigma^{-1} g^{ab} = \underline{g}^{ab} \in \Gamma(TM(-2))$

  $\rightarrow$ metric on $T^*M(1)$

- $(\omega^i{}_a)$ orthonormal dual frame $\longrightarrow J_A^i = Z_A^a \omega_a^i$

- $I_A = D_A \sigma \in \Gamma(T^*(1))$, $I^2 = H^{AB} I_A I_B$,

- H^{AB} non-degenerate $\Rightarrow \sigma^{-1} I^2 \neq 0$ on $\overline{M}$.

Therefore one can define :

$$\boxed{J_A^0 = \frac{\sigma^{-\frac{1}{2}}}{\sqrt{|\sigma^{-1} I^2|}} I_A} \quad , \quad \boxed{\Omega^{\mathcal{T}} = J^0 \wedge J^1_A \cdots \wedge J^n}$$

$$\boxed{\Omega^{\mathcal{T}} \overset{\nabla_g}{=} \begin{pmatrix} \dfrac{2\sigma^{\frac{n+1}{2}} \omega_g}{\sqrt{|\sigma^{-1} I^2|}} \\ 0 \end{pmatrix}}$$

A general formula for the Hodge star operator (B'21)

$$\Lambda^k \mathcal{T}^* \overset{\nabla}{\cong} \Lambda^{k-1} T^*M(k) \oplus \Lambda^k T^*M(k)$$

$$F_{A_1 \dots A_k} \overset{\nabla}{=} \begin{pmatrix} \mu_{a_2 \dots a_k} \\ \xi_{a_1 \dots a_k} \end{pmatrix} \quad \text{then :}$$

$$\boxed{\star F \overset{\nabla}{=} \begin{pmatrix} \dfrac{2\sigma^{\frac{1}{2}}}{\sqrt{|\sigma^{-1} I^2|}} \left((-1)^k \ast \xi + T \lrcorner (\ast \mu) \right) \\ \dfrac{\sigma^{-\frac{3}{2}} I^2}{2\sqrt{|\sigma^{-1} I^2|}} \ast \mu - \dfrac{2\sigma^{\frac{1}{2}}}{\sqrt{|\sigma^{-1} I^2|}} \left[(-1)^k T^\flat \wedge (\ast \xi) + T^\flat \wedge T \lrcorner (\ast \mu) \right] \end{pmatrix}}$$

where : $$T^a = -\frac{1}{n+1} \nabla_c \mathbf{g}^{ca} .$$

An operator algebra (B'21)

- $\mathcal{D}^* = (-1)^{(n+2)(k+1)+1} s\varepsilon * \mathcal{D} *$ $\qquad \varepsilon = \mathrm{sgn}(\sigma^{-1} I^2)$, $\quad s = \mathrm{sgn}(\det g^{ab})$
- $\mathcal{J} : F \mapsto I \wedge F$ où $I = D\sigma$
- $\mathcal{J}^* = (-1)^{(n+2)(k+1)+1} s\varepsilon * \mathcal{J} *$, $\quad (F \mapsto -I \lrcorner F)$

Lemma:

$$\boxed{\begin{gathered} \{\mathcal{J}, \mathcal{J}^*\} = -\sigma^{-1} I^2 h \,, \quad \mathcal{J}^2 = \mathcal{J}^{*2} = 0 \\ [\mathcal{D}, \sigma] = \mathcal{J} \,, \quad [\mathcal{D}^*, \sigma] = \mathcal{J}^* \end{gathered}}$$

$h = \underline{w} + \frac{n+2}{2}$

"WEIGHT OPERATOR"

$\Gamma(\mathcal{T}(w)) \ni T \mapsto wT$

If (M,g) Einstein:

$$\boxed{\begin{gathered} \{\mathcal{D}, \mathcal{J}^*\} = -\frac{\sigma^{-1} I^2}{2}(\underline{w} + \underline{k}) \,, \\ \{\mathcal{D}^*, \mathcal{J}\} = -\frac{\sigma^{-1} I^2}{2}(\underline{w} + n + 2 - \underline{k}) \,. \end{gathered}}$$

Corollary 1*: If (M,g) is Einstein then:

$$\begin{cases} x = \sigma \\ y = \dfrac{1}{\sigma^{-1} I^2}\{\mathcal{D}, \mathcal{D}^*\} \\ h = \underline{w} + \dfrac{n+2}{2} \end{cases}$$

form a $\mathfrak{sl}_2$ triple.

