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Par
Jack BORTHWICK

Scattering analytique et projectif sur des espaces-temps avec constante cosmologique positive

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Rapporteurs avant soutenance :

Michał Wrochna	Professeur des Universités, CY Cergy Paris Universités
A. Rod Gover	Professeur, Auckland University

Composition du Jury :

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	Michał Wrochna	Professeur des Universités, CY Cergy Paris Universités.
Dir. de thèse :	Jean-Philippe NICOLAS	Professeur des Universités, Université de Bretagne Occidentale, LMBA

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PREFACE

This work is an approach to the asymptotic analysis of classical fields in General Relativity, a field of research at the intersection between a large variety of[1] mathematics. The main problem under study can be stated vaguely as follows: given an equation on a Lorentzian manifold, what can be said « at large times » of the behaviour of a solution, supposing some initial data?

Over the years, the theory has developed from the relentless study of a handful of examples and has lead to a number of techniques to formulate mathematically and address the general problem. One can perhaps quote four predominant domains, the spectral analysis of self-adjoint operators on Hilbert spaces and extensions thereof, microlocal analysis, vector field methods and geometric compactifications.

It is of course no surprise that the actual geometry and equation under consideration condition largely to what extent certains ideas can be applied, and no one formalism, as of yet, provides any systematic treatment of the general problem. The study of specific examples is therefore an important part of research in the field. Black-hole geometries are of particular interest, not only for their physical significance, but also because they have features that illustrate concrete obstructions to the applications of some of these techniques. A classical example of this is the phenomenon of super-radiance which prevents the classical energy functional of a Klein-Gordon field from being positive-definite, thus complicating the setup for an analytical scattering problem.

The first major project of this thesis was to study the example of Dirac fields on an extremal black-hole background; the equation under study is the Dirac equation. The precise black-hole model is that of extreme Kerr-de Sitter, which is a rotating black-hole in a universe with positive cosmological constant. This model is studied in detail in Chapter 2. Besides the rotation, the particularity of the extreme model is the coincidence of two of the so-called horizons. This causes technical difficulties for a scattering theory due to the appearance of long-range potentials. It is shown in this thesis that one can formulate the problem as a Schrödinger type equation on a Hilbert space for which one can identify the asymptotic dynamics and construct an analytical scattering theory; this

1. beautiful and fascinating

is the object of Chapter 3.

The second major project, discussed in Chapter 4, will take the reader into the realm of geometric compactifications. In this approach, the idea is to explicitly construct and append, to the bulk manifold, a geometric boundary that is identified with « infinity ». Of course, this has to be done in such a way that there is enough compatibility with the geometric structure of the original manifold so that it is possible to extend objects defined on the bulk to the boundary. The hope then is that these extensions can be used to infer information about their asymptotic behaviour. In this work, the emphasis will be on *projective* compactifications. We will construct projectively invariant versions of the Klein-Gordon and Proca equations on so-called projective tractor bundles and establish results that are parallel to those available for conformally compact manifolds. In particular, we will show in both cases that there is a natural boundary calculus that leads to a formal solution operator which produces correctly the dominant asymptotic profiles on asymptotically de-Sitter manifolds as shown by microlocal techniques.

In the course of this thesis, the author has been introduced to an entirely different way of thinking about differential geometry than what is taught in most undergraduate courses. However, the material may not be particularly well known to everyone in the field. Consequently, the reader will find introductions in Chapters 1 and 4.

Table des matières

Chapter 1

Introduction

Although this work is mainly about linear partial differential equations that arise in physics and their solutions, I have as often as possible endeavoured to see things through the eyes of a geometer, rather than those of an analyst. In this first chapter, the reader will find a short introduction to the geometric langage used throughout this text and that may be unfamiliar to those having dabbled less in geometry. In particular, we discuss the basic theory of Principal Fibre Bundles and connections. It should be noted that Chapter 3 is relatively independent from this section, save the short preamble on spinors, as unfortunately the techniques used there have eluded as of yet any geometric understanding on my part. The present section is nevertheless a pre-requisite for Chapter 4.

Throughout this section, we will work in the category of smooth (C^∞) manifolds. In this context, all maps will be assumed smooth and the terms « smooth » and « differentiable » may be used interchangeably. Unless stated otherwise, until the end of this section, M will be a smooth manifold of dimension $n \in \mathbb{N}^*$. Last of all, the identity element of a generic group G will be written e.

1.1 Vocabulary and notation

1.1.1 The abstract index notation

For most of the tensor calculus in this text we will make use of Penrose's abstract index notation. Here we give a short overview and refer the reader to [PR84] for a thorough account. The main idea is to keep track of the nature of a tensor by appending indices to the label that represent its arguments [1]. These indices should be understood as distinct from coordinate indices used in physics literature. More precisely, let E be a vector bundle over M and $\Gamma(E)$ the $\mathcal{C}^\infty(M)$-module of smooth sections of E. We consider isomorphic copies of the module $\Gamma(E)$, that we denote by $E^a, E^b, \dots$. In this way to each $X \in \Gamma(E)$

1. In finite dimensions, we can think of a tensor as a multilinear form

there are corresponding elements $X^a \in E^a, X^b \in E^b, \dots$ The dual modules will be written $E_a, E_b, \dots$ and a general tensor product $E^{a_1} \otimes \cdots \otimes E^{a_p} \otimes E_{b_1} \otimes \cdots \otimes E_{b_q}$: $E^{a_1,\dots,a_p}_{b_1,\dots,b_q}$.

Let us now describe how this is used to write out the usual elementary operations of tensor calculus. As a first example, the tensor product $X \otimes Y$ of two vector fields will be written $X^a Y^b$. We can see now why it was necessary to introduce an infinite number of modules, one can write: $X \otimes X$ unambiguously $X^a X^b$.

Contraction will be indicated by the use of repeated indices as in the following examples:

- $Y_a X^a$ is the scalar function $Y(X)$,
- if T is a simple tensor:

$$T = X_1 \otimes \cdots \otimes X_j \otimes \cdots \otimes X_p \otimes Y^1 \otimes \cdots \otimes Y^i \ \cdots \otimes Y^q \in \Gamma(E)^{\otimes p} \otimes \Gamma(E^*)^{\otimes q},$$

then the contraction:

$$C^i_j T = Y^i(X_j) X_1 \otimes \cdots \otimes X_{j-1} \otimes X_{j+1} \otimes \cdots \otimes X_p \otimes Y^1 \otimes \cdots \otimes Y^{i-1} \otimes Y^{i+1} \otimes \cdots \otimes Y^q,$$

will be written:

$$T^{a_1 \dots a_{i-1} c a_{i+1} \dots a_n}_{b_1 \dots b_{j-1} c b_{j+1} \dots b_n} = X^{a_1}_1 \cdots X^{a_{j-1}}_{j-1} X^c_j X^{a_{j+1}}_{j+1} \cdots X^{a_p}_p Y^1_{b_1} \cdots Y^{i-1}_{b_{i-1}} Y^i_c Y^{i+1}_{b_{i+1}} \cdots Y^q_{b_q}.$$

δ^a_b will denote the identity map, i.e. $\delta^a_c X^c = X^a$, $\delta^c_a Y_c = Y_a$, and, last of all, symetrisation/antisymetrisation will be written as follows:

$$T_{(a_1,\dots,a_n)} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} T_{a_{\sigma(1)},\dots a_{\sigma(n)}},$$
$$T_{[a_1,\dots,a_n]} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) T_{a_{\sigma(1)},\dots a_{\sigma(n)}}.$$

1.1.2 Concrete indices

Since abstract indices are not relative to any local frame $e^a_1, \dots, e^a_n$ of the vector bundle, we will need another notation for the components in such a frame. The solution we will adopt is to use either bold characters, the latin letters i, j, k or Greek letters. Unlike abstract indices, concrete indices do have integer values, hence, using Einstein's summation convention:

$$X^a = X^{\boldsymbol{a}} e^a_{\boldsymbol{a}} = \sum_{i=1}^{n} X^i e^a_i.$$

In the above $X^{\boldsymbol{a}}$ (or X^i) are scalar functions. Finally, if $e^{\boldsymbol{a}}_a$ is the dual local frame, then:

$$e^{\boldsymbol{a}}_a X^a = X^{\boldsymbol{a}},$$
$$e^{\boldsymbol{a}}_a e^a_{\boldsymbol{b}} = e^{\boldsymbol{a}}_a \delta^a_b e^b_{\boldsymbol{b}} = \delta^{\boldsymbol{a}}_{\boldsymbol{b}},$$

where $\delta^{\boldsymbol{a}}_{\boldsymbol{b}}$ is, consequently, the identity matrix.

Remark 1.1.1. Despite the resemblance with the summation convention, there is no implicit sum in the notation: $X^a Y_a$ for contraction with abstract indices, nevertheless:

$$X^a Y_a = X^{\boldsymbol{a}} e^a_{\boldsymbol{a}} e^{\boldsymbol{b}}_a Y_{\boldsymbol{b}} = X^{\boldsymbol{a}} \delta^{\boldsymbol{a}}_{\boldsymbol{b}} Y_{\boldsymbol{b}} = X^{\boldsymbol{a}} Y_{\boldsymbol{a}},$$

and in the last equality there is implicit summation over the index $\boldsymbol{a}$.

1.1.3 The vocabulary of Lorentzian geometry

A real vector space V is said to be « Lorentzian » if it is equipped with a symmetric bilinear form g of signature $(n, 1)$, also written, $(-, +, \dots, +)$ [2]. Due to its significance in physics, a specific language has developed for Lorentzian signature bilinear forms. Vectors are classifed according to the sign of $g(x, x), x \in V$:

1. $x \neq 0$ is causal if $g(x, x) \leq 0$,
2. x is time-like if $g(x, x) < 0$,
3. $x \neq 0$ is isotropic, null or light-like, if $g(x, x) = 0$,
4. x space-like si $g(x, x) > 0$.

Similarly, subspaces of V are classified according to the signature of the induced bilinear form. In particular, if F is a vector subspace of V, F is said to be:

1. time-like if $g|_F$ is Lorentzian,
2. isotropic, null or light-like if $g|_F$ is degenerate,
3. spacelike if $g|_F$ is positive definite.

2. In Chapter 3, however we will adopt the opposite signature convention $(1, n)$.

The lightcone is the hypersurface $\mathscr{C}$ defined by the equation $g(x,x) = 0$. A time-orientation of V is a choice between one of the two connected components of $\mathscr{C}\backslash\{0\}$: all of the non-zero causal vectors inside the chosen component will be said future-oriented; correspondingly, those in the other component will be said past-oriented. A time-like vector $x \in V$ and a causal vector $y \in V$ have same time-orientation i.e. are inside the same component of the punctured lightcone, if and only if $g(x,y) < 0$. Note also that two linearly independent causal vectors cannot be orthogonal. Incidently a time-orientation can be thought of as choice of non-zero causal vector.

A Lorentzian manifold is a pseudo-Riemannian manifold (M, g) such that the metric tensor g has Lorentzian signature. A time-orientation for M is a choice of a continuous non-vanishing causal vector field. Finally a spacetime is a time-oriented Lorentzian manifold.

1.2 Principal fibre bundles

1.2.1 Definition

Principal fibre bundles offer a unified perspective of many of the geometric concepts that we will encounter. Spinors, tractors, densities and even usual tensors can all be described in a particularly efficient manner in terms of a fibre bundle. They can also be used to give a more general definition of the notion of connection. Without further a-due, here is the definition.

Definition 1.2.1. Let G be a Lie group and M a smooth manifold. A G-principal fibre bundle with base M, written (P, π, M) or (P, G, π, M), is a smooth manifold P equipped with a smooth surjective map $\pi : P \to M$ and a smooth right-action of G on P that satisfies:

— For any $x \in M$, there is an open neighbourhood U of x and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times G$ such that:

$$\phi(p \cdot g) = \phi(p) \cdot g,$$

where, on the right-hand side, $\phi(p) \cdot g$ is the canonical right action of G on $U \times G$. Such an open neighbourhood U will be called a « trivialising neighbourhood » and the couple (ϕ, U) « bundle chart ».

— If $\pi_U : U \times G \to U$ denotes the canonical projection, then:

$$\pi_U(\phi(p)) = \pi(p).$$

When the base manifold is clear from the context, we will simply write (P, π) or (P, G, π). Finally, when $x \in M$, the set $\pi^{-1}(\{x\})$ is called the « fibre above x ».

Before discussing a fundamental and illuminating example, let us first quote the following properties:

Proposition 1.2.1. *In the notation of the definition:*

1. *The right-action of G on P preserves the fibres in the sense that for any $p \in P, g \in G$, we have: $\pi(pg) = \pi(p)$.*
2. *G acts freely on P.*
3. *$\pi^{-1}(\{\pi(p)\})$ is the orbit of p under the action of G.*

Proof. 1. Let $p \in P$, and choose a bundle chart (ϕ, U) near $\pi(p)$, then:

$$\phi(p)g = \phi(pg),$$

so $pg \in \pi^{-1}(U)$. Moreover:

$$\pi_U(\phi(p)g) = \pi_U(\phi(p)) = \pi(p),$$
$$\pi_U(\phi(p)g) = \pi_U(\phi(pg)) = \pi(pg),$$

which proves the first point.

2. Let $p \in P$, $g \in G$ that stabilises p and (ϕ, U) a bundle chart. By assumption, $pg = p$ so $\phi(pg) = \phi(p)$, hence: $\phi(pg) = \phi(p)g = \phi(p)$ and g fixes $\phi(p)$ in $U \times G$. Since the canonical right-action of G on $U \times G$ is free, it follows that $g = e$.
3. Fix $p \in P$, since $\pi(pg) = \pi(p)$, $\{pg, g \in G\} \subset \pi^{-1}(\{\pi(p)\})$. For the other inclusion, choose $r \in \pi^{-1}(\{\pi(p)\})$ and (ϕ, U) a bundle chart near $\pi(p) = \pi(r) = x$. Write, $\phi(r) = (x, g')$ and $\phi(p) = (x, g), g, g' \in G$, then: $\phi(r) = \phi(p)g^{-1}g' = \phi(pg^{-1}g')$. Thus, since ϕ is injective, $r = pg^{-1}g'$.

The final point of the preceding Proposition implies that each of the fibres of P is diffeomorphic (as a manifold), in a non-canonical way, to the group G. Just like in affine space there is no privileged choice of identity and the fibres do not have a canonical group structure. To clarify further this structure, we shall now introduce a model example.

1.2.2 The frame bundle of a vector bundle

In this section we will discuss how to naturally associate with any vector bundle (E, π_E, V) with base M and m-dimensional fibre V, a $GL(V)$-principal fibre bundle $L(E)$ over M, called the « frame bundle » of E. A generic fibre $L(E)_x$ over a point $x \in M$ will intuitively be the set of all linear frames of the fibre $E_x = \pi_E^{-1}(\{x\})$. Formally, we set:

$$L(E) = \{(x, u_x), x \in M, u_x \in GL(V, E_x)\}\,. \tag{1.1}$$

The projection map for the principal bundle, π, is defined by $\pi(x, u_x) = x$ and we shall make $GL(V)$ act from the right on $L(E)$ according to the equation:

$$(x, u_x)g = (x, u_x \circ g), g \in GL(V).$$

In order to make $L(E)$ into a smooth fibre bundle, we need to ascribe it a topology and differential structure. To this end, consider a bundle atlas $\mathscr{A}$ of E, i.e. a family of (vector) bundle charts (ψ, U) where $\psi : \pi_E^{-1}(U) \to U \times V$ is a diffeomorphism such that for each $x \in U$ the restriction of ϕ to the fibre E_x above x induces a vector-space isomorphism between E_x et V, and, where U runs over an open cover of the base M. For each chart (ψ, U), define a family of isomorphisms $\alpha_x \in GL(V, E_x), x \in U$ by $v \mapsto \phi^{-1}(x, v)$ and set:

$$\tilde{\psi} : \begin{array}{ccc} \pi^{-1}(U) & \longrightarrow & U \times GL(V) \\ (x, u_x) & \longmapsto & (x, \alpha_x^{-1} \circ u_x) \end{array}. \tag{1.2}$$

$\tilde{\psi}$ is easily seen to be bijective, and:

$$\mathscr{B} = \left\{ \tilde{\psi}^{-1}(O), (\psi, U) \in \mathscr{A}, O \subset U \times GL(V) \text{ open } \right\},$$

is a basis for a topology on $L(E)$ which promotes each $\tilde{\psi}$ to a homeomorphism. About each point $p \in L(E)$ one can shrink U so that is the domain of some coordinate chart (ϕ, U) on M and construct a local chart $\bar{\psi} : \psi^{-1}(U) \to \phi(U) \times GL(V)$ by composing $\tilde{\psi}$ with:

$$\begin{array}{rcl} \phi \times \mathrm{Id}_{GL(V)} : U \times G & \longrightarrow & \phi(U) \times G, \\ (x, g) & \longmapsto & (\phi(x), g). \end{array}$$

This specifies a differential atlas that makes the $\tilde{\psi}$ diffeomorphisms.

It follows from our discussion that on each smooth n-dimensional manifold M, one can construct a $GL_n(\mathbb{R})$-principal fibre bundle: the linear frame bundle of its tangent bundle TM. In the sequel, we will see how one can describe the usual tensor bundles over TM using $GL_n(\mathbb{R})$ and how to construct new vector bundles.

1.3 Principal connections

We come now to the important topic of connections on a principal fibre bundle. In theoretical physics, fibre bundles are used to write down gauge theories, and the notion of connection we now seek to define enables the definition of a covariant derivative for particle fields. In geometry, connections (on the frame bundle) are used to define geodesics and curvature. There are at least three equivalent definitions of a connection, each with its own conceptual or computational advantages. In order to introduce the first definition, we shall first translate into the language of fibre bundles the more familiar notion of affine connection that is used in classical texts on Differential Geometry. We recall first, for instance from [Car92], the following definition:

Definition 1.3.1. An affine connection on the tangent bundle is a map: $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ such that:

1. For any $Y \in \Gamma(TM)$, $X \mapsto \nabla_X Y$ is a linear endomorphism of the $C^\infty(M)$-module $\Gamma(TM)$ of vector fields over M,
2. For each $X, Y \in \Gamma(TM)$ and $f \in C^\infty(M)$, $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

Using Penrose's abstract index notation [PR84], a connection is therefore an operator ∇_a that, for any vector field X^b and smooth function f, satisfies:

$$\nabla_a(fX^b) = (\nabla_a f)X^b + f\nabla_a X^b,$$

where $\nabla_a f$ is simply the differential of f.

1.3.1 Affine connection and local moving frames

Consider now an open set $U \subset M$ and $(e_1, \ldots, e_n)$ an n-tuple of smooth vector fields defined on U such that for each $x \in U$, $(e_1(x), \ldots, e_n(x))$ is a basis of the tangent space T_xM. We will now switch to the notation described in 1.1.1. Each vector field X^a on U

can be written $X^a = X^{\boldsymbol{a}} e^a_{\boldsymbol{a}}$, where the $X^{\boldsymbol{a}}$ are smooth functions on U. Let us calculate $\nabla_b X^a$:

$$\begin{aligned}\nabla_b X^a = \nabla_b (X^{\boldsymbol{a}} e^a_{\boldsymbol{a}}) &= (\nabla_b X^{\boldsymbol{a}}) e^a_{\boldsymbol{a}} + X^{\boldsymbol{a}} \nabla_b e^a_{\boldsymbol{a}}, \\ &= (\nabla_b X^{\boldsymbol{a}}) e^a_{\boldsymbol{a}} + e^{\boldsymbol{a}}_c \nabla_b e^c_{\boldsymbol{d}} X^{\boldsymbol{d}} e^a_{\boldsymbol{a}}.\end{aligned} \tag{1.3}$$

It transpires from the final equation that the local action of the affine connection can be described in terms of the matrix of differential forms $\omega^{\boldsymbol{a}}{}_{\boldsymbol{d}b} = e^{\boldsymbol{a}}_c \nabla_b e^c_{\boldsymbol{d}}$. This can equivalently be seen as a differential form with values in $M_n(\mathbb{R})$ [3] that we will call a: « local connection form ».

