Scattering of Dirac Fields near an Extremal Kerr-de Sitter Black-Hole

Jack Borthwick
Laboratoire de Mathématiques de Besançon

14e Rencontres GDR Dynamique Quantique

I - Kerr - de Sitter spacetimes

$$\boxed{g = \frac{\Delta_r}{\Xi^2 \rho^2}\left[dt - a\sin^2\theta\, d\varphi\right]^2 - \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{\Delta_\theta \sin^2\theta}{\rho^2 \Xi^2}\left[(r^2+a^2)\,d\varphi - a\,dt\right]^2}$$

$$\underbrace{\ell^2 = \frac{\Lambda}{3}}_{\Lambda:\ \text{cosmological constant}} ; \quad \Xi = 1 + a^2\ell^2 ; \quad \Delta_\theta = 1 + \underbrace{a^2}_{a:\ \text{angular momentum per unit mass}}\ell^2\cos^2\theta ; \quad \underbrace{\rho^2}_{\{\rho=0\}\ \text{"ring singularity"}} = r^2 + a^2\cos^2\theta$$

Last (but not least):

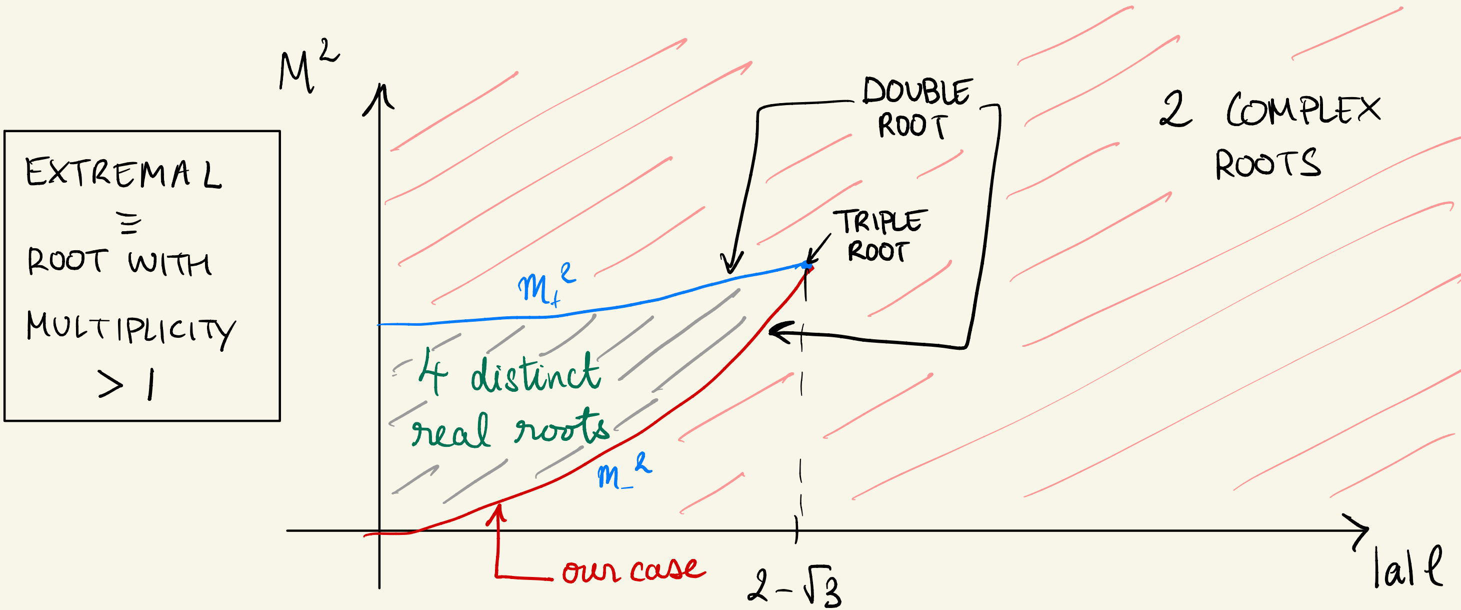
$$\boxed{\Delta_r = r^2 - 2\underbrace{M}_{\text{"mass" of black-hole}} r + a^2 - \ell^2 r^2 (r^2 + a^2)}$$

Remark: Coefficients are independent of t and φ $\Rightarrow$ $\partial_t, \partial_\varphi$ Killing.

Proposition:

$$\gamma = (a^2\ell^2 - 1)^2 - 12a^2\ell^2,$$

$$M_\pm^2 = \frac{1}{54\ell^2}\left[(1-a^2\ell^2)(a^4\ell^4 + 34a^2\ell^2 + 1) \pm \gamma^{3/2}\right]$$



Structure of the roots of Δ_r according to the parameters (a, ℓ, M)

Context of scattering problem.