Corollary 2: If $w \neq 0$ then the cohomology spaces of the cochain complex:

$$\Gamma(\mathcal{E}(w)) \xrightarrow{\mathcal{D}} \mathcal{E}_{A_1}(w-1) \xrightarrow{\mathcal{D}} \cdots \xrightarrow{\mathcal{D}} \mathcal{E}_{[A_1, \ldots, A_{n+1}]}(w-(n+1))$$

are all trivial

* Generalises the case $(k=0)$ established in S. Porath's PhD thesis without the Einstein assumption. (Boundary operators on Klein manifolds)

Application to the Proca equation :

Tractor version :

$$\begin{cases} \mathcal{D} F = 0 , \\ \mathcal{D}^* F = 0 . \end{cases} \qquad (\mathcal{S})$$

According to Corollary 2 , $\mathcal{D} F = 0 \Rightarrow \exists A \ / \ F = \mathcal{D} A$

Using the gauge symmetry we can set : $\mathcal{D}^* A = 0$ (Lorenz)

$$\boxed{(\mathcal{S}) \Longrightarrow \tilde{y} A = 0}$$

Why "Proca"?

If ∇_g is the Levi-Civita connection:

$$\boxed{\mathscr{D}^* A \overset{\nabla_g}{=} \begin{pmatrix} -\delta\mu \\ \\ \delta\xi - (w+n+1-k)\dfrac{\sigma^{-2}I^2}{4}\mu \end{pmatrix}}$$

$$\boxed{\mathscr{D}^*\mathscr{D}A \overset{\nabla_g}{=} \begin{pmatrix} \delta d\mu - (w+k)\delta\xi \\ \\ \delta d\xi + \dfrac{\sigma^{-2}I^2}{4}(w-1+n-k)\,d\mu - \dfrac{\sigma^{-2}I^2}{4}\underbrace{(w-1+n-k)(w+k)}_{m^2}\xi \end{pmatrix}}$$

- This generalises Proca on M:

$$\phi_{a_1\dots a_k} \in \Gamma(\Lambda^k T^*M) \longrightarrow \phi_{a_1\dots a_k}\underset{\substack{\uparrow \\ \text{parallel for } \nabla_g}}{\sigma^{\frac{w+n}{2}}} \in \Gamma(\Lambda^k T^*M(w+k)) \longrightarrow \underset{\Lambda^k \mathcal{T}^{*\Psi}(w)}{\phi_{a_1\dots a_k}\sigma^{\frac{w+n}{2}} Z^{a_1}_{A_1}\cdots Z^{a_k}_{A_k}}$$

Using the $\mathfrak{sl}_2$ algebra: formal solution operator (Gover-Waldron)

Consider the problem: $\tilde{y}f = 0$ on M and $f = f_0$ on ∂M.

Using the commutation rules we can formally find a solution of the form:

$$\boxed{f = \left(x^{\nu} \sum_{k \in \mathbb{N}} \alpha_k x^k y^k\right) f_0 = x^{\nu} F(z) f_0}$$

$$F(z) = \sum_{k=0}^{+\infty} \alpha_k z^k$$

$$z = :(xy)^k: = x^k y^k .$$

Where:

$$\begin{cases} (zF')' - (h_0 - 1) F' + F = 0 \\ \nu (h_0 + \nu - 1) = 0 \quad , \quad \text{où} \quad h f_0 = h_0 f_0 . \end{cases}$$

Prop: In De-Sitter, thanks to the symmetries, the resolution is exact.

Perspectives and future projects

- How to treat the Ricci-flat ($\alpha = 1$) case in which the boundary calculus degenerates ?
- How to give an analytical meaning to the formal series ? (Symbol of a Fourier integral operator ?) (with M. Capoferri (Heriot Watt) S. Murro (Gênes))
- Can we construct a tractor version of the Dirac equation, in particular in the case $\alpha = 1$?