To picture this, one can imagine the data contained in the vector fields $(e^a_{\boldsymbol{a}})$ as attaching a local frame of the tangent space to each point $x \in U$ that « moves » smoothly as x varies. The « infinitesimal » change in the frame as it moves from x to a point $x + \delta x$, to first order in δx, is described by an element of $M_n(\mathbb{R}) = \mathfrak{gl}_n(\mathbb{R})$, the Lie algebra of $GL_n(\mathbb{R})$, which corresponds to: $(\omega^{\boldsymbol{a}}{}_{\boldsymbol{d}})$.

A natural question arising now is: given two non-disjoint open sets U and V and two moving frames (e_i) and $(\tilde{e}_i)$ on U and V respectively, what relation exists between the two corresponding local connection forms $\omega = (\omega^i{}_j)$ and $\tilde{\omega} = (\tilde{\omega}^i{}_j)$ on $U \cap V$? To answer this, let us define for each $x \in U \cap V$, $P(x) = (P^i_j(x))$ the change of basis matrix from $(e_i(x))$ to $(\tilde{e}_i(x))$, and denote by $Q^i_j(x)$ its inverse. Let X be a fixed vector field defined on $U \cap V$, one has:

$$\begin{aligned}\tilde{\omega}^i{}_j(X) = \tilde{e}^i(\nabla_X \tilde{e}_j) &= Q^i_m e^m \nabla_X (P^k_j e_k), \\ &= Q^i_m X(P^k_j) \underbrace{e^m(e_k)}_{\delta^m_k} + Q^i_m \omega^m_k(X) P^k_j, \\ &= Q^i_m X(P^m_j) + Q^i_m \omega^m_k(X) P^k_j.\end{aligned} \tag{1.4}$$

Denoting by $dP(x)$ the matrix whose coefficients are the differentials of those of $P(x)$ we can rewrite (1.4):

$$\tilde{\omega}(X) = P^{-1} dP(X) + P^{-1}\omega(X)P. \tag{1.5}$$

This short analysis shows that an affine connection in the sense of Definition 1.3.1 can be encoded in a family of matrix valued forms, satisfying the compatibility condition (1.5), defined on a bundle atlas of M. It is easily seen that if (e_i) is a coordinate basis $\frac{\partial}{\partial x_i}$ associated to some local chart, then (ω^i_j) encodes the familiar Christoffel symbols.

Furthermore, the two terms on the right-hand side of (1.5) have a natural interpreta-

3. i.e. a section of the vector bundle $\Lambda^1(T^*U) \otimes (M \times M_n(\mathbb{R}))$, see also Appendix C.

tion in terms of the multiplication in the Lie group $GL_n(\mathbb{R})$. If $g \in GL_n(\mathbb{R})$ then define the maps:

— $L_g : A \in GL_n(\mathbb{R}) \mapsto gA$

— Ad_g: $A \mapsto gAg^{-1}$,

— $\mathfrak{ad}_g \in L(\mathfrak{gl}(\mathbb{R}))$ the differential at Id of Ad_g.

The map L_g gives rise to a canonical $\mathfrak{gl}_n(\mathbb{R})$-valued differential form [4] on $GL_n(\mathbb{R})$ defined by:

$$\theta(X)(g) = dL_{g^{-1}g}(X_g),$$

for any vector field $X \in \Gamma(TGL_n(\mathbb{R}))$. This form is known as the « Maurer-Cartan form ». We note at this point that none of these objects are specific to $GL_n(\mathbb{R})$ and can be defined *mutatis mutandis* on any Lie group G. Nevertheless, for matrix Lie groups, for which the tangent spaces can be identified with vector subspaces of $M_n(\mathbb{R})$, these maps are easily determined. In particular, (1.5) can in fact be written for any $x \in U \cap V$:

$$\tilde{\omega}_x = (P^*\theta)_x + \mathfrak{ad}_{P^{-1}(x)}(\omega_x). \tag{1.6}$$

In the above formula, P^* is the pullback of the Maurer-Cartan form θ by the map $x \mapsto P(x)$, i.e. $(P^*\theta)_x(X_x) = \theta_{P(x)}(dP_x \cdot X_x)$, for each $x \in U \cap V, X_x \in T_xM$. We justify the first term: let $X_x \in T_xM$ and γ be a smooth curve such that $\gamma(0) = x, \dot{\gamma}(0) = X_x$ then:

$$dP_x(X_x) = \frac{\mathrm{d}}{\mathrm{d}t}P(\gamma(t))\bigg|_{t=0}.$$

Furthermore: $t \mapsto P(\gamma(t))$ is a curve $\tilde{\gamma}$ in $GL_n(\mathbb{R})$ that satisfies $\tilde{\gamma}(0) = P(x)$ and $\dot{\tilde{\gamma}}(0) = \frac{\mathrm{d}}{\mathrm{d}t}P(\gamma(t))\Big|_{t=0}$, therefore:

$$(P^*\theta)_x(X_x) = dL_{P(x)^{-1}P(x)}(dP_x(X_x)) = \frac{\mathrm{d}}{\mathrm{d}t}\left(P(x)^{-1}P(\gamma(t))\right) = P(x)^{-1}dP_x(X_x).$$

Up to now, we have yet to discuss what any of this has to do with $L(TM)$. We first need to introduce a little extra material.

4. which also gives a global trivialisation of the tangent bundle of $GL_n(\mathbb{R})$

1.3.2 Local sections of a principal fibre bundle, transition functions

Definition 1.3.2. Let (P, G, π, M) be a principal fibre bundle, and U an open subset of M. A local section of P on U is a smooth map $\sigma_U : U \to P$ such that for each $x \in U$, $\pi(\sigma_U(x)) = x$.

Proposition 1.3.1. *Retaining the notation introduced in the previous definition, any local section on U determines a local bundle chart (ψ, U) and, conversely, every local bundle chart (ψ, U) determines a local section.*

Proof. If σ_U is a local section, define $\psi : \pi^{-1}(U) \to U \times G$ by:

$$\phi(\sigma_U(x) \cdot g) = (x, g).$$

In the above equation, $\sigma_U(\pi(p))$ is used to identify the fibre above $\pi(p) = x$ to G by mapping $\sigma_U(\pi(p))$ to e. Conversely, if (ψ, U) is a local bundle chart, then $\sigma_U(x) = \psi^{-1}(x, e)$ is a local section.

Local sections of $L(TM)$ are closely related to the intuitive notion of moving frames used earlier. To see this, suppose that σ_U is such a local section of $L(TM)$. By definition of $L(TM)$, this is equivalent to a family $(u_x)_{x\in U}$ of linear maps $u_x \in GL(\mathbb{R}^n, T_xM)$. Such a family determines a basis of each T_xM: the image of the canonical basis of $\mathbb{R}^n$. It is clear that a moving frame is equivalent to such a family and therefore to a local section.

Now, if (ϕ, U) and (ψ, V) are two local bundle charts with $U \cap V \neq \emptyset$ and we write $\phi(p) = (\pi(p), s_U(p))$ and $\psi(p) = (\pi(p), s_V(p))$, then for each $x \in U \cap V$, the element $s_V(p)(s_U(p))^{-1} \in G$ is in fact independent of the choice of p in the fibre above x, $\pi^{-1}(\{x\})$. Indeed, if $r = pg$ is another element in the fibre then:

$$s_V(r)s_U(r)^{-1} = s_V(pg)s_U(pg)^{-1} = s_V(p)g(s_U(p)g)^{-1} = s_V(p)s_U(p)^{-1}.$$

This leads to the following definition:

Definition 1.3.3. Using the same notation as above, the smooth map defined for each $x \in U \cap V$ by:

$$g_{VU}(x) = s_V(p)(s_U(p))^{-1},$$

where $p \in \pi^{-1}(\{x\})$ is arbitrary, is called a *transition function.*

The transition function describes how to change the local bundle chart. This can be seen through inspection of the map $\psi \circ \phi^{-1}$ on $(U \cap V) \times G$ given by:

$$\psi \circ \phi^{-1}(x, g) = (x, g_{VU}(x)g).$$

Indeed, for $p = \phi^{-1}(x, e)$, $g_{VU}(x) = s_V(\phi^{-1}(x, e))$ and:

$$\psi \circ \phi^{-1}(x, g) = (x, s_V(\phi^{-1}(x, g))) = (x, s_V(\phi^{-1}(x, e)g)) = (x, s_V(\phi^{-1}(x, e))g).$$

The transition functions satisfy the following properties:

Proposition 1.3.2.

$$g_{UU} = e, \quad g_{UV} = g_{VU}^{-1}, \quad g_{UV}g_{VW}g_{WU} = e. \tag{1.7}$$

It is in fact the case that the transition functions contain all of the information of the fibre bundle; the above properties guarantee that local bundle charts can be glued together appropriately. They are easily interpreted in terms of the local sections σ_U and σ_V determined by (ϕ, U) and (ψ, V): on $U \cap V$, $\sigma_U g_{UV} = \sigma_V$. On $L(TM)$, thinking of local sections as moving frames, g_{UV} is just the change of basis matrix from $\sigma_U(x)$ to $\sigma_V(x)$. We can now reformulate our previous conclusions as follows: an affine connection on $L(TM)$ is equivalent to the data consisting of a matrix-valued differential form ω_U on each local bundle chart (ϕ, U) of $L(TM)$. On the intersection of any two local bundle charts U et V, the forms ω_U and ω_V must satisfy:

$$\omega_V = (g_{UV}^* \theta) + \mathfrak{ad}_{g_{UV}^{-1}}(\omega_U). \tag{1.8}$$

This leads to the first definition of a connection on a principal fibre bundle:

Definition 1.3.4. Let (P, π, M) be a G-principal fibre bundle over M, a connection on P is the data consisting of a differential form ω_U with values in the Lie algebra $\mathfrak{g}$ of G for each local section σ_U of P, such that if two local sections are related by $\sigma_V = \sigma_U g_{UV}$ then:

$$\omega_V = (g_{UV}^* \theta_G) + \mathfrak{ad}_{g_{UV}^{-1}}(\omega_U), \tag{1.9}$$

where θ_G is the Maurer-Cartan form of G.

1.3.3 Vertical vectors

The above definition is very useful for local computations on manifolds, and is used extensively in the Physics literature. However, from a conceptual perspective it is interesting to point out that these local connection forms actually stem from a global object defined on P. To see this we first need to study the projection map π of a G-principal fibre bundle (P, π, M) in more detail.

According to Definition 1.2.1, $\pi : P \to M$ is a smooth surjection. In fact, π is a submersion since, if $p \in P$, one can always find a local section σ_U (cf. Proposition 1.3.1), that satisfies $\sigma_U(\pi(p)) = p$. Such a section is a differentiable right inverse of π and the tangent map $d\sigma_{U x}$ is a right inverse of $d\pi_{\sigma_U(x)} = d\pi_p$. Consequently, $(V_p)_{p\in P}$, where $V_p = \ker d\pi_p$ is a smooth distribution of TP.

Elements of $V_p, p \in P$ will be referred to as *vertical vectors*. Intuitively, they are the vectors in T_pP that have no component in the direction of the basis. In other words, they are the tangent vectors to curves entirely contained in one of the fibres of P.

Definition 1.3.5. Denote the Lie algebra of G by $\mathfrak{g}$ and let $A \in \mathfrak{g}$. A gives rise to a vector field, called the *fundamental field*, A^*, according to the formula:

$$A^*_p = \frac{\mathrm{d}}{\mathrm{d}t}(p\exp(At))\bigg|_{t=0}. \tag{1.10}$$

For fixed p, A^*_p is nothing more than the image of A under the tangent map at e of the map $\mathscr{L}_p : G \to P$, given by $\mathscr{L}_p(g) = pg$. Hence, for fixed p, $A \mapsto A^*_p$ is linear, and we claim that it is injective: suppose that $A^*_p = 0$ for some $A \in \mathfrak{g}$, and let $s \in \mathbb{R}$, then:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(p\exp(tA)\exp(sA)\right)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(p\exp((t+s)A)\right)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(p\exp(tA)\right)\bigg|_{t=s}$$

It follows that $s \mapsto p\exp(sA)$ is the constant map $s \mapsto p$. Since the group G acts freely on P necessarily $\exp(sA) = e$ for every $s \in \mathbb{R}$. However, $\exp : \mathfrak{g} \to G$ is a local diffeomorphism near 0, so $A = 0$. Last of all, $A^*_p \in V_p$ for each p, since:

$$d\pi_p(A^*_p) = \frac{\mathrm{d}}{\mathrm{d}t}(\pi(p\exp(At)))\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\pi(p)\bigg|_{t=0} = 0.$$

Consequently, this induces a vector space isomorphism $V_p \cong \mathfrak{g}$, and at each $p \in P$ we have the following short exact sequence:

$$0 \to \mathfrak{g} \to T_pP \to T_{\pi(p)}M \to 0.$$

A local section σ_U right splits the sequence:

$$0 \to \mathfrak{g} \to T_{\sigma_U(x)}P \to T_xM \to 0,$$

so that we have the isomorphism: $T_{\sigma_U(x)}P \cong \mathfrak{g} \oplus T_xM$.

1.3.4 Connection as a $\mathfrak{g}$-valued differential form on P

We can now introduce the second definition of a connection that we alluded to at the beginning of the preceding paragraph. To illustrate the construction, we return to the example of an affine connection; our arguments are, however, completely independent of this choice and can be applied to any connection as defined by Definition 1.3.4.

Suppose that we have a local connection form ω_U on each local bundle chart (ϕ, U) of $L(TM)$ and that they are compatible in the sense of (1.8) on the intersection of any two charts. For any given chart (ϕ, U), let σ_U denote the section of $L(TM)$ given by $\sigma_U(x) = \phi^{-1}(x, e)$. In light of the concluding remarks of the previous paragraph, for each $x \in U$ we define a linear map: $\omega_{\sigma_U(x)} : T_{\sigma_U(x)}L(M) \to M_n(\mathbb{R})$ by:

$$\omega_{\sigma_U(x)}(d\sigma_{Ux}(Y) + A^*_{\sigma_U(x)}) = \omega_{Ux}(Y) + A,\ \ Y \in T_xM. \tag{1.11}$$

We then extend the definition to other points in the fiber above x by imposing the equivariance:

$$R_g^*\omega = \mathfrak{ad}_{g^{-1}}\omega, \tag{1.12}$$

where R_g is the smooth map $p \mapsto pg$. This means that for any $p \in \pi^{-1}(\{x\}), g \in GL_n(\mathbb{R})$:

$$\omega_{pg}(dR_g(X)) = \mathfrak{ad}_{g^{-1}}\omega_p(X),\ \ X \in T_pP. \tag{1.13}$$

In this way we get a smooth matrix valued differential form on the bundle $\pi^{-1}(U)$, that, additionally, satisfies:

$$\omega(A^*) = A. \tag{1.14}$$

This can be seen in the following manner. Let $p \in \pi^{-1}(U)$ and set $x = \pi(p)$. One can find

$g \in GL_n(\mathbb{R})$ such that $p = \sigma_U(x)g$, hence:

$$\omega_p(A^*_p) = \mathfrak{ad}_{g^{-1}}\omega_{\sigma_U(x)}(dR_{g^{-1}}{}_p(A^*_p)).$$

Moreover, for any $A \in \mathfrak{g}$:

$$\begin{aligned} dR_{g^{-1}}{}_p(A^*p) = \frac{\mathrm{d}}{\mathrm{d}t}\left(p\exp(At)g^{-1}\right)\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(pg^{-1}g\exp(At)g^{-1}\right)\Big|_{t=0}, \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(pg^{-1}\exp(\mathfrak{ad}_g At)\right)\Big|_{t=0}, \\ &= (\mathfrak{ad}_g A)^*_{\sigma_U(x)}. \end{aligned}$$

Consequently: $\omega_p(A^*_p) = \mathfrak{ad}_{g^{-1}}\mathfrak{ad}_g A = A$.

In order to complete the construction of a differential form on $L(TM)$, it remains to show that the definition is consistent on the intersection of any two bundle charts U, V. Let (ϕ, U) and (ψ, V) be two local bundle charts and denote by σ_U, σ_V the associated local sections. Finally, we call ω^U and ω^V the forms on $\pi^{-1}(U)$ and $\pi^{-1}(V)$ respectively, constructed as described above from local connection forms ω_U and ω_V on U and V respectively.

The invariance (1.12) implies that it is sufficient to check that for $x \in U \cap V$: $\omega^V_{\sigma_V(x)} = \omega^U_{\sigma_V(x)}$. Additionally, (1.14) implies that we only need to show that for any vector field $Y \in T_xM$:

$$\omega^V_{\sigma_V(x)}(d\sigma_V{}_x(Y)) = \omega^U_{\sigma_V(x)}(d\sigma_V{}_x(Y)).$$

Let us evaluate the right-hand side:

$$\begin{aligned} \omega^U_{\sigma_V(x)}(d\sigma_V{}_x(Y)) &= \omega^U_{\sigma_U(x)g_{UV}(x)}(d(\sigma_U g_{UV})_x(Y)), \\ &= \omega^U_{\sigma_U(x)g_{UV}(x)}\left(d\mathscr{L}_{\sigma_U(x)}{}_{g_{UV}(x)}(dg_{UV}{}_x(Y)) + dR_{g_{UV}(x)}{}_x(d\sigma_U{}_x(Y))\right), \\ &= \mathfrak{ad}_{g_{UV}(x)^{-1}}\omega^U_{\sigma_U(x)}(d\sigma_U{}_x(Y)) + \omega^U_{\sigma_U(x)g_{UV}(x)}\left(d\mathscr{L}_{\sigma_U(x)}{}_{g_{UV}(x)}(dg_{UV}{}_x(Y))\right). \end{aligned}$$

In the last two lines we recall that $\mathscr{L}_p, p \in L(TM)$ is defined by: $g \mapsto pg$. Let us focus our attention on the second term of the final equation. Choose a curve γ on M such that

$\gamma(0) = x$ and $\dot{\gamma}(0) = Y$, and note that:

$$\begin{aligned} d\mathscr{L}_{\sigma_U(x) g_{UV(x)}}(dg_{UV\,x}(Y)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sigma_U(x) g_{UV}(\gamma(t))\right)\Big|_{t=0}, \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sigma_V(x) g_{UV}(x)^{-1} g_{UV}(\gamma(t))\right)\Big|_{t=0}. \end{aligned} \tag{1.15}$$

From the final equation we see the left-hand side is actually a tangent vector to a curve in the fibre of $L(M)$ containing $\sigma_V(x)$; in other words it is a vertical vector field. To determine to which element $A \in \mathfrak{gl}_n(\mathbb{R})$ it corresponds, we only need to calculate the derivative at 0 of the curve in $GL_n(\mathbb{R})$ given by $t \mapsto g_{UV}(x)^{-1} g_{UV}(\gamma(t))$. The result is: $dL_{g_{UV}(x)^{-1} g_{UV}(x)}(dg_{UV\,x}(Y)) = (g_{UV}^* \theta)_x(Y)$. Therefore, using (1.8):

$$\begin{aligned} \omega^U_{\sigma_V(x)}(d\sigma_{V\,x}(Y)) &= \mathfrak{ad}_{g_{UV}(x)^{-1}} \omega_{U\,x}(Y) + (g_{UV}^* \theta)_x(Y), \\ &= \omega_{V\,x}(Y). \end{aligned} \tag{1.16}$$

Since ω^U et ω^V coincide on their common domain of definition, we have thus shown that they can be glued together to yield a matrix valued differential form on $L(TM)$. This leads to the following definition:

Definition 1.3.6. Let (P, G, π, M) be a smooth principal fibre bundle. A connection on P is a smooth $\mathfrak{g}$-valued differential form ω that satisfies:

1. $\omega(A^*) = A$,
2. $R_g^* \omega = \mathfrak{ad}_{g^{-1}} \omega$.