Where ?

double horizon　　　　　　cosmological horizon

r	$-\infty$		r_-		0		r_e		r_+		$+\infty$
Δ_r		$-$	0	$+$			0	$+$	0	$-$	

"exterior" region : $\mathcal{B}$

Boyer-Lindquist blocks of an extremal KdS black hole

Observer: Stationary observer with trajectory of the form:

$$\boxed{r = r_0 \quad \theta = \theta_0 , \ \varphi = \omega t + \varphi_0 , \ \omega \in \mathbb{R}, \ r_0 \in]r_e, r_+[, \ \theta_0 \in]0,\pi[, \ \varphi_0 \in]0,2\pi[}$$

According to the observer's clock, the time necessary for a photon to reach the horizon following a principal null geodesic:

$$\boxed{\Delta \tau \propto \int_{r_0}^{r_+} \frac{\Xi (r^2+a^2)}{\Delta_r} dr = +\infty}$$

$\Rightarrow$ Horizons are asymptotic regions

First tangible consequence of a double horizon

We will use the "Regge Wheeler" coordinate :

$$\boxed{r^* = \int \frac{\Xi (r^2+a^2)}{\Delta_r}\, dr}$$

$$r_+ - r \underset{r^* \to +\infty}{\sim} e^{-\frac{2\ell}{\Xi r_+}\sqrt{\frac{M}{r_e}}\, r^*}$$

"standard" behaviour at normal horizon

$$r - r_e \underset{r^* \to +\infty}{\sim} \frac{r_e^2 (r_e^2 + a^2)}{3Mr - 4a^2} \, \frac{1}{r^*}$$

The Dirac equation

$(\phi_A, \chi^{A'})$ Dirac spinor

$$\begin{cases} \nabla^{AA'}\phi_A = \frac{m}{\sqrt{2}}\chi^{A'}, \\ \nabla_{AA'}\chi^{A'} = -\frac{m}{\sqrt{2}}\phi_A . \end{cases}$$

- <u>conserved current</u>: $j_a = j_{AA'} = \phi_A \bar{\phi}_{A'} + \chi_{A'} \bar{\chi}_A$
- <u>conserved charge</u>: $Q = \int_{\Sigma_{t_0}} N^a j_a \, \omega_{g,\Sigma_{t_0}}$ $\longrightarrow$ scalar product on each slice $\Sigma_{t_0} = \{t = t_0\}$

$\downarrow$

they are all isometric

The Dirac equation ... in coordinates

After an appropriate choice of spin-frame and working with a spinor density, the Dirac equation can be expressed as the evolution equation:

$$i\frac{d\Phi}{dt} = H\Phi \quad \text{on} \quad \mathcal{H} = L^2(\mathbb{R}\times S^2)\otimes\mathbb{C}^4 \quad \text{where } H \text{ is the operator:}$$

$$\boxed{\begin{aligned} H = {} & \frac{\Delta_r\sqrt{\Delta_\theta}}{\Xi\sigma}\Gamma^1 D_r + \frac{\sqrt{\Delta_r\Delta_\theta}}{\Xi}\mathcal{D}_{S^2} + \frac{aq^2\rho^2}{\sigma^2}D_\varphi + \frac{\sqrt{\Delta_r\Delta_\theta}}{\sigma\sin\theta}\left(\frac{\rho^2}{\sigma} - \frac{\sqrt{\Delta_\theta}}{\Xi}\right)\Gamma^3 D_\varphi \\ & - \frac{i\sqrt{\Delta_r\Delta_\theta}}{\sigma\Xi}e\tilde{V}_1 + \frac{\sqrt{\Delta_r\Delta_\theta}}{\sigma\Xi}\rho m\Gamma^0 \end{aligned}}$$

$\mathcal{D}_{S^2}$: Dirac operator on the Euclidean sphere

$D_{x_i} = \frac{1}{i}\partial_{x_i}$

The scattering theory (B', 19)

(To appear in Annales de l'Institut Fourier)

Prop 1: H is a self-adjoint operator and has purely absolutely continuous spectrum.