As remarked above, our previous computations generalise without essential modification to the case of an arbitrary Lie group G with Lie algebra $\mathfrak{g}$. Furthermore, they are the hard part of the proof that Definition 1.3.6 is equivalent to Definition 1.3.4. To obtain ω_U from ω, it suffices to pullback along the local section σ_U [5]: $\omega_U = \sigma_U^* \omega$.

1.3.5 Principal connection as a G-equivariant horizontal distribution

To close this discussion on principal connections, we would like to mention that there is a third equivalent definition, which is conceptually useful:

5. This is quite natural given the short exact sequences at the end our discussion on vertical vectors.

Definition 1.3.7. A principal connection on principal fibre bundle (P, G, π, M) is a smooth G-equivariant distribution complementary to the vertical vector distribution i.e. a family of vector subspaces $(H_p)_{p \in P}, H_p \subset T_p P$ such that for any $p \in P, g \in G$.

1. $T_p P = H_p \oplus V_p$,
2. $(dR_g)(H_p) = H_{pg}$. (equivariance)
3. The association $p \mapsto H_p$ is smooth in the sense that on a neighbourhood of any point $p \in P$ one can find n vector fields that generate H_q at each point q in the neighbourhood.

For a proof of the equivalence with our previous definitions, we refer the reader to [Nab10; Ble05].

1.4 Associated bundles

Throughout this work we will tend to think of principal bundles as a general frame bundle. The following construction further justifies this point of view and is probably one of the most important applications of principal fibre bundles used in this thesis.

To introduce our point of view, let us borrow, as we often do, a situation from Physics. Imagine that we would like to describe the movement of a particle in Newton's absolute space. To do this, we set up a frame of reference and a system of cartesian coordinates (x, y, z). With respect to this frame of reference, we measure a certain number of physical quantities, of different nature such as the position, velocity or angular momentum, associated with the movement of the particule. These quantities can each be represented as one or a collection of real numbers. It is natural to ask, how can we compare our measurements with those carried out in another frame of reference ? Do these numbers change, and if so, how ? To simplify things a little, assume that the reference frames are anchored in the same point so that the transformation group is simply the familiar group of rotations $SO(3)$. In this situation, we will find that if we rotate our frame using $A \in SO(3)$, then the 3 real numbers that describe the velocity $(v_i)_{i\in\{1,2,3\}}$ change according to the following rule: $\tilde{v}^i = (A^{-1})^i_j v^j$. In view of this, we will say that the velocity is a « vector » quantity; similarly, the mass, which will not change, will be said to be a « scalar » quantity. In other words, we classify quantities according to the representation of the rotation group to which they correspond.

In General Relativity, tensor quantities of this type are still used to describe systems, the difference being that the transformation group applies to local frames. The machinery that we are now going to describe, enables us to construct a vector bundle over a manifold M, given any G-principal fibre bundle over M - which contains the information about local frames - and a linear representation (V, ρ) of the group G. We will denote this vector bundle by $P \times_G V$, the fiber above any point of this bundle is a vector space of quantities with the transformation rule prescribed by the representation. More generally, an adaptation of our discussion enables the definition of associated bundles $P \times_G X$ for a number of differentiable actions of the group G on a manifold X.

1.4.1 Constructing an associated vector bundle

Let (P, G, π, M) be a G-principal bundle over M and (V, ρ) a smooth finite dimensional linear representation (to simplify: $V \approx \mathbb{K}^n, \mathbb{K} = \mathbb{R}, \mathbb{C}$) of G. The construction of $P \times_G V$

is largely inspired by our usual experience of tensor fields. Consider the product manifold $P \times V$ and the smooth left action of G on $P \times V$ given by:

$$\begin{array}{ccc} G \times (P \times V) & \longrightarrow & P \times V \\ (g,(r,v)) & \longmapsto & (rg, \rho(g)^{-1}v). \end{array} \tag{1.17}$$

$P \times_G V$ is simply defined as the quotient space $(P \times V)/G$.

To motivate this choice, it is illuminating to think of a pair (r, v) as the data consisting of a local frame r and the coordinates of a vector v in that frame. Multiplying from the right a frame by an element $g \in G$ corresponds to changing frame (g is the change of basis matrix) and, accordingly, the components of the vector v should change according to the usual change of basis rule in linear algebra.

The proof of the fact that $P \times_G V$ is a vector bundle over M with fibre V is often left as an exercise in the literature. For the benefit of impatient readers, we sketch the proof below. To simplify notations a little we introduce the notation $E = P \times_G V$. Let $\mathsf{p} : P \times V \longrightarrow P \times_G V$ be the canonical projection. Let (ϕ, U) be a local bundle chart and suppose that $p \in U, \phi(p) = (\pi(p), s_u(p))$. Shrinking, if necessary, U we can assume that it is the defining domain of a local chart (x, U) on M. Set now:

$$\begin{array}{cccc} \overline{x}: & \pi^{-1}(U) \times V & \longrightarrow & x(U) \times V \\ & (r,v) & \longmapsto & (x(\pi(r)), \rho(s_u(r))v). \end{array}$$

Observe that it factors to a map: $\tilde{\overline{x}} : \mathsf{p}(\pi^{-1}(U) \times V) \longrightarrow x(U) \times V$. Indeed, if $(r, v) \sim (r', v')$ then one can find $g \in G$ such that $r' = rg$ and $v' = \rho(g)^{-1}v$. Hence:

$$\begin{cases} x(\pi(r')) = x(\pi(rg)) = x(\pi(r)), \\ \rho(s_u(r'))v' = \rho(s_u(rg))\rho(g^{-1})v = \rho(s_u(r)g)\rho(g)^{-1}v = \rho(s_u(r))\rho(g)\rho(g)^{-1}v = \rho(s_u(r))v. \end{cases}$$

We will now show that the maps $\tilde{\overline{x}}$ are a bundle atlas for E. This relies on two lemmata:

Lemma 1.4.1. *Let G be a topological group and $(X, \mathscr{T})$ a topological space on which G acts continuously from the left, i.e. the map $(g, x) \mapsto g \cdot x$ is continuous for the product topology. In this case, if O is an open set in X and $A \subset G$, then the set $A \cdot O = \{a \cdot x, a \in A, x \in O\}$ is open.*

Proof. For any fixed $g \in G$, the map $x \mapsto g \cdot x$ is a homeomorphism with inverse $x \mapsto g^{-1} \cdot x$; in particular it is an open map. Hence, $g \cdot O$ is open. The result follows for an arbitrary subset $A \subset G$ by an arbitrary union of open sets.

Lemma 1.4.2. *The canonical projection map* p *is open.*

Proof. Let O be an open subset of $P \times V$. By definition of the quotient topology the set $\mathsf{p}(O)$ is open if and only if $\mathsf{p}^{-1}(\mathsf{p}(O))$ is open, but:

$$\begin{aligned}\mathsf{p}^{-1}(\mathsf{p}(O)) &= \{(r,v) \in P \times V, \mathsf{p}(r,v) \in \mathsf{p}(O)\ \}\,,\\ &= \{(r,v) \in P \times V, \exists (r',v') \in O, \mathsf{p}(r,v) = \mathsf{p}(r',v')\}\,,\\ &= \left\{(r,v) \in P \times V, \exists (r',v') \in O, \exists g \in G, (r,g) = (r'g, \rho(g)^{-1}v)\right\}\,,\\ &= g \cdot O.\end{aligned}$$

Since the action defined by (1.17) is continuous, we conclude by the previous Lemma.

It follows that the defining domain of $\tilde{\bar{x}}$, $\mathsf{p}(\pi^{-1}(U) \times V)$, is an open subset of E. Define now the surjective map:

$$\begin{array}{rccc}\bar{\pi}: & P \times V & \longrightarrow & M\\ & (p,v) & \longmapsto & \pi(p),\end{array}$$

which factors, like $\bar{x}$, to a surjective map: $\tilde{\bar{\pi}} : E \longrightarrow M$. Moreover:

$$\bar{\pi}^{-1}(U) = \mathsf{p}^{-1}(\tilde{\bar{\pi}}^{-1}(U)).$$

Therefore, since p is surjective, one has:

$$\mathsf{p}(\bar{\pi}^{-1}(U)) = \tilde{\bar{\pi}}^{-1}(U),$$

furthermore, $\bar{\pi}^{-1}(U) = \pi^{-1}(U) \times V$, thus:

$$\tilde{\bar{\pi}}^{-1}(U) = \mathsf{p}(\pi^{-1}(U) \times V).$$ [6]

Overall: $\tilde{\bar{x}} : \tilde{\bar{\pi}}^{-1}(U) \longrightarrow x(U) \times V \subset \mathbb{R}^n \times V$. To show that $\tilde{\bar{x}}$ is itself a homeomorphism, we describe explicitly its inverse. Choose an arbitrary element $g \in G$ and set:

$$\phi : \begin{array}{ccc} x(U) \times V & \longrightarrow & \tilde{\bar{\pi}}^{-1}(U) = p(\pi^{-1}(U) \times V)\\ (a,b) & \longmapsto & \mathsf{p}(\phi^{-1}(x^{-1}(a), g), \rho(g)^{-1} \cdot b)).\end{array} \tag{1.18}$$

6. This is actually sufficient to prove that it is open.

As a composition of continuous functions, ϕ is continuous and is actually independent of the choice of g [7]. Moreover:

$$\phi(\tilde{\bar{x}}(p(r,v))) = \phi((x(\pi(r)), \rho(s_u(r))v)) = p(\phi^{-1}(\pi(r),g), \rho(g)^{-1}\rho(s_u(r))v)).$$

Define $r' = \phi^{-1}(\pi(r),g)$, clearly: $r' \in \pi^{-1}(\pi(r))$ so r' and r are in the same orbit, hence we can find $g' \in G$ satisfying $r' = rg'$ i.e. $r = r'g'^{-1}$. By consequence:

$$\begin{aligned}
\mathsf{p}(\phi^{-1}(\pi(r),g), \rho(g)^{-1}\rho(s_u(r))v)) &= \mathsf{p}(r', \rho(g)^{-1}\rho(s_u(r'g'^{-1})v) \\
&= \mathsf{p}(rg', \rho(g)^{-1}\rho(\underbrace{s_u(r')}_{g})\rho(g'^{-1})v), \\
&= \mathsf{p}(rg', \rho(g')^{-1}v).
\end{aligned}$$

Hence:

$$\phi(\tilde{\bar{x}}(\mathsf{p}(r,v))) = p(rg', \rho(g')^{-1}v) = \mathsf{p}(r,v),$$

and ϕ is a right inverse of $\tilde{\bar{x}}$. Let us compute now $\tilde{\bar{x}}(\phi(a,b))$:

$$\begin{aligned}
\tilde{\bar{x}}(\phi(a,b)) &= \tilde{\bar{x}}(\mathsf{p}(\phi^{-1}(x^{-1}(a),g), \rho(g)^{-1}b)), \\
&= \bar{x}(\phi^{-1}(x^{-1}(a),g), \rho(g)^{-1}b), \\
&= (a, \rho(g)\rho(g)^{-1}b) = (a,b).
\end{aligned}$$

Consequently, ϕ is the inverse map to $\tilde{\bar{x}}$, which is a homeomorphism. If we consider the maximal atlas containing all charts constructed in this manner, it is not difficult to see that this is a smooth structure for E which is hence a smooth manifold of dimension $\dim M + \dim V$. Furthermore, studying the form of these charts, one can see that E is a vector bundle over M with model fibre V. For instance, a fibre $V_q = \tilde{\bar{\pi}}^{-1}(\{x\}) = \mathsf{p}(\bar{\pi}^{-1}(\{q\})) = \mathsf{p}(\pi^{-1}(\{q\}) \times V), q \in M$ is naturally equipped with a vector space structure induced by that of $\{r\} \times V$ for each $r \in \pi^{-1}(\{x\})$. Indeed, if $\mathfrak{v}_1 = \mathsf{p}(r,v)$ and $\mathfrak{v}_2 = \mathsf{p}(r',v')$ are two elements of V_q, then, since $r' \in \pi^{-1}(\{\pi(r)\} = \{q\})$, one can find $g \in G$ such that $r = r'g$. In particular: $\mathfrak{v}_2 = (r, \rho(g)^{-1}v')$. Consequently we set:

$$\mathfrak{v}_1 + \mathfrak{v}_2 = \mathsf{p}((r,v) + (r, \rho(g)^{-1}v')) = \mathsf{p}(r, v + \rho(g)^{-1}v'),$$

7. It would have been simpler to take directly $g = e$.

this is independent of the choice of representative and so we have a well-defined addition $+$. Analogously, if $\lambda \in \mathbb{K}$, we set:

$$\lambda \cdot \mathfrak{v}_1 = \mathsf{p}(\lambda \cdot (r, v)) = \mathsf{p}((r, \lambda v)).$$

Once more, linearity of the representation can be used to show that this definition is independent of the choice of (r, v) in the class $\mathsf{p}(r, v)$:

$$\mathsf{p}((r, \lambda v)) = \mathsf{p}(rg, \rho(g)^{-1}(\lambda v)) = \mathsf{p}(rg, \lambda \rho(g)^{-1} v).$$

This linear structure is chosen so that $\tilde{\phi}_q = \tilde{\tilde{x}}_{|\tilde{\pi}^{-1}(\{q\})} : V_q \longrightarrow \{x(q)\} \times V$ becomes a vector space isomorphism:

$$\begin{aligned}
\tilde{\phi}_q(\lambda \mathfrak{v}_1 + \mu \mathfrak{v}_2) = \tilde{\tilde{x}}(\mathsf{p}(r, \lambda v_1 + \mu v_2)) &= (x(q), \rho(s_u(r))(\lambda v_1 + \mu v_2)), \\
&= (x(q), \lambda \rho(s_u(r)).v_1 + \mu \rho(s_u(r)) v_2)), \\
&= \lambda(x(q), \rho(s_u(r)).v_1) + \mu(x(q), \rho(s_u(r)) v_2), \\
&= \lambda \tilde{\phi}_q(\mathfrak{v}_1) + \mu \tilde{\phi}_q(\mathfrak{v}_2); \quad \lambda, \mu \ \in \mathbb{K}, \mathfrak{v}_i \in V_q.
\end{aligned}$$

We omit the proof that the action is proper which implies that $P \times_G V$ is indeed Hausdorff.

1.4.2 Examples of associated bundles

We will now give some examples of associated bundles, the first of which are the usual tangent and cotangent bundles TM et T^*M:

1. TM is the associated bundle to $L(TM)$ corresponding to the fundamental representation of $GL_n(\mathbb{R})$,
2. T^*M is the associated bundle $L(TM)$ corresponding to the dual or contragredient representation: $P \mapsto {}^tP^{-1}$,
3. more generally, the bundle of tensors of type (p, q) is the associated bundle to $L(TM)$ corresponding to the representation $(\rho, \otimes^{p+q}\mathbb{R}^n)$, defined by:

$$\rho(P) v_1 \otimes \cdots \otimes v_p \otimes l^1 \otimes \cdots \otimes l^q = (Pv_1) \otimes \cdots \otimes (Pv_p) \otimes ({}^tP^{-1}l^1) \otimes \cdots \otimes ({}^tP^{-1}l^q).$$

The following example is important for Chapter 4, so we present it as a definition:

Definition 1.4.1. Let $\omega \in \mathbb{C}$, a projective density of weight ω [8] is a smooth section of the bundle $\mathcal{E}(\omega)$ associated to $L(TM)$ with the representation of $GL_n(\mathbb{R})$ on $\mathbb{R}$ given by:

$$\begin{array}{rcl} GL_n(\mathbb{R}) \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (A, v) & \longmapsto & |\det A|^{\frac{\omega}{n+1}} v. \end{array}$$

If $\mathscr{B}$ is any fibre bundle on M, we will write $\mathscr{B}(\omega)$ for $\mathscr{B} \otimes \mathcal{E}(\omega)$.

Note that $\mathcal{E}(\omega)$ is a trivial line bundle; one can construct a non-zero global section using a partition of unity.

1.4.3 Covariant derivative on an associated vector bundle

A principal connection ω on a principal fibre bundle (P, G, π, M) induces an affine connection in the sense of Definition 1.3.1 on any associated vector bundle. In Physics literature this is often referred to as a covariant derivative. The following definition will help us simplify the construction:

Definition 1.4.2. Let (P, G, π) and (P', G', π') be two fibre bundles over M and $\phi : P \to P'$ be a smooth map. ϕ is a principal bundle morphism if:

1. there is a group homomorphism $\varphi : G \to G'$ such that for any $p \in P, g \in G$:

$$\phi(pg) = \phi(p)\varphi(g),$$

2. the following diagram is commutative:

$$\begin{tikzcd} P \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd} .$$

Presently, an interesting example is given by the following lemma:

Lemma 1.4.3. *Let (P, G, π, M) be a principal fibre bundle and $P \times_G V$ an associated vector bundle corresponding to a representation (V, ρ) of G. In this case, there is a principal bundle morphism $P \longrightarrow L(P \times_G V)$ where $L(P \times_G V)$ is the frame bundle of $P \times_G V$*

8. We may also speak of densities of projective weight ω

(cf. Paragraph 1.2.2) given by:

$$\begin{aligned} \Phi: \quad P &\longrightarrow L(P \times_G V) \\ r &\longmapsto (\pi(r), u_r). \end{aligned} \tag{1.19}$$

In the above, $u_r \in GL(V, (P \times_G V)_{\pi(r)})$ is defined by $v \mapsto p(r, v)$ and p is the canonical projection from $P \times V$ onto $P \times_G V$.

Proof. We simply check that: $\Phi(rg) = \Phi(r)\rho(g)$; for which it is sufficient to show that $u_{rg} = u_r \circ \rho(g)$. For each $r \in P, g \in G$: $u_{rg}(v) = p(rg, v) = p(r, \rho(g)v) = u_r(\rho(g)(v))$.

It follows that a local section σ_U of P, gives rise to a local section of $L(P \times_G V)$ according to the formula $\tilde{\sigma}_U = \Phi \circ \sigma_U$. Furthermore, the tangent map to the representation morphism ρ at the identity of G induces a Lie algebra representation of $\mathfrak{g}$ that we will call ρ_*. The local connection form $\omega_U = \sigma_U^* \omega$ induces a local connection form on $L(P \times_G V)$ according to: $\tilde{\omega}_U = \rho_* \omega_U$.