Thm 1: There is a bounded operator P_+ with spectrum $\sigma(P_+) = \{-1, 1\}$ such that for any $J \in C_\infty(\mathbb{R})$:

$$J(P_+) = s\text{-}\lim_{t \to +\infty} e^{iHt} J\left(\frac{r}{t}\right) e^{-iHt}$$

The scattering theory

Thm 2: Define $X_{in} = \mathbb{1}_{\{-1\}}(P_+)$ and $X_{out} = \mathbb{1}_{\{+1\}}(P_+)$ then:

$\mathcal{H} = X_{in} \oplus X_{out}$ and there is a unitary operator W^+ such that

$$\lim_{t \to +\infty} \| e^{-itH} \phi_0 - \mathcal{U} e^{-itH_{-\infty}} W^+ \phi_0 \| = 0 \quad \text{if} \quad \phi_0 \in X_{in}$$

↑ Dollard modification

$$\lim_{t \to +\infty} \| e^{-itH} \phi_0 - e^{-itH_{+\infty}} W^+ \phi_0 \| = 0 \quad \text{if} \quad \phi_0 \in X_{out}$$

with:

$$\begin{cases} H_{+\infty} = \Gamma^1 D_{r^*} + \dfrac{a}{r_+^2 + a^2} D_\phi \\ H_{-\infty} = \Gamma^1 D_{r^*} + \dfrac{a}{r_e^2 + a^2} D_\phi \end{cases}$$

The proof (at a glance) : Global strategy

- We impose a global rotation of our coordinates: $\varphi' = \varphi - \frac{a}{r_e^2+a^2}t$
- Working on components of stable decomposition into eigenspaces of D_ϕ.

$$H - \frac{a}{r_e^2+a^2}D_\phi = H'$$

$$H' \xrightarrow[|r^*|\to+\infty]{} H_1 = H_0 + h^{-1}V_c h$$

$$H_1 \xrightarrow{r^*\to+\infty} H_0 \xrightarrow{r^*\to+\infty} \Gamma^1 D_{r^*} + \left(\frac{ap}{r_+^2+a^2} - \frac{ap}{r_e^2+a^2}\right)$$

can be reduced to a 1D problem

$$H_1 \xrightarrow{r^*\to-\infty} \underbrace{H_0 + g(r^*)F(0)}_{H_e} \xrightarrow{r^*\to-\infty} \Gamma^1 D_{r^*}$$

Dollard modified dynamics

- Main tool: Mourre theory
  - → absence of singular continuous spectrum
  - → (weak) minimal velocity estimate

A few words on Mourre theory.

Let $I \subset \mathbb{R}$.

We seek a (local) "conjugate" operator A that satisfies:

- H evolves sufficiently smoothly under the dynamics of A:

$$\boxed{\forall u \in \mathcal{H},\ \forall z \in \rho(H),\quad s \longmapsto e^{isA}(H-z)^{-1}e^{-isA}u \text{ is } C^2.}$$

- There is a compact operator and $\mu > 0$ such that:

$$\boxed{\mathbb{1}_I(H)\, i[H,A]\, \mathbb{1}_I(H) \geq \mu\, \mathbb{1}_I(H) + K}$$

+ technical conditions on the quadratic forms $i[H,A]$, $[A,[A,H]]$.

Choice of conjugate operator (at the double horizon)

- All our operators are perturbations of H_0:

$$\boxed{\begin{gathered} H_0 = \Gamma^1 D_{r^*} + g(r^*)\mathcal{D} + f(r^*) \\ \mathcal{D} = \Delta_0^{1/4}\, \mathcal{D}_{S^2}\, \Delta_0^{1/4} \end{gathered}}$$

$$g(r^*) = \frac{\sqrt{\Delta_r}}{\Xi(r^2+a^2)}$$

$$f(r^*) = \frac{ap}{r^2+a^2} - \frac{ap}{\underbrace{r_e^2+a^2}_{c_0}}$$

Formally similar to $$\cancel{H} = \underbrace{\Gamma^1 D_{r^*} - \frac{c}{r^*}\mathcal{D}}_{\substack{\text{massless Dirac operator for} \\ \text{metric of a ``crinkled'' asymptotically flat metric} \\ \eta = dt^2 - dr^2 - \left(\frac{r^*}{c}\right)^2 \frac{1}{\Delta_0} d\sigma^2 \text{ on } \mathbb{R}^*_- \times S^2}} + c_-$$

treat like spacelike infinity $\Leftarrow$

$$\boxed{A \simeq \frac{1}{2}\{r^*, D_{r^*}\} + \Gamma^1 r^* c_-}$$

Why the rotated coordinates ?