To convince ourselves that this is indeed a connection, let us verify that (1.8) holds for $\tilde{\omega}$. Let σ_V be another local section such that $U \cap V \neq \emptyset$. Since Φ is a principal fibre bundle morphism, $\tilde{\sigma}_V = \Phi(\sigma_U g_{UV}) = \Phi(\sigma_U)\rho(g_{UV})$ and:

$$\tilde{\omega}_V = \rho_* \omega_V = \rho_*(g_{UV}^* \theta) + \rho_* \mathfrak{ad}_{g_{UV}^{-1}}(\omega_U).$$

The result follows from:

$$\begin{aligned} \rho_* \mathfrak{ad}_{g_{UV}^{-1}}(\omega_U) &= d\rho_e \circ d(Ad_{g_{UV}^{-1}})_e(\omega_U), \\ &= d(\rho \circ Ad_{g_{UV}^{-1}})_e(\omega_U), \\ &= d(Ad_{\rho(g_{UV})^{-1}} \circ \rho)_e(\omega_U) = \mathfrak{ad}_{\rho(g_{UV})^{-1}}(\tilde{\omega}_U), \end{aligned}$$

and:

$$\begin{aligned} \rho_* g_{UV}^* \theta &= d\rho_e \circ dL_{g_{UV}^{-1}\, g_{UV}}(dg_{UV}), \\ &= d(\rho \circ L_{g_{UV}^{-1}})_{g_{UV}}(dg_{UV}), \\ &= dL_{\rho(g_{UV})^{-1}}{}_{\rho(g_{UV})}(d\rho_{g_{UV}} \circ dg_{UV}), \\ &= dL_{\rho(g_{UV})^{-1}}{}_{\rho(g_{UV})}(d\rho(g_{UV})). \end{aligned}$$

1.4.4 Fibre bundle reductions and geometric properties

The final topic that we wish to touch upon briefly in this walkthrough on Principal fibre bundles is that of fibre bundle reductions. Our interest in this is mainly conceptual: orientation or metric tensors can be reinterpreted as reductions of $L(M)$ to an H-principal fibre bundle where $H \subset GL_n(\mathbb{R})$ is a closed subgroup.

Definition 1.4.3. Let H be a closed subgroup of a Lie group G, (P, π, M) a G-principal fibre bundle and $\iota : H \to G$ the canonical injection. A reduction of P is a H-principal fibre bundle (P', π', M) and a principal bundle morphism ϕ such that: $\phi(qh) = \phi(q)\iota(h), q \in P', h \in H$. In this case: ϕ is an immersion and we have the isomorphism $P' \times_H G \cong P$ where H acts on G by left multiplication.

The existence of a reduction is generally subject to topological constraints. For instance, if $G = GL_n(\mathbb{R})$, $H = \{\mathrm{Id}\}$ and $P = L(TM)$ then there is a reduction P' if and only if M is parallelisable. In a similar fashion, if $H = GL_n(\mathbb{R})^+$, the subgroup of matrices with positive determinant, then the existence of a reduction is equivalent to orientability. This is more easily seen when thinking in terms of an orientation atlas: one can construct the bundle P' by imitating the method described in Paragraph 1.2.2 but restricting to positively oriented frames: the orientation atlas is exactly what is needed to guarantee that this is possible. In terms of transition functions, a reduction translates to the possibility to restrict their values to a subgroup. Finally, a choice of metric is equally equivalent to a reduction of the frame bundle which restricts to orthonormal frames: one can choose a covering such that the transition functions take their values in $O(p, q)$. If furthermore M is orientable, one can work with positively oriented orthonormal frames.

Chapter 2

Kerr-de Sitter spacetimes

2.1 Preamble

The original aim of this part of my work was to give a classification of the different types of geometries within the Kerr-de Sitter family in order to formulate a precise definition of the so-called « extreme » case. This was in preparation for the scattering theory to be constructed in Chapter 3 and to the author's knowledge the result given here was absent from the literature.

As with all such classifications for analytical black-holes it is based on the study of the roots of a polynomial depending on the physical parameters of the family. In the case of Kerr-de Sitter spacetimes, excluding the scalar-flat case, the polynomial is of degree 4. The black-hole is called « extreme » when one of the roots is of multiplicity greater than one; we determine here the conditions on the parameters for this to occur. This is achieved solely through algebraic methods.

The significance of these roots is that they correspond to poles in the coefficients of the black-hole metric. However, these poles are not true physical singularities since it is possible to analytically extend the manifold through them, obtaining hypersurfaces referred to as « horizons ». The author's interest in understanding precisely what this means caused the work to grow into a larger article that addressed the extension problem that we describe below. Although its solution is considered known by the community, no mathematical account existed in the literature and this work fills that hole.

Often, a black-hole geometry arises as a solution to Einstein's field equations with apparent singularities in the metric. For instance, the Schwarzschild blackhole (with mass m) is $M = \mathbb{R}_t \times]0, +\infty[_r \times S^2$, equipped with the metric g:

$$g = \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\sigma^2 \tag{2.1}$$

where $\mathrm{d}\sigma^2$ is the usual round metric on S^2.

The expression of g is clearly problematic at points $p \in M$ such that $r(p) = 0$ or $r(p) = 2m$. So at this stage, it is only really sensible to consider it separately on the open subsets $U_1 = \mathbb{R}\times]0, 2m[\times S^2$ and $U_2 = \mathbb{R}\times\ ]2m, +\infty[\times S^2$ of $\mathbb{R} \times \mathbb{R}_+^* \times S^2$. We therefore have two Lorentzian manfiolds $(U_1, g_1 = g|_{U_1})$ and $(U_2, g_2 = g|_{U_2})$ which, to begin with, are unrelated. It is nevertheless natural to wonder if it is possible to find another Lorentzian manifold $(N, \tilde{g})$, in which we can isometrically embed each of the manifolds $(U_i, g_i), i \in \{1, 2\}$ and such that the metric $\tilde{g}$ extends g to one or several points where it is not defined. Ideally, we would also like the values of $\tilde{g}$ to be completely determined by those of g near these points. We will study this extension problem in the category of analytical [1] Lorentzian manfiolds. Consequently, if there is a solution, it is unique by analytical continuation.

In the case of Schwarzschild's metric (2.1), the extension problem posed above has a solution for points where $r = 2m$. It can be constructed by performing the following change of coordinates :

$$\begin{array}{rcl} \mathbb{R}\times]2m, +\infty[\times S^2 & \longrightarrow & \mathbb{R}\times]2m, +\infty[\times S^2 \\ (t, r, \vartheta) & \longmapsto & (t - F(r), r, \vartheta) \end{array}, \tag{2.2}$$

where $F(r) = r + 2m \ln(|r - 2m|)$. In these new coordinates (t', r', ϑ') - known as the outgoing Eddington-Finkelstein chart - the metric g has the form :

$$\left(1 - \frac{2m}{r'}\right) \mathrm{d}t'^2 + 2\mathrm{d}r'\mathrm{d}t' - r'^2 \mathrm{d}\sigma^2, \tag{2.3}$$

which is regular at all points $\{r' = 2m\}$. One can then define the Lorentzian manifold $N = \mathbb{R} \times \mathbb{R}^*{}_+ \times S^2$, with metric given by (2.3) and it is easily seen that (U_1, g_i) and (U_2, g_i) can be isometrically embedded into N. The function r' analytically extends r, and, in N, $\{r' = 2m\}$ is an isotropic hypersurface, which is referred to as an « event horizon ». The situation is not as favourable at $r = 0$: inspecting geodesics, one can show that there is no analytic solution to the extension problem there and in this sense it is a true geometric singularity.

Besides the apparent singularities in the metric, the Schwarzschild blackhole as defined above has another drawback : a number of its geodesics are incomplete, i.e. there are maximal solutions to the geodesic equation that are not defined over all of $\mathbb{R}$. Given their classical physical interpretation, it seems natural to attempt to extend again the manifold

1. We restrict to analytical transition maps, and assume that the metric components are analytical

to remedy this, at least for geodesics that do not run into the singularity. If one can find an extension that satisfies this additional condition, it will be said to be a *maximal* extension. Our extension N above, is not maximal in this sense, but there are global coordinates that do lead to a maximal extension. They are known as the Kruskal-Szekeres coordinates, the interested reader can find the details of this in [Wal84].

This work lead to a publication [Bor18] in *Classical and Quantum Gravity* available at : `https://iopscience.iop.org/article/10.1088/1361-6382/aae3dc`.

Maximal Kerr-de Sitter spacetimes

The following is the author's final preprint of the eponymous article [Bor18] published in Vol. 35 No. 21 of Classical and Quantum Gravity.

2.2 Introduction

Over the past decade or so, there has been increasing interest in asymptotically de Sitter spacetimes, as opposed to the better-studied asymptotically flat spacetimes, notably due to the experimental evidence that our universe is actually in expansion, and that this expansion is accelerating. De Sitter spacetime, named after the Dutch mathematician and astronomer Willem de Sitter, is one of the simpler models of such a universe. It can be seen as the submanifold of equation $-x_0^2 + \sum_i^{n+1} x_i^2 = \alpha^2, \alpha \in \mathbb{R}$ in $(n+2)$-dimensional Minkowski space and is a maximally symmetric vacuum solution to Einstein's equation with positive cosmological constant $\Lambda = \frac{3}{\alpha^2}$; the parameter α is also related to the Ricci scalar by $R = \frac{n(n-1)}{\alpha^2}$. In this paper, we are interested in 4-dimensional Kerr-de Sitter spacetimes describing a rotating black hole on a de-Sitter background. These solutions where first discussed by Brandon Carter [Car09], but more thorough studies of them, and in particular of the structure of the roots of the polynomial Δ_r according to the values of the parameters a, l and M, have been delayed, until recently, due to its supposed more geometrical than physical significance. In recent articles, several authors have shown interest in Kerr-de Sitter spacetimes, and a numerical study is proposed in [AM11].

In this work we give complete and relatively simple characterisations of the Kerr-de Sitter analogs of "fast", "extreme" and "slow" Kerr spacetime and describe in detail the construction of a maximal analytical extension of the Kerr-de Sitter solution in each case. The text is organised as follows: in Section 2.3 we give a succinct description of the geometric properties of the Kerr-de Sitter metric in Carter's Boyer-Lindquist like coordinates; the principal result of interest is the computation of the curvature forms ${\Omega^i}_j$. Following [GH77; AM11], the sign convention for Λ is opposite to that in Carter's original work. In Section 2.4, we discuss the root structure of the family of polynomials Δ_r according to the values of the parameters (a, l, M). After writing this article, we discovered that a similar study had already been lead in [LZ15]; our results confirm and complete theirs. In Section 2.5, we describe the construction of maximal Kerr-de Sitter spacetimes, the criterion for maximality being the completeness of all principal null geodesics that do not run into a curvature singularity. The results of Section 2.3 confirm the fact that only minor adaptations of the methods used in [ONe14] are required, however, some of the proofs are repeated and complements are provided in appendices so that the text is as self-contained as possible. We decided not to discuss more general geodesics than the principal nulls used in the construction of maximal extensions, but found that recent articles had ventured into this terrain: a classification of null geodesics is proposed in [CS17] and a discussion on all causal geodesics is given in [ZS17].

The signature convention used in this work is $(-,+,+,+)$ and, when units are relevant, formulae are written in geometric units where $G=1$ and $c=1$.

2.3 The Kerr-de Sitter metric

In this section we will define the Kerr-de Sitter (KdS) metric g and calculate the curvature forms ${\Omega^i}_j$ on each of the so-called "Boyer-Lindquist blocks" in an appropriate frame. The algebraic structure of the curvature tensor encoded in these forms will show that, like that of the Kerr metric, the Weyl tensor of the Kerr-de Sitter metric is of Petrov type D at each point of these blocks.

The components g_{ij} of the Kerr-de Sitter metric on the connected components of the manifold $(\mathbb{R}_t \times \mathbb{R}_r) \times S^2 \setminus \Sigma \cup \mathcal{H}$, $\mathcal{H} = \{\Delta_r = 0\}, \Sigma = \{\rho^2 = 0\}$, referred to as the Boyer-Lindquist (BL) blocks, are given in table 2.1; some useful alternative expressions are also given in appendix A.4. When $l = 0$, these expressions reduce to those of the usual Kerr metric. The coordinates (t, r, θ, ϕ) will be referred to as Boyer-Lindquist(-like) coordinates. The parameters a, M and Λ have their usual physical interpretation: M is the

	Kerr metric	Kerr-de Sitter Metric
g_{tt}	$-1+\frac{2rM}{\rho^2}$	$\frac{\Delta_\theta a^2 \sin^2\theta - \Delta_r}{\rho^2 \Xi^2}$
g_{rr}	$\frac{\rho^2}{\Delta}$	$\frac{\rho^2}{\Delta_r}$
$g_{\theta\theta}$	ρ^2	$\frac{\rho^2}{\Delta_\theta}$
$g_{\phi\phi}$	$\left[r^2+a^2+\frac{2rMa^2\sin^2\theta}{\rho^2}\right]\sin^2\theta$	$\left[\Delta_\theta(r^2+a^2)^2 - \Delta_r a^2 \sin^2\theta\ \right]\frac{\sin^2\theta}{\rho^2\Xi^2}$
$g_{\phi t}$	$-\frac{2rMa\sin^2\theta}{\rho^2}$	$\frac{a\sin^2\theta}{\Xi^2\rho^2}\left(\Delta_r - \Delta_\theta(r^2+a^2)\right)$
Other	All zero	All zero
$l^2=\frac{\Lambda}{3}$ $\quad\Xi = 1+l^2a^2$ $\quad\Delta_\theta = 1+l^2a^2\cos^2\theta$		
$\Delta_r = \Delta - l^2r^2(r^2+a^2)$ $\quad\rho^2 = r^2+a^2\cos^2\theta$ $\quad\Delta = r^2-2Mr+a^2$		

Table 2.1 – Metric tensor elements in Boyer-Lindquist like coordinates

mass of the black hole, a its angular momentum per unit mass and Λ is the cosmological constant.

As in the case of the Kerr metric, the Kerr-de Sitter metric line element can be divided into two parts that clearly have an unique analytic extension to all of $(\mathbb{R}_t \times \mathbb{R}_r) \times S^2 \setminus \Sigma \cup \mathcal{H}$ (whereas the expressions in table 2.1 are a priori only valid at points where $\sin\theta \neq 0$).

More precisely we have $\mathrm{d}s^2 = g_{rr}\mathrm{d}r^2 + Q + Q'$ where Q and Q' are the two quadratic forms given by:

$$\begin{aligned} Q &= g_{tt}\mathrm{d}t^2 + 2g_{\phi t}\,\mathrm{d}\phi\mathrm{d}t, \\ &= -\frac{\Delta_\theta}{\Xi^2}\mathrm{d}t^2 + \frac{1}{\Xi^2}\left(l^2(r^2+a^2) + \frac{2Mr}{\rho^2}\right)\left(\left[\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi\right]^2 - a^2\sin^4\theta\mathrm{d}\phi^2\right), \end{aligned} \tag{2.4}$$

$$Q' = g_{\theta\theta}\mathrm{d}\theta^2 + g_{\phi\phi}\mathrm{d}\phi^2 = \frac{\rho^2}{\Delta_\theta}\mathrm{d}\sigma^2 + \left(\frac{\Xi}{\Delta_\theta}\left(1 - l^2r^2\right) + \frac{2Mr}{\rho^2}\right)\frac{a^2\sin^4\theta}{\Xi^2}\mathrm{d}\phi^2. \tag{2.5}$$

In the last expression $\mathrm{d}\sigma^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$ is the usual line element of the sphere, which is naturally extendable to the poles. Moreover, the form $a\sin^2\theta\mathrm{d}\phi$ is well defined [2] on all of S^2. Hence, the above expressions have unique analytic extensions to the points of the "axis" $\mathcal{A} = \mathbb{R}^2 \times \{p_\pm\}$ where $p_\pm$ are the poles of the sphere.

The set Σ is the ring singularity of the Kerr-de Sitter spacetime and the zeros of Δ_r will give us the number of Boyer-Lindquist blocks as well as the position of the horizons when we construct a maximal analytical extension of the Boyer-Lindquist blocks in section 2.5. Its sign will also be of importance since, as seen from the expression in table 2.1, it determines the nature [3] of the coordinate vector fields $\partial_t, \partial_r, \partial_\phi$. The properties of Δ_r will be studied in section 2.4. For now, we write $\varepsilon = \mathrm{sgn}(\Delta_r)$ and define an orthonormal frame $(E_i)_{i\in\,\{0,\ldots,3\}}$ on each Boyer-Lindquist block as follows:

$$\begin{aligned} E_0 &= \frac{V\Xi}{\rho\sqrt{\varepsilon\Delta_r}}, \quad & E_1 &= \frac{\sqrt{\varepsilon\Delta_r}}{\rho}\partial_r, \\ E_2 &= \frac{\sqrt{\Delta_\theta}}{\rho}\partial_\theta, \quad & E_3 &= \frac{\Xi W}{\sin\theta\sqrt{\Delta_\theta}\rho}. \end{aligned} \tag{2.6}$$

The choice of vector fields $V = (r^2+a^2)\partial_t + a\partial_\phi$ and $W = \partial_\phi + a\sin^2\theta\partial_t$ to replace ∂_t and ∂_ϕ reduces the indeterminacy of the nature of the vectors to the sign of Δ_r which

2. In cartesian coordinates it is $a(x\mathrm{d}y - y\mathrm{d}x)$
3. space-like $g(v,v) > 0$, time-like $g(v,v) < 0$, light-like or isotropic $g(v,v) = 0$

will be constant on each Boyer-Lindquist block. It is identical to that in [ONe14] for the Kerr metric, where they play an important role; this will also be the case for the Kerr-de Sitter metric. The dual frame is readily determined from (2.6):

$$\begin{gathered}\omega^0 = \frac{\sqrt{\varepsilon\Delta_r}}{\Xi\rho}\mathrm{d}t - \frac{a\sin^2\theta\,\sqrt{\varepsilon\Delta_r}}{\rho\Xi}\mathrm{d}\phi, \quad \omega^1 = \frac{\rho}{\sqrt{\varepsilon\Delta_r}}\mathrm{d}r, \quad \omega^2 = \frac{\rho}{\sqrt{\Delta_\theta}}\mathrm{d}\theta, \\ \omega^3 = \frac{(r^2+a^2)\sqrt{\Delta_\theta}\sin\theta}{\rho\Xi}\mathrm{d}\phi - \frac{a\sqrt{\Delta_\theta}\sin\theta}{\rho\;\Xi}\;\mathrm{d}t.\end{gathered} \tag{2.7}$$

This furnishes a more compact expression of the line element:

$$\begin{aligned}\mathrm{d}s^2 &= -\varepsilon(\omega^0)^2 + \varepsilon(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2, \\ &= -\frac{\Delta_r}{\Xi^2\rho^2}\left[\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi\right]^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{\Delta_\theta\sin^2\theta}{\rho^2\Xi^2}\left[(r^2+a^2)\mathrm{d}\phi - a\mathrm{d}t\right]^2.\end{aligned} \tag{2.8}$$

From these expressions one can determine the connexion forms [4] $v \mapsto \omega^i{}_j(v) = \omega^i(\nabla_v E_j)$, characterised uniquely by the first structural equation $\mathrm{d}\omega^i = -\sum_m \omega^i{}_m \wedge \omega^m$, and the curvature forms $\Omega^i{}_j = d\omega^i{}_j + \sum_m \omega^i{}_m \wedge \omega^m_j$. The curvature forms are:

$$\begin{aligned}\Omega^0{}_1 &= \varepsilon(2I+l^2)\omega^0\wedge\omega^1 + 2\varepsilon J\omega^3\wedge\omega^2, \\ \Omega^0{}_2 &= -\varepsilon J\omega^1\wedge\omega^3 + (I-l^2)\omega^2\wedge\omega^0, \\ \Omega^0{}_3 &= \varepsilon J\omega^1\wedge\omega^2 - (I-l^2)\omega^0\wedge\omega^3, \\ \Omega^1{}_2 &= -(I-l^2)\omega^1\wedge\omega^2 - \varepsilon J\omega^0\wedge\omega^3, \\ \Omega^1{}_3 &= -(I-l^2)\omega^1\wedge\omega^3 + \varepsilon J\omega^0\wedge\omega^2, \\ \Omega^2{}_3 &= 2J\omega^0\wedge\omega^1 + (2I+l^2)\omega^2\wedge\omega^3.\end{aligned} \tag{2.9}$$

where: $I = \frac{Mr}{\rho^6}(r^2 - 3a^2\cos^2\theta)$ and $J = \frac{Ma\cos\theta}{\rho^6}(3r^2 - a^2\cos^2\theta)$. When $l = 0$ these formulae coincide with those in [ONe14] [5]. It is surprising to find that the additional contribution due to the presence of a positive cosmological constant Λ is completely separate from that of the curvature due to the black hole.

The curvature forms are related to the Riemann curvature tensor by:

$$\omega^a(R(E_c,E_d)E_b) = R^a{}_{bcd} = \Omega^a{}_b(E_c,E_d). \tag{2.10}$$

4. given in appendix A.1
5. It should be noted that there is a small error in the expression of Ω^0_3 given on page 98 of [ONe14], it should read: $\Omega^0{}_3 = -I\omega^0\wedge\omega^3 + \varepsilon J\omega^1\wedge\omega^2$

As in the case of Kerr metric, the presence of the factor ρ^{-6} in these formulae indicates that the loci of $\rho^2 = 0$ is a real curvature singularity and that there is no sensible extension of the Boyer-Lindquist block containing Σ to include these points. Using (2.9) we find that the Ricci tensor is given by:

$$R_{ab} = 3l^2 g_{ab} = \Lambda g_{ab}, \tag{2.11}$$

and so the Kerr-de Sitter metric is indeed a vacuum solution to Einstein's field equations with cosmological constant:

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0. \tag{2.12}$$

The relative simplicity of (2.9) is reflected in the algebraic decomposition of the Riemann curvature tensor. In particular, we find that the conformal Weyl tensor [6] is given by:

$$C_{abcd} = R_{abcd} - l^2 (g_{ac} g_{bd} - g_{ad} g_{bc}). \tag{2.13}$$

We can deduce from this that the conformal properties of the KdS-Boyer-Lindquist blocks are exactly those of the Kerr Boyer-Lindquist blocks ($l = 0$). In particular:

Proposition 2.3.1.

1. *At each point of the Boyer-Lindquist blocks the Weyl tensor has Petrov type D.*
2. *The principal null directions are determined by the rays of $E_0 \pm E_1$ or equivalently, $\pm \partial_r + \frac{\Xi}{\Delta_r} V$.*

Remark 2.3.1. The normalisation chosen here is different from that in [AM11], our choice is justified by the following lemma.

Proposition 2.3.1 is a statement about the algebraic structure of the Weyl tensor. On a four dimensional Lorentzian manifold (M, g), Hodge duality can be used to define a complex structure on the $\Lambda^2(T_x M)$ at each point $x \in M$. Exploiting its symmetries and trace free property, the Weyl tensor at a given point can be interpreted as a symmetric linear map $\boldsymbol{C}_x$ on $\Lambda^2(T_x M)$ (with respect to g_x), that is $\mathbb{C}$-linear with respect to this structure. The Petrov classification is based on a discussion on the eigenvalues of $\boldsymbol{C}_x$; Petrov type D is the case in which $\boldsymbol{C}_x$ is diagonalisable and has exactly two distinct eigenvalues. In this case, the Principal Null Directions are determined by specific eigenvectors. We

6. $C_{abcd} = R_{abcd} - \frac{1}{2}(g_{ac}R_{bd} - g_{ad}R_{bc} + R_{ac}g_{bd} - R_{ad}g_{bc}) + \frac{R}{6}(g_{ac}g_{bd} - g_{ad}g_{bc})$.

refer to [ONe14, Chapter 5] for more details. We note also that there is a shorter root to defining the Principal Null Directions provided by Penrose's spinor formalism, we refer to either [Wal84, Chapter 13] or [PR88, Chapter 8].

Lemma 2.3.1. *On each Boyer-Lindquist block the integral curves of* $\pm\partial_r + \frac{\Xi}{\Delta_r}V$ *are geodesics.*

Proof. This is actually a consequence of the Petrov type of the Weyl tensor C [7], but since we have at our disposition all of the connection forms, we can also verify it directly. The geodesic equations are given in appendix A.2. Consider an integral curve $\gamma : I \mapsto KdS$ of $\partial_r + \frac{\Xi}{\Delta_r}V$. It satisfies for $t \in I$:

$$\dot{\gamma}(t) = \frac{\rho}{\sqrt{\varepsilon\Delta_r}}|_{\gamma(t)}E_1(t) + \frac{\varepsilon\rho}{\sqrt{\varepsilon\Delta_r}}|_{\gamma(t)}E_0(t). \tag{2.14}$$

Setting $\Gamma^3 = \Gamma^2 = 0$ in the left-hand side of the equations in the appendix, shows that the last one is trivial and the remaining three reduce to:

$$\dot{\Gamma^0}(t) = -\frac{\partial}{\partial r}\left(\frac{\sqrt{\varepsilon\Delta_r}}{\rho}\right)\Bigg|_{\gamma(t)}\Gamma^0(t)\Gamma^1(t), \tag{2.15}$$

$$\dot{\Gamma^1}(t) = -\frac{\partial}{\partial r}\left(\frac{\sqrt{\varepsilon\Delta_r}}{\rho}\right)\Bigg|_{\gamma(t)}\left(\Gamma^0(t)\right)^2, \tag{2.16}$$

$$(\Gamma^0(t))^2 = (\Gamma^1(t))^2. \tag{2.17}$$

Equation (2.17) is clearly satisfied and, substituting the expressions of Γ^0 and Γ^1 into the right-hand side of (2.15) equation, we find:

$$\begin{aligned}
-\frac{\partial}{\partial r}\left(\frac{\sqrt{\varepsilon\Delta_r}}{\rho}\right)\Bigg|_{\gamma(t)}\Gamma^0(t)\Gamma^1(t) = -\varepsilon\,\frac{\partial}{\partial r}\left(\frac{\sqrt{\varepsilon\Delta_r}}{\rho}\right)\frac{\rho^2}{\varepsilon\Delta_r}\Bigg|_{\gamma(t)} &= \varepsilon\frac{\partial}{\partial r}\left(\frac{\rho}{\sqrt{\varepsilon\Delta_r}}\right)\Bigg|_{\gamma(t)}, \\
&= \mathrm{d}r_{\gamma(t)}(\dot{\gamma}(t))\,\varepsilon\frac{\partial}{\partial r}\left(\frac{\rho}{\sqrt{\varepsilon\Delta_r}}\right)\Bigg|_{\gamma(t)}, \\
&= \dot{\Gamma^0}(t).
\end{aligned}$$

7. cf. Goldberg-Sachs theorem [GS09].

Similarly, for the right-hand side of (2.16):

$$-\frac{\partial}{\partial r}\left(\frac{\sqrt{\varepsilon\Delta_r}}{\rho}\right)\Bigg|_{\gamma(t)}\left(\Gamma^0(t)\right)^2 = -\frac{\partial}{\partial r}\left(\frac{\sqrt{\varepsilon\Delta_r}}{\rho}\right)\frac{\rho^2}{\varepsilon\Delta_r}\Bigg|_{\gamma(t)} = \mathrm{d}r_{\gamma(t)}(\dot\gamma(t))\frac{\partial}{\partial r}\left(\frac{\rho}{\sqrt{\varepsilon\Delta_r}}\right)\Bigg|_{\gamma(t)},$$
$$= \dot\Gamma^1(t).$$

The remaining case is similar.

2.4 Fast, Extreme and Slow Kerr-de Sitter

In this section we study the structure of the roots of the family of polynomials:

$$\Delta_r(a,l,M) = r^2 - 2Mr + a^2 - l^2r^2(r^2+a^2). \tag{2.18}$$

Throughout the following discussion we will assume that all of the parameters are non-zero, this guarantees that we are really on a de Sitter background and excludes Schwarzchild-de Sitter which is studied in [AM11]. Moreover, we assume $a > 0$, $l > 0$. There is no loss of generality in assuming $a > 0$ as all of the results of this section remain valid under the substitution $a \leftrightarrow |a|$, alternatively, we can always reverse the orientation of the axis of rotation. The restriction $l \neq 0$ also guarantees that $\deg\, \Delta_r = 4$. In the analytical extensions constructed in section 2.5, each root of Δ_r will give rise to a totally geodesic null hypersurface, that we will refer to as a horizon. Under the hypothesis that $l \neq 0$, it is clear that :

$$\Delta_r = r^2 - 2Mr + a^2 - l^2r^2(r^2+a^2) = 0 \Leftrightarrow r^4 - \frac{1-l^2a^2}{l^2}r^2 + 2\frac{M}{l^2}r - \frac{a^2}{l^2} = 0. \tag{2.19}$$

To simplify notations we introduce $A = \frac{a}{l}$ and $m^2 = \frac{M}{l^2}$, and will therefore study the structure of the roots of the degree 4 polynomial with real coefficients:

$$P = X^4 - \frac{1-l^4A^2}{l^2}X^2 + 2m^2X - A^2. \tag{2.20}$$

Let us call (x_1, x_2, x_3, x_4) the (not necessarily distinct) complex roots of P. Writing out the Vieta formulae for this polynomial we know that the roots of P must satisfy the following system:

$$\left\{\begin{array}{ll} x_1+x_2+x_3+x_4=0, & (i)\\ x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4=\frac{A^2l^4-1}{l^2}, & (ii)\\ x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4=-2m^2, & (iii)\\ x_1x_2x_3x_4=-A^2, & (iv)\end{array}\right. \tag{2.21}$$

We can deduce immediately from equation (iv) that for all positive real values of the parameters A, m^2, l the polynomial P will always have at least two distinct *real* roots with opposite sign; these are the cosmological horizons. In particular, there is always a horizon "inside" the singularity ($r < 0$). Moreover, the multiplicity of any root is at most 3 and there is at most one root with multiplicity > 1.

2.4.1 Extreme Kerr-de Sitter

For the usual Kerr metric, extreme Kerr corresponds to the case where the polynomial Δ_r has a double root, i.e. the two black hole horizons coincide. A necessary and sufficient condition for this is that $M^2 = a^2$. In this section we characterise the analogous case for the KdS metric. In fact, we find that there are three cases where horizons coincide:

1. Three horizons situated in the region $r > 0$ coincide.
2. The two black hole horizons coincide.
3. The outer black hole horizon coincides with the outer cosmological horizon.

We begin by proving the following proposition:

Proposition 2.4.1. *Let $a, M, l \in \mathbb{R}_+^*$ and P be defined by (2.20). P has a root with multiplicity exactly 2 if and only if the parameters satisfy both of the following conditions:*

(i) $al < 2-\sqrt{3}$,

(ii) $M^2 = \dfrac{(1-a^2l^2)(a^4l^4+34a^2l^2+1)\pm\sqrt{\delta}}{54l^2}$.

$$\delta = (al-(2-\sqrt{3}))^3(al+2+\sqrt{3})^3(al+2-\sqrt{3})^3(al-(2+\sqrt{3}))^3.$$

Furthermore: [P has a root with multiplicity 3] $\Leftrightarrow \left\{\begin{array}{l} al = 2-\sqrt{3},\\ M^2 = \frac{16}{9}\sqrt{3}a^3l.\end{array}\right.$

Proof. Firstly, a necessary and sufficient condition for the polynomial P to have a root with multiplicity > 1 is that its discriminant, $\Delta(P)$, should vanish. We recall that the discriminant is related to the resultant [8] $R(P, P')$ of P and its formal derivative P' by:

$$\Delta(P) = \frac{(-1)^{\frac{n(n-1)}{2}}}{a_n} R(P, P'). \tag{2.22}$$

In the above formula, n is the degree of the polynomial, and a_n is the coefficient of the leading term. Here:

$$\begin{aligned}\Delta(P) &= -\frac{16}{l^{10}}\Big(a^{10}l^8 + 4\,a^8l^6 + 6\,a^6l^4 + 27\,M^4l^2 + 4\,a^4l^2 \\ &\qquad\qquad + \Big(a^6l^6 + 33\,a^4l^4 - 33\,a^2l^2 - 1\Big)M^2 + a^2\Big), \\ &= -\frac{16}{l^{10}}\Big(27M^4l^2 + (a^2l^2-1)(a^4l^4+34a^2l^2+1)M^2 + a^2(a^2l^2+1)^4\Big).\end{aligned}$$

Thus:

$$\Delta(P) = 0 \Leftrightarrow 27M^4l^2 + (a^2l^2-1)(a^4l^4+34a^2l^2+1)M^2 + a^2(a^2l^2+1)^4 = 0. \tag{2.23}$$

This is a second order polynomial equation in M^2. We require that the roots be real and at least one of the roots be positive. However, as $a^2(a^2l^2+1)^4 > 0$ if one root is positive both of them are. Moreover, since the sum of the roots is given by $-(a^2l^2-1)(a^4l^4+34a^2l^2+1)$ when the roots exist and are real, they are both positive if and only if $al < 1$.

The solutions are real if and only if the discriminant δ of the order two polynomial

$$Q = 27X^2l^2 + (a^2l^2-1)(a^4l^4+34a^2l^2+1)X + a^2(a^2l^2+1)^4,$$

is positive. We find that:

$$\begin{aligned}\delta = &\Big[(1-a^2l^2)(a^4l^4+34a^2l^2+1) - 6\sqrt{3}al(a^2l^2+1)^2\Big] \\ &\qquad\qquad \times\Big[(1-a^2l^2)(a^4l^4+34a^2l^2+1) + 6\sqrt{3}al(a^2l^2+1)^2\Big].\end{aligned}$$

Assuming as necessary $al < 1$ we see that δ has the same sign as:

$$\phi(al) = (1-a^2l^2)(a^4l^4+34a^2l^2+1) - 6\sqrt{3}al(a^2l^2+1)^2.$$

8. The definition of the resultant is recalled in appendix A.3

Defining $y = al$, we are therefore interested in the sign of $\phi(y)$ for $y \in]0,1[$. One can check [9] that $2-\sqrt{3}$ and $2+\sqrt{3}$ are a roots of ϕ and that

$$\phi(y) = -(y-(2-\sqrt{3}))^3(y+2+\sqrt{3})^3.$$

For $y \geq 0$, we find that $\phi(y)$ has opposite sign to $y-(2-\sqrt{3})$ and so is positive if and only if $y \leq (2-\sqrt{3}) < 1$.

Therefore, we have shown that P has a root with multiplicity > 1 if and only if :

$$al \leq (2-\sqrt{3}) \text{ and } M^2 = \frac{(1-a^2l^2)(a^4l^4+34a^2l^2+1) \pm \sqrt{\delta}}{54l^2}.$$

We will now show that when P has a root with multiplicity > 1 it is of multiplicity 3 if and only if $al = (2-\sqrt{3})$.

Suppose now that P has a root x with multiplicity > 1. In particular the above conditions are satisfied. x is of multiplicity at least two, and so, we can assume $x_3 = x_4 = x$. Vieta's formulae (2.21) then reduce to:

$$\begin{cases} x_1 + x_2 = -2x, & (i') \\ x_1x_2 - 3x^2 = \frac{A^2l^4-1}{l^2}, & (ii') \\ x_1x_2x - x^3 = -m^2, & (iii') \\ x_1x_2x^2 = -A^2. & (iv') \end{cases} \tag{2.24}$$

Equation (iv') show that as $A > 0$ no root is zero so the system (2.24) is equivalent to:

$$\begin{cases} x_1 + x_2 = -2x, & (i') \\ 3x^4 + \frac{A^2l^4-1}{l^2}x^2 + A^2 = 0, & (ii'') \\ x^4 - m^2x + A^2 = 0, & (iii'') \\ x_1x_2x^2 = -A^2. & (iv') \end{cases} \tag{2.25}$$

Finally combining (ii'') and (iii'') we see that (2.25) is equivalent to:

$$\begin{cases} x_1 + x_2 = -2x & (i') \\ \frac{A^2l^4-1}{l^2}x^2 + 3m^2x - 2A^2 = 0 & (ii''') \\ x^4 - m^2x + A^2 = 0 & (iii'') \\ x_1x_2x^2 = -A^2 & (iv') \end{cases} \tag{2.26}$$

9. either by direct calculation or assuming simply $a^2l^2 + 2\sqrt{3}al - 1 = 0$

We assume now that $al = 2 - \sqrt{3}$. It follows that $\delta = 0$, furthermore, noting that $a^2l^2 + 2\sqrt{3}al - 1 = 0$, it is straightforward to verify that:

$$a^4l^4 + 34a^2l^2 + 1 = 48a^2l^2, \tag{2.27}$$

and therefore:

$$M^2 = \frac{16}{9}a^3l\sqrt{3}. \tag{2.28}$$

Consider now (ii'''), which, written in terms of a is:

$$\frac{a^2l^2 - 1}{l^2}x^2 + 3m^2x - 2\frac{a^2}{l^2} = 0. \tag{2.29}$$

We find that the equation has one double root given by:

$$x = \frac{m^2l\sqrt{3}}{4a}. \tag{2.30}$$

Now, the other two roots x_1, x_2, are the roots of the polynomial

$$R = X^2 - (x_1 + x_2)X + x_1x_2.$$

By (2.26) one has:

$$R = X^2 + 2xX - \frac{a^2}{l^2x^2}, \tag{2.31}$$

the reduced discriminant δ' of R is given by:

$$\delta' = x^2 + \frac{a^2}{l^2x^2}.$$

Since:

$$x^2 = \frac{3}{16}m^4\frac{l^2}{a^2} = \frac{\sqrt{3}}{3}\frac{a^3}{l^3}\frac{l^2}{a^2} = \frac{\sqrt{3}}{3}\frac{a}{l},$$

it follows that:

$$\frac{1}{x^2}\frac{a^2}{l^2} = \sqrt{3}\frac{l}{a}\frac{a^2}{l^2} = \sqrt{3}\frac{a}{l} = 3x^2.$$

Hence: $\delta' = 4x^2$ and the roots of R are x and $-3x$. The roots of P and their multiplicities are then $(x, 3), (-3x, 1)$. Conversely, assume that P has a root of multiplicity 3, say, without loss of generality: $x_1 = x$ and $x_2 = x_3 = x_4 = y$, Vieta's formulae (2.21) reduce this time to:

$$\begin{cases} x = -3y, & (a) \\ \frac{A^2l^4-1}{l^2} = 3xy + 3y^2, & (b) \\ 3xy^2 + y^3 = -2m^2, & (c) \\ xy^3 = -A^2. & (d) \end{cases} \tag{2.32}$$

As before, equation (d) forbids that one of the roots be zero so (2.32) is equivalent to:

$$\begin{cases} x = -3y, & (a') \\ 6y^2 = \frac{1-A^2l^4}{l^2}, & (b') \\ 4y^3 = m^2, & (c') \\ 3y^4 = A^2. & (d') \end{cases} \tag{2.33}$$

Equation (c') shows that $y^3 > 0$ and so $y > 0$ too, hence equation (b') gives:

$$y = \frac{\sqrt{1 - A^2l^4}}{\sqrt{6}l}.$$

Equations (c') and (d') are compatibility equations, using the expression for y we find that:

$$A^2 = \frac{1}{12l^4}(1 - A^2l^4)^2, \tag{2.34}$$

$$m^2 = \frac{2}{3}\frac{(1 - A^2l^4)}{\sqrt{6}l^3}\sqrt{1 - A^2l^4}. \tag{2.35}$$

As $m^2 > 0$ there is no loss of information in squaring (2.35) to find that:

$$m^4 = \frac{2}{27}\frac{(1 - A^2l^4)^3}{l^2},$$

or, in terms of M and a:

$$M^2 = \frac{2}{27}\frac{(1 - A^2l^4)^3}{l^2}. \tag{2.36}$$

Expanding (2.34) yields a second order equation for A^2:

$$12A^2l^4 = (1 - A^2l^4)^2 \Leftrightarrow (A^2l^4 + 2\sqrt{3}Al^2 - 1)(A^2l^4 - 2\sqrt{3}Al^2 - 1) = 0. \tag{2.37}$$

The equation $0 = A^2l^4 - 2\sqrt{3}Al^2 - 1 = a^2l^2 - 2\sqrt{3}al - 1$ cannot give any solutions compatible with the condition $al \leq 2 - \sqrt{3} < 1$ as in this case

$$a^2l^2 = 2\sqrt{3}al + 1 \geq 1$$

Consequently, we consider only the solutions of $A^2l^4 + 2\sqrt{3}Al^2 - 1 = 0$. They are:

$$A \in \{\frac{2-\sqrt{3}}{l^2}, -\frac{2+\sqrt{3}}{l^2}\}.$$

As we assume $A > 0$ the second solution is excluded so A must equal $\frac{2-\sqrt{3}}{l^2}$ which gives:

$$al = 2 - \sqrt{3}. \tag{2.38}$$

Using the equation $a^2l^2 + 2\sqrt{3}al - 1 = 0$ we see that (2.36) becomes:

$$M^2 = \frac{2}{27}\frac{(1-A^2l^4)^3}{l^2} = \frac{2}{27}\frac{(1-a^2l^2)^3}{l^2} = \frac{2}{27}\frac{(2\sqrt{3}al)^3}{l^2} = \frac{16}{9}a^3l\sqrt{3}. \tag{2.39}$$

Comparing (2.39) and (2.28) we see that the condition $\Delta(P) = 0$ is satisfied, which concludes the proof.