- If $c_- \neq 0$ then $i[\mathcal{H}, A] = \mathcal{H} + C c_- \underbrace{[\Gamma', D]}_{\substack{\neq 0 \\ \text{no control}}}$

- Another idea :

  (i) perform the unitary transformation : $U = e^{-\frac{i}{2}\{Dr^0, r^0\} \ln|D|}$

  (ii) try : $A = \Gamma' r^+$

$$i\left[U^* \mathcal{H} U, \Gamma' r^+\right] = \frac{1}{|D|} - C \underbrace{\overset{\neq 0}{\left[\frac{D}{|D|}, \Gamma'\right]}}_{\text{no decay.}}$$

- In the rotated coordinate system : $A = \frac{1}{2}\{r^+, D r^-\}$ is a good candidate.

Velocity estimates

- Since $H \in \mathcal{C}^2(A)$, the Mourre estimate can be used to show:

For any $\chi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\chi \cap \{0\} = \emptyset$, $\exists\, \varepsilon_\chi, C \in \mathbb{R}_+^*$ /

$$\forall \psi \in \mathcal{H}, \quad \int_1^{+\infty} \left\| \mathbb{1}_{[0,\varepsilon_\chi]}\left(\frac{|\Lambda^\circ|}{t}\right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leqslant C \|\psi\|^2$$

Furthermore: $\displaystyle \operatorname*{s-lim}_{t \to +\infty} \mathbb{1}_{[0,\varepsilon_\chi]}\left(\frac{|\Lambda^\circ|}{t}\right) \chi(H) e^{-itHt} = 0$

- We also have a maximal velocity estimate, but lack information in between...

Reducing to 1D – the angular perturbations

- The main "obstruction" is the lack of spherical symmetry.

First: $\mathcal{D} = \Delta_\theta^{1/4} D_{S^2} \Delta_\theta^{1/4} = \begin{pmatrix} \tilde{S} & 0 \\ 0 & -\tilde{S} \end{pmatrix}$; $(\Delta_\theta = 1 + a^2\ell^2\cos^2\theta)$

Nevertheless, since $|a\ell| < 2-\sqrt{3} < 1$, perturbation arguments show that:

Prop:
- $\tilde{S}$ is self adjoint with compact resolvent
- $-\sigma(\tilde{S}) = \sigma(\tilde{S})$
- $\sigma(\tilde{S}) \cap]-1,1[= \emptyset$.

Corollary: $\mathcal{H} = \overset{\perp}{\bigoplus_{k,\ell}} \mathcal{H}_{k,n}$, where $\mathcal{H}_{k,n}$ is stable under H_0 and

$$\boxed{H_0^{k,n} = \Gamma^1 D_{r^*} - \lambda_{k,n}\, g(r^*)\, \Gamma^2 + f(r^*).}$$

At the double horizon: a "miracle"

- H is a short range perturbation of:

$$H_e = \Gamma' D_{r^*} + g(r^*)\boxed{\big(D + \underbrace{F(0)}\big)} + f(r^*)$$

D_e ⚠ not purely geometric as contains mass terms

- $F(0)$ breaks the block-diagonal form of D.

- Nevertheless: • D_e has compact resolvent and anti-commutes with Γ'
  $\Rightarrow$ spectrum is symmetric
  • Viewing the eigenvalue equation as an ODE on an open subset of the Riemann surface $\{(z,w) \in \mathbb{C}^2,\ z^2+w^2=1\}$ one can show that the eigenspaces are at most 2 dimensional.

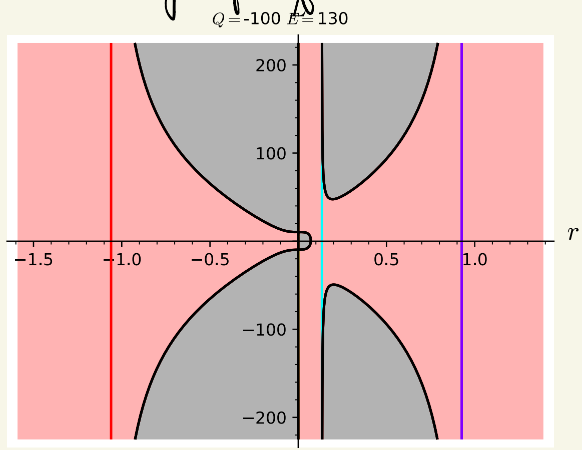
$\Rightarrow$ Réduction analogous to D.

Concluding remarks

- Can we avoid the réduction to 1D ?
- Does the asymptotic symmetry group of the extreme near horizon geometry play a role ?

A brief word on geodesics.

- r-L diagrams (<u>The Geometry of Kerr blackholes</u> - O'Neill)
  (work touched upon with a group of students « Les cordées de la réussite »)



- Can the tools of projective differential geometry simplify the study?
  "Distinguished curves and integrability in Riemannian, conformal and projective geometry"
  (2020) – D. Snell, R. Gover, Taghavi-Chabert