We have now characterised all the cases where P has a root with multiplicity > 1, in the case of the double root we can also show:

Proposition 2.4.2. *If P has a root x with multiplicity exactly 2 and*

$$M^2 = \frac{(1-a^2l^2)(a^4l^4 + 34a^2l^2 + 1) + \varepsilon\sqrt{\delta}}{54l^2}, \qquad \varepsilon \in \{-1, 1\}, \tag{2.40}$$

then:

$$x = \frac{12a^2l^2 + (1-a^2l^2)(1-a^2l^2+\varepsilon\sqrt{\gamma})}{18m^2l^4} = \frac{12a^2l^2 + (1-a^2l^2)(1-a^2l^2+\varepsilon\sqrt{\gamma})}{18Ml^2}, \tag{2.41}$$

where:

$$\gamma = (a^2l^2-1)^2 - 12a^2l^2 = (a^2l^2 - 2\sqrt{3}al - 1)(a^2l^2 + 2\sqrt{3}al - 1).$$

Proof. To find the expression of x, solve equation (ii'') of (2.25) for x^2, and then use equation (ii''') of (2.26) to find x. To decide which root to take for x^2, introduce $\varepsilon' \in \{-1, 1\}$ in front of the radical in the expression for x^2 and then square the expression obtained for x. Injecting into this new expression those of M^2 and x^2, it is straightforward to obtain an expression for $\varepsilon\sqrt{\delta}$. After simplification we find that $\varepsilon\sqrt{\delta} = \varepsilon'\gamma\sqrt{\gamma}$. Hence, using the lemma below: $\varepsilon' = \varepsilon$.

Lemma 2.4.1. $\delta = \gamma^3$

Using this result, we can study the relative position of the double root x with respect to the other two roots; the above expression (2.41) shows immediately that $x > 0$. As before, the other roots are those of the polynomial:

$$X^2 + 2xX - \frac{a^2}{l^2 x^2}.$$

As expected one of the roots (x_-) will be negative and the other positive, the positive root is given by:

$$x_+ = -x + \sqrt{x^2 + \frac{a^2}{l^2 x^2}}. \tag{2.42}$$

We see that $x_+ > x$ if and only if $\sqrt{x^2 + \frac{a^2}{l^2x^2}} > 2x > 0$. This holds if and only if:

$$\frac{a^2}{l^2 x^2} > 3x^2,$$

Or, equivalently:

$$x^4 < \frac{1}{3}\frac{a^2}{l^2}$$

As $x^4 = m^2 x - \frac{a^2}{l^2}$, we deduce that:

$$x_+ > x \Leftrightarrow x < \frac{4}{3}\frac{a^2}{M} \tag{2.43}$$

Note that $x = \frac{4}{3}\frac{a^2}{M}$ corresponds to the case where there is a triple root. Rewriting (2.41) we have:

$$x = \frac{4}{3}\frac{a^2}{M} + \frac{\gamma + (1 - a^2 l^2)\varepsilon\sqrt{\gamma}}{18Ml^2}, \tag{2.44}$$

So if $\varepsilon = 1$ then $\frac{\gamma+(1-a^2l^2)\varepsilon\sqrt{\gamma}}{18Ml^2} > 0$ and so $x_+ < x$. In this case the outer black hole horizon coincides with the cosmological horizon.

If $\varepsilon = -1$ we show that $\frac{\gamma-(1-a^2l^2)\sqrt{\gamma}}{18Ml^2} < 0$ and so $x_+ > x$; the two black hole horizons coincide. This is the closest Kerr-de Sitter analog of extreme Kerr.

In order to show that: $\frac{\gamma-(1-a^2l^2)\sqrt{\gamma}}{18Ml^2} \leq 0$ we only need to study the sign of $\sqrt{\gamma}-(1-a^2l^2)$.

i.e. the sign of:

$$f(y) = \sqrt{(1-y^2)^2 - 12y^2} - (1-y^2),$$

when $0 \leq y \leq 2-\sqrt{3}$. However, $f(y)$ has same sign as :

$$\begin{aligned} f(y)(\sqrt{(1-y^2)^2 - 12y^2} + (1-y^2)) &= (1-y^2)^2 - 12y^2 - (1-y^2)^2, \\ &= -12y^2 < 0. \end{aligned}$$

To summarise, we have found three cases where horizons coincide:

Proposition 2.4.3. *Let $(a, l, M) \in \mathbb{R}_+^*$, then 2 horizons coincide if and only if the both of the following conditions are satisfied:*

(*i*) $al < 2-\sqrt{3}$,

(*ii*) $M^2 = \dfrac{(1-a^2l^2)(a^4l^4 + 34a^2l^2 + 1) \pm \sqrt{\delta}}{54l^2} = m_\pm^2.$

More precisely:

— *If $M^2 = m_+^2$ then the outer black hole horizon coincides with the the other cosmological horizon.*

— *If $M^2 = m_-^2$ then the two black hole horizons coincide.*

Finally, if $al = 2-\sqrt{3}$ and M^2 satisfies (ii) then all three horizons situated in the region $r > 0$ coincide.

2.4.2 Fast and slow Kerr-de Sitter

We will now move on to study the Kerr-de Sitter equivalents to the usual so-called "fast" and "slow" Kerr black holes. Fast Kerr usually correspond to the case where there are no horizons. It owes its name to the fact that when $l = 0$, it is completely characterised by the condition $a^2 > M^2$. "Slow" Kerr, on the other hand, is characterised when $l = 0$ by the condition $a^2 < M^2$. In terms of the roots of the polynomial these cases correspond respectively, when $l = 0$, to Δ_r having no roots, or Δ_r having two distinct real roots. As we have already noted, there are always two distinct roots with opposite sign in the case $l > 0$ of Kerr-de-Sitter which correspond to the cosmological horizons inside and outside the singularity. Hence, in terms of roots the natural analogs for the Kerr-de Sitter metric are:

— P has 4 distinct real roots ("Slow" Kerr-de Sitter),

— P has a complex root ("Fast" Kerr-de Sitter).

A further accommodating consequence of the necessary existence of two distinct real roots is that we can distinguish between the above cases using the sign of $\Delta(P)$. Indeed, let us denote the roots of P by x_1, x_2, x_3, x_4 and assume, without loss of generality, that x_1 and x_2 are both real and distinct.

From Proposition A.3.3 of appendix A.3 we can write (in $\mathbb{C}$):

$$\Delta(P) = (x_1 - x_2)^2(x_1 - x_3)^2(x_1 - x_4)^2(x_2 - x_3)^2(x_2 - x_4)^2(x_3 - x_4)^2.$$

From this expression we see that if $x_3 \in \mathbb{R}$, $\Delta(P) \geq 0$ [10]. If, however, $x_3 = z \in \mathbb{C} \setminus \mathbb{R}$ then $x_4 = \bar{z}$, hence:

$$\begin{aligned}\Delta(P) &= (x_1 - x_2)^2(x_1 - z)^2(x_1 - \bar{z})^2(x_2 - z)^2(x_2 - \bar{z})^2(2i\Im(z))^2,\\ &= -4\Im(z)^2(x_1 - x_2)^2|x_1 - z|^2|x_2 - z|^2 < 0.\end{aligned}$$

Therefore, P has two conjugate complex roots if and only if $\Delta(P) < 0$. We recall the expression of $\Delta(P)$ of the previous section:

$$\Delta(P) = -\frac{16}{l^{10}}\left(27M^4l^2 + (a^2l^2 - 1)(a^4l^4 + 34a^2l^2 + 1)M^2 + a^2(a^2l^2 + 1)^4\right). \tag{2.45}$$

The expression $27M^4l^2 + (a^2l^2 - 1)(a^4l^4 + 34a^2l^2 + 1)M^2 + a^2(a^2l^2 + 1)^4$ is a second order polynomial in M^2 whose discriminant is given by:

$$\delta = \gamma^3 = (y - (2 - \sqrt{3}))^3(y + 2 + \sqrt{3})^3(y + 2 - \sqrt{3})^3(y - (2 + \sqrt{3}))^3,$$

where $y = al$. From this factorisation we deduce the sign of δ given in table 2.2, and the following cases:

(i) $0 \leq al \leq 2 - \sqrt{3}$:

In this case $\Delta(P) = -\frac{432}{l^8}(M^2 - m_-^2)(M^2 - m_+^2)$ where $0 \leq m_-^2 \leq m_+^2$. It follows that if $M^2 \in [m_-^2, m_+^2]$ then $\Delta(P) \geq 0$ otherwise, $\Delta(P) < 0$.

(ii) $2 - \sqrt{3} < al < 2 + \sqrt{3}$:

Here $\Delta(P)$ never vanishes for any value of M^2. Since for $M^2 = 0$, $\Delta(P) < 0$ and $\Delta(P)$ is a continuous function of M^2, $\Delta(P) < 0$ for all values of M^2.

10. The discussion in the previous section shows that necessarily $x_4 \in \mathbb{R}$ too.

(*iii*) $al \geq 2+\sqrt{3}$:

$\Delta(P) = -\frac{432}{l^8}(M^2+m_-^2)(M^2+m_+^2)$ where $0 \leq m_+^2 \leq m_-^2$ Therefore, for all values of $M \geq 0$, $\Delta(P) < 0$.

Table 2.2 – Sign of δ

$y = al$	0		$2-\sqrt{3}$		$2+\sqrt{3}$		$+\infty$
Sign of δ		$+$	0	$-$	0	$+$	

Combined with the results of the previous section and preserving the terminology introduced at the beginning of this section, we have thus shown:

Proposition 2.4.4.

— *"Slow" Kerr de Sitter is characterised by the following conditions on the parameters* $(a,l,M) \in \mathbb{R}_+^*$:

(*i*) $al < 2-\sqrt{3}$,

(*ii*) $M^2 \in]m_-^2, m_+^2[$ *where* $m_\pm^2 = \frac{(1-a^2l^2)(a^4l^4+34a^2l^2+1)\pm\sqrt{\delta}}{54l^2}$.

— *"Fast" Kerr-de Sitter corresponds to the cases:*

▷ $0 \leq al \leq 2-\sqrt{3}$ *and* $M^2 \notin [m_-^2, m_+^2]$ *where* $m_\pm^2 = \frac{(1-a^2l^2)(a^4l^4+34a^2l^2+1)\pm\sqrt{\delta}}{54l^2}$. *This is the case that most ressembles the usual fast Kerr spacetime.*

▷ $al > 2-\sqrt{3}$.

In the above proposition we see the black hole horizons exist on a de Sitter background only under relatively strict conditions on the parameters, we have notably, for a given value of Λ, upper *and* lower bounds on the mass, as well as a restriction on the rotation parameter a of the black hole. Let us concentrate for a moment on the upper bound for the mass for a given values of $a, l, al < 2-\sqrt{3}$ of a slow KdS spacetime. According to condition (*ii*), we must have:

$$M^2 \leq \frac{(1-a^2l^2)(a^4l^4+34a^2l^2+1)+\sqrt{\delta}}{54l^2}. \tag{2.46}$$

Despite our assumption that $a > 0$, setting $a = 0$ and taking the square root furnishes a well known result in Schwarzschild-de Sitter spacetime [SH99]:

$$M < \frac{1}{3\sqrt{\Lambda}}. \tag{2.47}$$

More generally, the map $y \mapsto (1-y^2)(y^4+34y^2+1)+\sqrt{\delta(y)}$, is well defined and continuous for $y \in [0, 2-\sqrt{3}]$ and attains a maximum at $y = 2-\sqrt{3}$. This yields a global bound on the mass: $M < \frac{C}{\sqrt{\Lambda}}$ where $C = \frac{4}{\sqrt{3}}\sqrt{26\sqrt{3}-45} \approx 0.4215$. Studying how the expression of the upper bound depends on a, it can be shown that in fact the minimum value is attained for $a = 0$: rotating black holes can be slightly more massive than non-rotating black holes and still maintain their horizon structure.

We conclude this section by addressing one last question regarding slow Kerr-de Sitter black hole: can there be more than one horizon inside the singularity, i.e. in the region $r < 0$? The answer is negative, as shown in the following lemma.

Lemma 2.4.2. *We suppose $a \neq 0$. In slow Kerr-de Sitter only one horizon lies in the region $r < 0$.*

Proof. It has already been noted that there must always be at least one negative root; an even number of both positive and negative roots is excluded again by equation (iv) in (2.21). The statement of the lemma is therefore equivalent to the fact that there cannot be 3 negative roots. As usual, denote by x_1, x_2, x_3, x_4 the 4 roots of Δ_r. By hypothesis, they are all real. Suppose, without loss of generality, $x_1 x_2 < 0$. It follows that $x_3 x_4 > 0$ from equation (iv) of (2.21). Call $P = x_3 x_4$ and $S = x_3 + x_4$. Equation (i) of (2.21) gives: $S = -(x_1 + x_2)$. Equation (iii) of (2.21) yields:

$$-\frac{A^2}{P}S - SP = -2m^2,$$

which is equivalent to:

$$S = \frac{2m^2 P}{A^2 + P^2} \geq 0.$$

Therefore $S = x_3 + x_4$ is always positive and thus x_3 and x_4 are both positive.

2.4.3 Boyer-Lindquist blocks

We are now in a position to give a more precise description of the Boyer-Lindquist blocks. We will do this first in the slow case, where there are four distinct roots, say, r_{--}, r_-, r_+, r_{++} ordered as:

$$r_{--} < 0 < r_- \leq r_+ \leq r_{++}.$$

In table 2.3 we give the sign of Δ_r as r varies and the chosen numbering for the Boyer-Lindquist blocks. We also give the sign of the diagonal metric tensor elements g_{ii}. The "•" means that the sign changes within the block. That $g_{\phi\phi} > 0$ for $r > 0$ is not clear from the initial expression of $g_{\phi\phi}$ given in table 2.1, however one can write:

$$g_{\phi\phi} = \left((r^2 + a^2) + \frac{2Mra^2 \sin^2\theta}{\rho^2 \Xi} \right) \frac{\sin^2\theta}{\Xi}. \tag{2.48}$$

r	$-\infty$		r_{--}		0		r_-		r_+		r_{++}		$+\infty$
Δ_r		$-$	0		$+$		0	$-$	0	$+$	0	$-$	
Boyer Lindquist blocks		V			IV			III		II		I	
g_{tt}		$+$			•			$+$		•		$+$	
g_{rr}		$-$			$+$			$-$		$+$		$-$	
$g_{\theta\theta}$		$+$			$+$			$+$		$+$		$+$	
$g_{\phi\phi}$		$-$		•		$+$		$+$		$+$		$+$	
$g(V,V)$		$+$			$-$			$+$		$-$		$+$	
$g(W,W)$		$+$			$+$			$+$		$+$		$+$	

Table 2.3 – Sign of Δ_r and Boyer-Lindquist blocks

Up to now, we have not addressed the question of the time-orientation [11] of the manifolds under consideration. The time-orientability of each Boyer-Lindquist block is clear from table 2.3, so each Boyer-Lindquist block can separately become a *spacetime*. For the usual Kerr metric and the Schwarzchild metric, the time parameter t coincides with the proper time of a distant stationary observer in the limit $r \to \infty$. In this case, time-

11. A time orientation of a Lorentzian manifold is a choice of a globally defined nowhere vanishing non-spacelike continuous vector field. A vector field is said to be *time-orientable* if such a vector field exists

orientation of the Boyer-Lindquist block that lies beyond all black hole horizons can be chosen naturally under the prescription that ∂_t is future-pointing when non-space-like. This interpretation of t fails for the Kerr-de Sitter metric, but we still have a number of partial results. First, under the assumption that our visible universe is not beyond a cosmological horizon and not between two black hole horizons, block II (cf table 2.3) is identified as the most physically relevant block. On this block t is still a "time function" in the following sense:

Lemma 2.4.3. *On block II, the hypersurfaces "$t = t_0$" are spacelike.*

Proof. At each point p of such a surface the tangent space is given by the kernel of dt_p, or, equivalently $(\nabla t(p))^\perp$. But, ∇t is timelike on block II (minus axes) since [12] $g(\nabla_t, \nabla_t) = g^{tt} = -\frac{g_{\phi\phi}\Xi^4}{\sin^2\theta\Delta_\theta\Delta_r}$. This also holds for points on the axes, as this expression extends continuously to such points.

Corollary 2.4.1. *Along any non-spacelike C^1 curve α in block II, $t \circ \alpha$ is strictly monotonic.*

The region in the Kerr-Boyer-Lindquist blocks where $g_{tt} > 0$ is known as the "ergosphere". It has interesting physical properties explored in [ONe14] in the Kerr case, the most notable of which being the possibility to extract energy from a Kerr black hole. In the case of the Kerr-de Sitter metric it is no longer guaranteed that the ergosphere does not cover all of block II, unless we impose further conditions:

Proposition 2.4.5. *Suppose $a^2l^2 < 1$, then a sufficient condition for there to be an interval $I \subset \mathbb{R}_+^*$ such that $g_{tt} \leq 0$ when $r \in I$ is that:*

$$27M^2l^2 \leq (1 - a^2l^2)^3. \tag{2.49}$$

Proof. Rewrite g_{tt} as:

$$g_{tt} = \frac{1}{\rho^2\Xi^2}\left(\underbrace{a^2\cos^2\theta(l^2a^2\sin^2\theta - 1)}_{\leq 0} + l^2r\left(r^3 + r\frac{(a^2l^2-1)}{l^2} + \frac{2M}{l^2}\right)\right),$$

12. Refer to lemma A.4.2,A.4.3 in appendix A.4

$a^2l^2 < 1$, hence $l^2a^2\sin^2\theta \leq 1$, so the first term is always non-positive. The sign of the second term is determined by that of the polynomial:

$$P = X^3 + X\frac{a^2l^2 - 1}{l^2} + \frac{2M}{l^2},$$

P can become negative on $\mathbb{R}_+^*$ if and only if there is a positive real root, hence its discriminant must be positive. This is because if there is only one real root, it must be negative as $\frac{2M}{l^2} > 0$. The discriminant of P is given by:

$$\Delta(P) = (1 - a^2l^2)^3 - 27M^2l^2,$$

it is positive if and only if $27M^2l^2 \leq (1 - a^2l^2)^3$ and in this case all roots are real, but they cannot all be negative since their sum must vanish.

t is nevertheless a "function of time" and, even though there are cases where ∂_t is always space-like, its gradient always furnishes on block II a time-like vector field that can be used to time-orient it. By analogy with the Kerr case, we choose to time-orient block II by specifying that $-\nabla t$ is future-pointing.

2.5 Maximal Kerr-de Sitter spacetimes

In this section we will cease to consider the Boyer-Lindquist blocks as separate spacetimes and construct analytical manifolds containing isometric copies of these blocks, of which the union is dense, and to which the Kerr-de Sitter metric extends analytically. In order for these manifolds to be spacetimes they will be constructed in such a way to ensure that they are time-orientable. The methods used here are adapted from [ONe14] and are still applicable due to the remarkable algebraic decomposition of the Riemann curvature tensor described in section 2.3.

2.5.1 KdS^* et $^*KdS$ spacetimes

The first two analytical manifolds will be constructed by choosing coordinates for the Boyer-Lindquist blocks in which one of the two null geodesic congruences generated by the vector fields

$$N_\pm = \pm\partial_r + \frac{\Xi}{\Delta_r}V, \tag{2.50}$$

are coordinate-lines. Recall from proposition 2.3.1 that at each point $p \in \mathcal{B}$ of any Boyer-Lindquist block $\mathcal{B}$ the rays generated by the vectors $N_{\pm}(p)$ define the principal null directions. The geometric significance of these directions justifies using them to construct an analytical extension.

Definition 2.5.1. We define KdS^* coordinates by:

$$\begin{cases} t^* = t + T(r), \\ r^* = r, \\ \theta^* = \theta, \\ \phi^* = \phi + A(r), \end{cases} \tag{2.51}$$

similarly, ${}^*KdS$ coordinates are defined by:

$$\begin{cases} {}^*t = t - T(r), \\ {}^*r = r, \\ {}^*\theta = \theta, \\ {}^*\phi = \phi - A(r), \end{cases} \tag{2.52}$$

where $T(r) = \int \frac{(r^2+a^2)\Xi}{\Delta_r} \mathrm{d}r$ and $A(r) = \int \frac{a\ \Xi}{\Delta_r} \mathrm{d}r$.

2.5.2 KdS^*

Proposition 2.5.1. *Let $\mathcal{B}$ be a Boyer-Lindquist block and $\mathcal{A} = \mathbb{R}_t \times \mathbb{R}_r \times \{p_\pm\}$; $p_\pm$ denote the poles of the S^2. Define: $\Phi^* : \mathcal{B} \setminus \mathcal{A} \longrightarrow \mathbb{R}_{t^*} \times \mathbb{R}_{r^*} \times S^2$ by:*

$$\Phi^*(t, r, \theta, \phi) = (t + T(r), r, \theta, \phi + A(r)),$$

then Φ^* is an analytic diffeomorphism of $\mathcal{B} \setminus \mathcal{A}$ onto an open subset of $\mathbb{R}_{t^*} \times \mathbb{R}_{r^*} \times S^2$.

Proof. That Φ^* is analytic is clear; fix $(t, r, \theta, \phi) \in \mathcal{B} \setminus \mathcal{A}$, then the Jacobian matrix is given by:

$$J(\phi)(t, r, \theta, \phi) = \begin{pmatrix} 1 & \frac{r^2+a^2}{\Delta_r}\Xi & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{a\Xi}{\Delta_r} & 0 & 1 \end{pmatrix}.$$

Thus, $\det J(\phi)(t, r, \theta, \phi) = 1$. It follows that Φ^* is a local analytic diffeomorphism at each point of $\mathcal{B} \setminus \mathcal{A}$. It suffices to show that Φ^* is injective to conclude that it is a global diffeomorphism. Injectivity is clear however, as, according to Definition 2.5.1:

$$\Phi^*(r, t, \theta, \phi) = \Phi^*(r', t', \theta', \phi') \Leftrightarrow \begin{cases} t + T(r) = t' + T(r'), \\ r = r', \\ \theta = \theta', \\ \phi + A(r), = \phi' + A(r'), \end{cases} \Leftrightarrow \begin{cases} t = t', \\ r = r', \\ \theta = \theta', \\ \phi = \phi'. \end{cases}$$

(t^*, r, θ, ϕ^*) are therefore coordinates functions on $\mathcal{B} \setminus \mathcal{A}$

Lemma 2.5.1. *The coordinate vector fields $\partial_{t^*}, \partial_{r^*}, \partial_{\theta^*}, \partial_{\phi^*}$ are given on each Boyer-Lindquist block by:*

$$\partial_{t^*} = \partial_t, \qquad \partial_{r^*} = \partial_r - \frac{\Xi}{\Delta_r} V, = -N_- \qquad \partial_{\theta^*} = \partial_\theta \qquad \partial_{\phi^*} = \partial_\phi. \tag{2.53}$$

Furthermore, in KdS^* coordinates the line element can be written:

$$ds^2 = g_{tt} dt^{*2} + g_{\theta\theta} d\theta^{*2} + g_{\phi\phi} d\phi^{*2} + \frac{2}{\Xi} dt^* dr^* - \frac{2a \sin^2\theta}{\Xi} dr^* d\phi^* + 2g_{\phi t} dt^* d\phi^*. \tag{2.54}$$

Corollary 2.5.1. *On each Boyer-Lindquist block $\mathcal{B}$ the integral curves of N_- are the coordinate lines of r^*.*

Inspecting the form of (2.54) and comparing with the discussion at the beginning of section 2.3 we deduce:

Corollary 2.5.2. *By analogy with the notations used in section 2.3, let:*

$$\Sigma^* = \{(t^*, r^*, \theta^*, \phi^*) \in \mathbb{R}_{t^*} \times \mathbb{R}_{r^*} \times S^2, r^{*2} + a^2 \cos^2\theta^* = 0\},$$

then the line element (2.54) extends analytically to all of $\mathbb{R}_{t^*} \times \mathbb{R}_{r^*} \times S^2 \setminus \Sigma^*$ as a non-degenerate metric tensor.

This last result leads us to define:

Definition 2.5.2. We call KdS^* the analytical manifold $\mathbb{R}_{t^*} \times \mathbb{R}_{r^*} \times S^2 \setminus \Sigma^*$ equipped with metric tensor g^* defined by (2.54) and time-oriented such that $-\partial_{r^*}$ is future-pointing.

Remark 2.5.1. — Time-orientation is chosen here so that the integral curves (and coordinate lines) of N_- are future-oriented.

— It is consistent with the choice that $-\nabla t$ is future-pointing on block II, since, using (2.54) and Lemma A.4.3 in appendix A.4, it is easily seen that $g^*(-\partial_{r^*}, -\nabla t) = g^*(\partial_{r^*}, \nabla t) = -\frac{\Xi}{\Delta_r}(r^2 + a^2) < 0$.

Define now the subsets $\mathcal{B}^*$ of KdS^* by the same inequalities as the corresponding Boyer-Lindquist blocks $\mathcal{B}$, then:

Lemma 2.5.2. *Φ^* has an analytic extension to a diffeomorphism of $\mathcal{B}$ onto $\mathcal{B}^*$.*

Proof. For $\alpha \in \mathbb{R}$, let $R_\alpha : S^2 \longrightarrow S^2$ be the restriction of the rotation of angle α about the z-axis in $\mathbb{R}^3$ to S^2. The map $\psi : \mathbb{R}_r \times S^2 \longrightarrow S^2$ defined by $\psi(r, q) = R_{A(r)}(q)$ is analytic everywhere except at values of r where $\Delta_r = 0$. Then:

$$\tilde{\Phi}^* : \begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}^* \\ (t, r, q \in S^2) & \longmapsto & (t + T(r), r, \psi(r, q)) \end{array}.$$

is the desired extension.

Corollary 2.5.3. *Each Boyer-Lindquist block $\mathcal{B}$ can be identified isometrically with an open subset of KdS^*.*

The vector fields $\partial_t, \partial_\theta, \partial_\phi$ are, a priori, only well defined on each $\mathcal{B}^*$, but, in view of equation (2.53), $\partial_{t^*}, \partial_{\theta^*}, \partial_{\phi^*}$ are analytic extensions of these fields to all of KdS^*. Hence, we define $\partial_t, \partial_\theta$ and ∂_ϕ by equation (2.53) on all of KdS^*.

The hypersurfaces $\mathscr{H}_i^*$ defined by the equations $r = r^* = r_i$ ($i \in \{--, -, +, ++\}$) are now well-defined submanifolds of KdS^*, it is easy to show that, as is custom with black hole horizons:

Proposition 2.5.2. *Each $\mathscr{H}_i^*$ is a totally geodesic null hypersurface of KdS^*. In particular, for $p \in \mathscr{H}_i^*$:*

$$T_p\mathscr{H}_i^* = V_p^\perp = \mathit{span}\left((\partial_t)_p, (\partial_\theta)_p, (\partial_\phi)_p\right) = \mathit{span}\left(V_p, (\partial_\theta)_p, (\partial_\phi)_p\right)$$

We shall now address the question of the integral curves of N_+ in KdS^*, the situation is not symmetrical with that of N_-, as, in terms of the KdS^* coordinate fields:

$$N_+ = \partial_{r^*} + \frac{2\Xi}{\Delta_r} V.$$

Thus, N_+ is still undefined on the horizons $\mathscr{H}_i$, moreover, N_+ is not always future-pointing since:

$$g^*(N_+, -\partial_{r^*}) = -\frac{2\rho^2}{\Delta_r}.$$

This can be remedied by considering reparametrisations of the integral curves of N_+ that are integral curves of $n_+ = \frac{\Delta_r}{2\Xi} N_+$. The integral curves of n_+ are all future-oriented since $g^*(n_+, -\partial_{r^*}) = \frac{-\rho^2}{\Xi^2} < 0$.

Definition 2.5.3. On KdS^* we will call:

1. "Ingoing principal null geodesics" the integral curves of the vector field N_- extended to all of KdS^* by (2.53).
2. "Outgoing principal null geodesics" geodesic reparametrisations of the integral curves of n_+. These curves coincide on $\mathcal{B}^*$ with the images of the principal null geodesics of the Boyer-Lindquist blocks by $\tilde{\Phi}^* \equiv i^*$.

In figure 2.1, we give a schematic representation of KdS^* spacetime that will be useful in the following. The principal null geodesics are represented by oriented line segments; horizontally, the "ingoing" principal null geodesics run from $r = +\infty$ to $r = -\infty$ - we will say that they are "complete" -, vertically, the "outgoing" principal null geodesics are confined within a given Boyer-Lindquist block. We have not represented the principal null geodesics that are confined within the horizons.

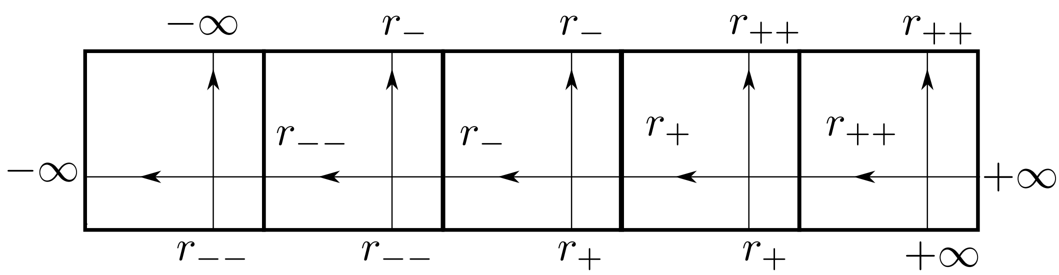


Figure 2.1 – Schematic representation of KdS^* spacetime: horizontally, the ingoing principal null geodesics run unimpeded from $r = +\infty$ to $r = -\infty$, vertically, the outgoing principal null geodesics are confined within a given Boyer-Lindquist block and on the horizons.

2.5.3 $^*KdS$

Repeating the above arguments, using instead $^*KdS$ coordinates, yields the following results:

Lemma 2.5.3.

1. *On each Boyer-Lindquist block $({}^*t, {}^*r, {}^*\theta, {}^*\phi)$ are well defined coordinate functions.*
2. *In these coordinates the line element can be written:*

$$ds^2 = g_{tt} d^*t^2 + g_{\theta\theta} d^*\theta^2 + g_{\phi\phi} d^*\phi^2 - \frac{2}{\Xi} d^*t d^*r + \frac{2a\sin^2\theta}{\Xi} d^*r d^*\phi + 2g_{\phi t} d^*t d^*\phi. \quad (2.55)$$

   This expression has an unique analytic extension to all points of $\mathbb{R}_{^*t} \times \mathbb{R}_{^*r} \times S^2 \setminus {}^*\Sigma$.
3. *The coordinate vector fields are:*

$$\partial_{^*r} = \partial_r + \frac{\Xi}{\Delta_r} V = N_+, \qquad \partial_{^*t} = \partial_t, \qquad \partial_{^*\theta} = \partial_\theta, \qquad \partial_{^*\phi} = \partial_\phi. \quad (2.56)$$

Proposition 2.5.3. *Define the Lorentizan manifold ${}^*KdS$ to be the analytic manifold $\mathbb{R}_{^*t} \times \mathbb{R}_{^*r} \times S^2 \setminus {}^*\Sigma$ equipped with the metric ${}^*g$ defined by equation (2.55) and time-oriented such that the globally defined vector field $\partial_{^*r}$ is future-pointing then:*

1. *The submanifolds ${}^*\mathscr{H}_i$ of equations $r = r_i, i \in \{--, -, +, ++\}$ are totally geodesic null hypersurfaces.*
2. *Defining ${}^*\mathcal{B}$ by the same inequalities as the Boyer-Lindquist block $\mathcal{B}$, then ${}^*\mathcal{B}$ and $\mathcal{B}$ are isometric, i.e. ${}^*KdS$ contains isometric copies of each Boyer-Lindquist block.*

Definition 2.5.4. On ${}^*KdS$ we will call:

1. "Outgoing principal null geodesics" the integral curves of the vector field N_+ extended to all of ${}^*KdS$ by (2.56).
2. "Ingoing principal null geodesics" geodesic reparametrisations of the integral curves of the everywhere future-pointing vector field $n_- = \frac{\Delta_r}{2\Xi} N_-$.

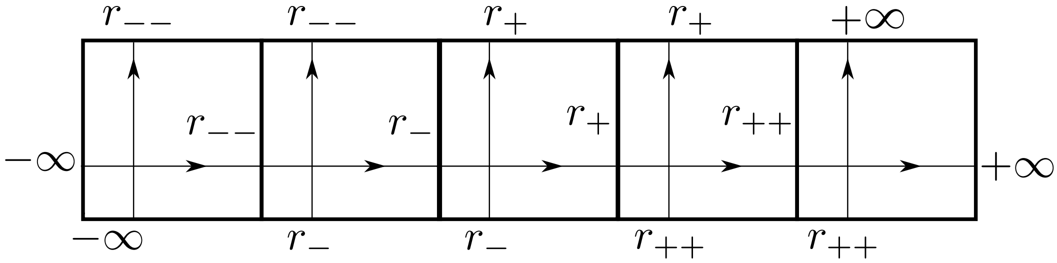


Figure 2.2 – Schematic representation of ${}^*KdS$ spacetime

In figure 2.2, we give the corresponding schematic representation of ${}^*KdS$. Again, the principal null geodesics are represented by oriented line segments. Here though, horizontally, are the *outgoing* principal null geodesics running from $r = -\infty$ to $r = +\infty$ and vertically, the *ingoing* principal null geodesics confined within a single Boyer-Lindquist block ${}^*\mathcal{B}$. Again, we have omitted the ingoing principal null geodesics trapped in the horizon.

The asymmetric treatment of the outgoing and ingoing principal null geodesics shows that ${}^*KdS$ and ${}^*KdS$ are certainly not the same spacetime. Nevertheless, there is a natural isometry μ between ${}^*\mathcal{B}$ and $\mathcal{B}^*$ for each Boyer-Lindquist block $\mathcal{B}$, in coordinates it can be written:

$$\mu({}^*t, {}^*r, {}^*\theta, {}^*\phi) = ({}^*t + 2T(r), {}^*r, {}^*\theta, {}^*\phi + 2A(r)), \tag{2.57}$$

from which we deduce that:

$$\mathrm{d}\mu(\partial_{{}^*r}) = \partial_r^* + \frac{2\Xi}{\Delta_r}V.$$

Hence:

$$g^*(-\partial_{r^*}, \mathrm{d}\mu(\partial_{{}^*r}\ )) = -\frac{2\rho^2}{\Delta_r}.$$

Therefore, μ preserves time-orientation on blocks II and IV (see table 2.3) but reverses it on blocks I, III and V.

We conclude this section defining two more spacetimes:

Definition 2.5.5. We define $KdS^{*\prime}$ and ${}^*KdS'$ to be the spacetimes obtained from KdS^* and ${}^*KdS$ respectively by reversing time orientation.

Lemma 2.5.4. *For each Boyer-Lindquist block $\mathcal{B}$, the isometries ${}^*\mathcal{B} \longrightarrow \mathcal{B}^{*\prime}$ and ${}^*\mathcal{B}' \longrightarrow \mathcal{B}^*$ defined in coordinates by (2.57) preserve time-orientation on blocks I, III and V, but reverse it on blocks II and IV.*

After reversing time-orientation, the principal null geodesics are now past-oriented. Their orientation should be reversed so that they are future-oriented, but because this changes the sign in front of ∂_r in the original expression, we also adapt terminology: an orientation reversed integral curve of ∂_{r^*} (resp. $\partial_{{}^*r}$) will become an outgoing principal null geodesics in $KdS^{*\prime}$ (resp. ${}^*KdS'$) and similarly for the integral curves of $n_\pm$. The reason for this is purely semantic, in the next section we will seek to extend the incomplete outgoing principal null geodesics by gluing together along the Boyer-Lindquist blocks combinations of the four manifolds of this section, the change of vocabulary ensures that we always extend outgoing principal null geodesics using outgoing principal null geodesics.

2.5.4 Maximal slow Kerr-de Sitter spacetime

In the previous section we constructed four isometric - but not identical - analytic extensions of the KdS-Boyer-Lindquist blocks. In one case, ingoing principal null geodesics are complete, and in the other outgoing principal null geodesics are complete. In this section, we seek an analytical extension of these spacetimes such that all principal null geodesics, save those that run into the singularity, are complete, i.e. a maximal extension of these curves is defined on all of $\mathbb{R}$. As for Kerr spacetime in [ONe14], the maximal extensions by "gluing" together the aforementioned manifolds in an elaborate fashion.

By "gluing" two semi-Riemannian manifolds X and Y, we mean that we construct a new manifold Q containing isometric copies of X and Y and equipped with a metric extending that of both X and Y. A natural way of doing this is to specify two open sets $U \subset X$ and $V \subset Y$ that are identified by an isometry $\phi : U \longrightarrow V$, in this case we denote the new manifold by $X \coprod_\phi Y$. It comes with two "canonical" embeddings $\bar{i} : X \longrightarrow Q, \bar{j} : Y \longrightarrow Q$ and $\bar{i}(X) \cap \bar{j}(Y) = \bar{i}(U) = \bar{j}(V)$. A brief outline of the construction is given in appendix A.5, however we note here that whilst most topological properties of the new space Q follow directly from those of X and Y, separation is not guaranteed. Nevertheless, we have a technical criterion- proved in appendix A.5 - that will suffice for all cases encountered in the sequel:

Lemma 2.5.5. *If X and Y are two manifolds and there is no sequence $(x_n)_{n\in\mathbb{N}}$ of points in U converging to a point in $\bar{U} \setminus U$ and such that $\phi(x_n)_{n\in\mathbb{N}}$ converges to a point in $\bar{V} \setminus V$, then Q is Hausdorff.*

Throughout this section, we assume that the conditions of slow KdS as described in section 2.4 are satisfied. In particular, we assume that Δ_r has four distinct roots. Whilst some of the more technical results in this section are independent of this hypothesis, the gluing pattern is dependent of this choice.

Kruskal domains

Rather than directly gluing the manifolds $KdS^*, {}^*KdS$ and their orientation reversed counterparts, the pattern is more conveniently described by first constructing smaller manifolds, called "Kruskal domains", from selected open sets of these manifolds. Four such domains are required, one per horizon; they are illustrated in figure 2.3 and are destined to be assembled by gluing along Boyer-Lindquist blocks sharing identical labels.

Unprimed labels indicate that the blocks are time-oriented according to KdS^*, primed labels are worn by blocks with the opposite time-orientation.

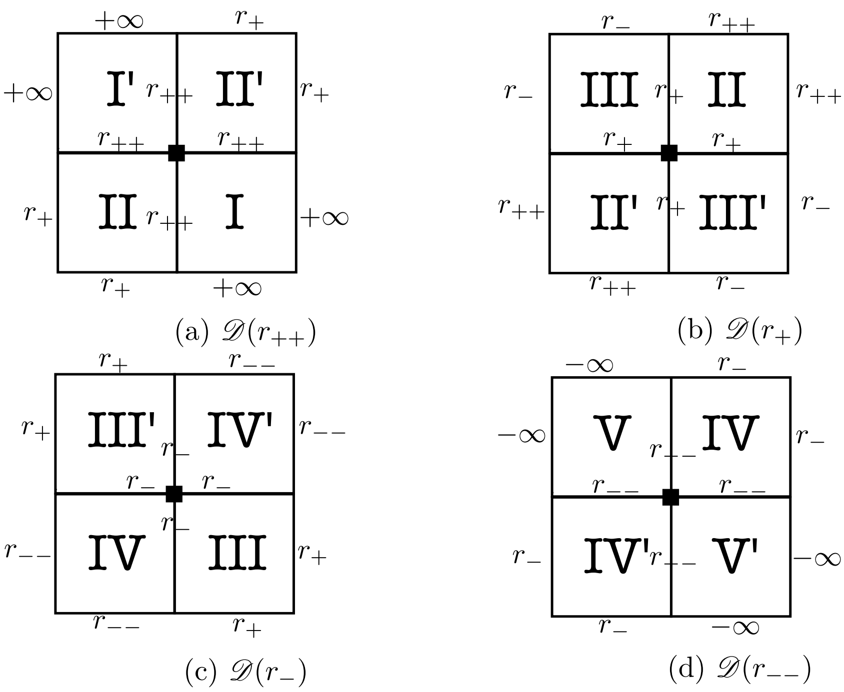


Figure 2.3 – Kruskal domains, the black square is the crossing-sphere (see section 2.5.4)

The Kruskal domains are also built in two stages. First, chosen open sets - that contain selected Boyer-Lindquist blocks - are glued together using the isometries discussed at the end of section 2.5.3; the result of this will be a manifold $\mathscr{D}_0(r_i)$. However, closer analysis of the principal null geodesics contained within the horizons of KdS^* and $^*KdS$ will show that $\mathscr{D}_0(r_i)$ does not complete all principal null geodesics as required and will also need to be extended.

Let us consider, as an example, $\mathscr{D}_0(r_{++})$; the other domains can be constructed similarly. $\mathscr{D}_0(r_{++})$ is built according to figure 2.4. The details are as follows:

1. Begin with the manifold K_1 consisting of the open set containing blocks I* and II* in KdS^*. The "outgoing" principal null geodesics of block I* are future-incomplete. In order to extend them, glue the open set of $^*KdS'$ containing blocks *II and *I onto K_1 using the time-orientation preserving isometry of section 2.5.3 to identify the blocks I* and *I. It is necessary to use $^*KdS'$ as opposed to $^*KdS$ to ensure that the isometry preserves time-orientation. It may surprise the reader that, according to our terminology, we are extending an outgoing principal null geodesic using an

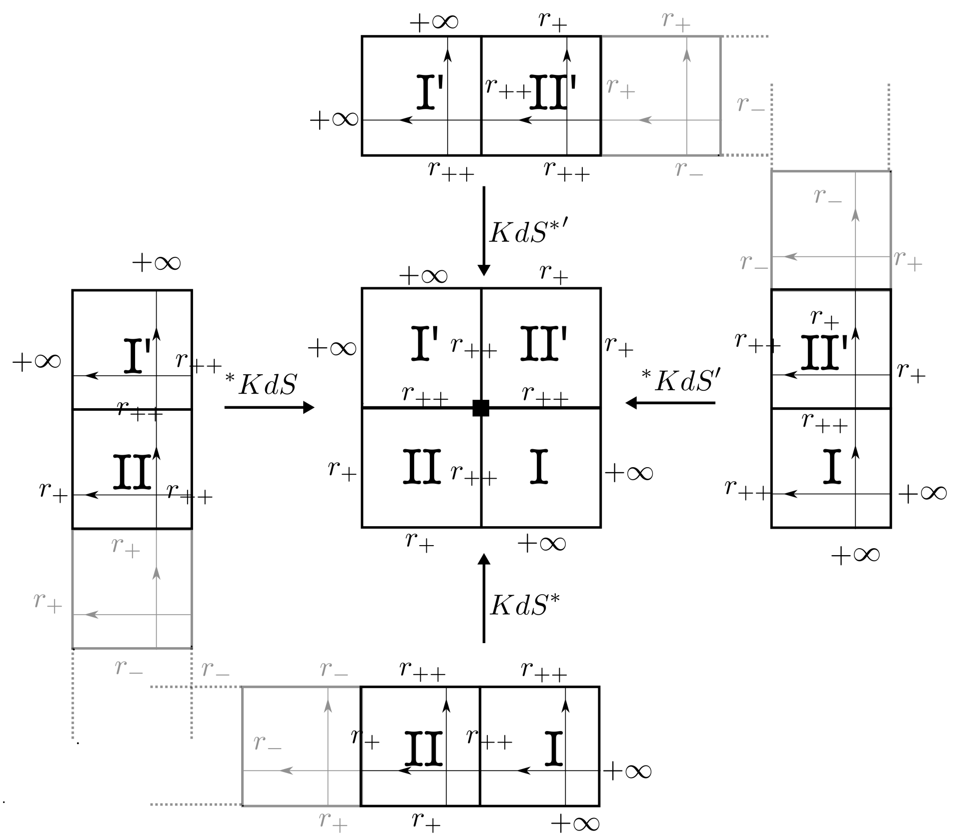


Figure 2.4 – Building $\mathscr{D}_0(r_{++})$

ingoing principal null geodesic. This is not really the case, as inspection of figure 2.1 reveals that the "outgoing" principle null geodesic of block I, is actually a badly named "ingoing" principle null geodesic, since $dr^*(n+) \leq 0$ on block I.

We verify briefly on this example that the condition of Lemma 2.5.5 is satisfied. Here the coordinate expression of $\phi : \mathrm{I}^* \longrightarrow {}^*\mathrm{I}$ is

$$\phi(t^*, r^*, \theta^*, \phi^*) = (t^* - 2T(r^*), r^*, \theta^*; \phi^* - 2A(r^*)).$$

Suppose that $(x_n)_{n\in\mathbb{N}} = (t^*_n, r^*_n, \theta^*_n, \phi^*_n)$ is a sequence of points in $U = I^*$ converging to a point on the horizon $r^* = r_{++}$, in particular the sequence $(t^*_n)_{n\in\mathbb{N}}$ has a finite limit, but $|T(r)| \underset{r\to r_{++}}{\longrightarrow} \infty$ so $(\phi(x_n))_{n\in\mathbb{N}}$ cannot converge.

2. Call K_2 the manifold obtained after step 1. We extend the outgoing principal null geodesics of block II in the same way, except that we use ${}^*KdS$, since on block II

time-orientation is preserved by the isometry of section 2.5.3.

3. Complete the manifold K_3 resulting from steps 1 and 2 by gluing the open set of $KdS^{*\prime}$ containing blocks I' and II' onto K_3 identifying, using the isometries of 2.5.3, I' *and* II' with those contained in K_3.

Crossing spheres

Our ambition is to construct a spacetime in which all principal null geodesics are complete (except those that run into the singularity). Until now, we have payed very little attention to those which are trapped in the horizons. To fix notations, consider KdS^*, but this discussion also holds with very minor modifications in ${}^*KdS$. Recall from section 2.5.2 that outgoing principal null geodesics are defined as geodesic reparametrisations of the integral curves of $n_+ = \frac{\Delta_r}{2\Xi}\partial_{r^*} + V$. For any point p on a horizon $\mathscr{H}$, $n_+(p) = V(p) \in T_p\mathscr{H}$.

Lemma 2.5.6. *Let $i \in \{--, -, +, ++\}$, then for any $p \in \mathscr{H}_i$:*

$$(\nabla_V V)|_p = \frac{1}{\Xi}\left(r_i - M - l^2 r_i(2r_i^2 + a^2)\right) V.$$

Lemma 2.5.7. *Call $k_i = \frac{r_i - M - l^2 r_i(2r_i^2+a^2)}{\Xi}, i \in \{--, -, +, ++\}$ then:*

$$k_{++} = -\frac{l^2}{2\Xi}(r_{++} - r_{--})(r_{++} - r_+)(r_{++} - r_-) < 0, \tag{2.58}$$

$$k_{+} = \frac{l^2}{2\Xi}(r_{+} - r_{--})(r_{++} - r_+)(r_{+} - r_-) > 0, \tag{2.59}$$

$$k_{-} = -\frac{l^2}{2\Xi}(r_{-} - r_{--})(r_{++} - r_-)(r_{+} - r_-) < 0, \tag{2.60}$$

$$k_{--} = \frac{l^2}{2\Xi}(r_{++} - r_{--})(r_{+} - r_{--})(r_{-} - r_{--}) > 0. \tag{2.61}$$

Proof. Follows immediately from the relation: $r_i - M - l^2 r_i(2r_i^2 + a^2) = \frac{1}{2}\frac{\partial}{\partial r}\Delta_r\Big|_{r=r_i}$ after factorisation of Δ_r: $\Delta_r = -l^2\prod_i(r - r_i)$.

Corollary 2.5.4. *Let $i \in \{--, -, +, ++\}$, then, if r_i is a root with multiplicity > 1 of Δ_r, then for any $p \in \mathscr{H}_i$:*

$$(\nabla_V V)|_p = 0.$$

Proposition 2.5.4.

1. *On horizons arising from a root of multiplicity* > 1 *of* Δ_r*, the integral curves of* V *are complete.*
2. *On the other horizons the integral curves of* V *are not complete.*

Proof. For the first point, according to Corollary 2.5.4 the integral curves of n_+ are already geodesically parametrised. Furthermore, since V is a constant linear combination of the coordinate fields $\partial_{t^*}, \partial_{\phi^*}$, its integral curves are complete (i.e. they can be extended so that the interval of definition is $\mathbb{R}$).

Assume now that r_i is a simple root of Δ_r, then according to the above: $k_i \neq 0$, and the integral curves of n_+ are not geodesically parametrised. A generic integral curve of n_+ on $\mathscr{H}_i$ is given in KdS^* coordinates by:

$$\gamma(s) = ((r_i^2 + a^2)s + t_0^*, r_i, \theta_0, as + \phi_0^*), s \in \mathbb{R}.$$

Since ∂_{ϕ^*} and ∂_{t^*} are global Killing fields on KdS^*, it suffices to consider the case where $t_0^* = \phi_0^* = 0$. When geodesically parametrised and the affine parameter chosen so that $\tilde{\gamma} = \gamma \circ s(\lambda)$ is future-oriented, we have:

$$\tilde{\gamma}(\lambda) = \left((r_i^2 + a^2)k_i^{-1} \ln(k_i\lambda), r_i, \theta_0, ak_i^{-1} \ln(k_i\lambda)\right), k_i\lambda > 0, \tag{2.62}$$

which cannot be extended though $\lambda \to 0$.

Remark 2.5.2. — On $KdS^{*\prime}$ where orientation is reversed, the future-oriented geodesic parametrisation of the integral curves is:

$$\tilde{\gamma}(\lambda) = \left((r_i^2 + a^2)k_i^{-1} \ln(-k_i\lambda), r_i, \theta_0, ak_i^{-1} \ln(-k_i\lambda)\right), k_i\lambda < 0. \tag{2.63}$$

— The formulae for ${}^*KdS$ et ${}^*KdS'$ are obtained by the substitution:

$$t^* \to {}^*t, \phi^* \to {}^*\phi.$$

Sending $\lambda \to 0$ in formulae (2.62),(2.63), it would seem that $\tilde{\gamma}(\lambda)$ approaches a point that would be located at the center of each of the diagrams of figure 2.3. We now seek to construct an analytic extension $\mathscr{D}(r_i)$ of each $\mathscr{D}_0(r_i)$ that contains such a limit point, this will be achieved by building a new system of coordinates.

Definition 2.5.6.

$$\begin{aligned}
A(r) &= \frac{a}{2\kappa_{--}}\ln|r-r_{--}| - \frac{a}{2\kappa_{-}}\ln|r-r_{-}| + \frac{a}{2\kappa_{+}}\ln|r-r_{+}| - \frac{a}{2\kappa_{++}}\ln|r-r_{++}|,\\
T(r) &= \frac{r_{--}^2+a^2}{2\kappa_{--}}\ \ln|r-r_{--}| - \frac{r_{-}^2+a^2}{2\kappa_{-}}\ln|r-r_{-}| + \frac{r_{+}^2+a^2}{2\kappa_{+}}\ln|r-r_{+}|\\
&\quad - \frac{r_{++}^2+a^2}{2\kappa_{++}}\ln|r-r_{++}|,\\
\kappa_i &= \operatorname{sgn}(k_i)k_i, \quad i\in\{--,-,+,++\}.
\end{aligned}$$

The proofs of the following technical lemmata are left to the reader:

Lemma 2.5.8. *For each* $i\in\{--,-,+,++\}, A(r)-\dfrac{a}{r_i^2+a^2}T(r)$ *is analytic at* r_i.

Lemma 2.5.9. *Let* $i\in\{--,-,+,++\}$*: On any Boyer-Lindquist block (minus points on the axis* $\mathcal{A}$*), the functions* $({}^*t, t^*, \theta, \phi^i)$*, where* $\phi^i = \dfrac{1}{2}\left({}^*\phi+\phi^* - \dfrac{a}{r_i^2+a^2}({}^*t+t^*)\right)$ *form a coordinate chart.*

We specialise now to $\mathscr{D}(r_{++})$:

Definition 2.5.7. Define maps U^{++}, V^{++} on $\mathscr{D}(r_{++})$ by:

$$\text{On I'}: \begin{cases} U^{++} = -\exp\left(\frac{\kappa_{++}{}^*t}{r_{++}^2+a^2}\right),\\ V^{++} = \exp\left(-\frac{\kappa_{++}t^*}{r_{++}^2+a^2}\right),\end{cases} \qquad \text{On II}: \begin{cases} U^{++} = -\exp\left(\frac{\kappa_{++}{}^*t}{r_{++}^2+a^2}\right),\\ V^{++} = -\exp\left(-\frac{\kappa_{++}t^*}{r_{++}^2+a^2}\right),\end{cases}$$

$$\text{On II'}: \begin{cases} U^{++} = \exp\left(\frac{\kappa_{++}{}^*t}{r_{++}^2+a^2}\right),\\ V^{++} = \exp\left(-\frac{\kappa_{++}t^*}{r_{++}^2+a^2}\right),\end{cases} \qquad \text{On I}: \begin{cases} U^{++} = \exp\left(\frac{\kappa_{++}{}^*t}{r_{++}^2+a^2}\right),\\ V^{++} = -\exp\left(-\frac{\kappa_{++}t^*}{r_{++}^2+a^2}\right).\end{cases}$$

Recall that on I,I' $r > r_{++}$ and on II,II' $r_+ < r < r_{++}$.

Lemma 2.5.10.

— U^{++}, V^{++}, θ *and* ϕ^{++} *have analytic extensions to all of* $\mathscr{D}_0(r_{++})\setminus\ \{$*axis points*$\}$ *(that we will denote by the same symbols). Furthermore* $\eta^{++} = (U^{++}, V^{++}, \theta, \phi^{++})$ *is a coordinate system on* $\mathscr{D}_0(r_{++})\setminus\{$*axis points*$\}$.

— η^{++} *has an analytic extension to a diffeomorphism of* $\mathscr{D}_0(r_{++})$ *onto* $\mathbb{R}^2\setminus\{(0,0)\}\times S^2$.

— r *has an analytic extension to all of* $\mathbb{R}_{U^{++}}\times\mathbb{R}_{V^{++}}\times S^2$.

— $r \mapsto G^{++}(r) = \frac{r-r_{++}}{U_{++}V_{++}}$ *is an analytic function of* $r \notin \{r_-, r_+, r_{--}\}$ *that never vanishes.*

Proposition 2.5.5. *In the coordinates* η^{++} *of* $\mathscr{D}_0(r_{++}) \setminus \{$*axis points*$\}$*, the line element can be expressed as:*

$$\begin{aligned}
ds^2 =& \frac{\Delta_r (r_{++}^2 + a^2) G^{++2}(r) a^2 \sin^2\theta}{4\kappa_{++}^2 (r - r_{++})(r^2+a^2)\Xi^2\rho^2} (r + r_{++}) \left(\frac{\rho^2}{r^2+a^2} + \frac{\rho_{++}^2}{r_{++}^2 + a^2} \right) \\
& \qquad \times \left(V^{++2} dU^{++2} + U^{++2} dV^{++2} \right) \\
&+ g_{\theta\theta} d\theta^2 + g_{\phi\phi}^2 d\phi^2 \\
&+ \frac{\Delta_r (r_{++}^2 + a^2)^2 G^{++2}(r)}{2\kappa_{++}^2 (r - r_{++})\rho^2 \Xi^2} \left(\frac{\rho^4}{(r^2+a^2)^2} + \frac{\rho_{++}^4}{(r_{++}^2+a^2)^2} \right) dU^{++}\ dV^{++} \\
&+ \frac{a \sin^2\theta G^{++}(r)}{\rho^2 \Xi^2 \kappa_{++}} \Bigg(\Delta_\theta (r + r_{++})(r^2 + a^2) \\
& \qquad - \frac{\Delta_r \rho_{++}^2}{r - r_{++}} \Bigg) d\phi^{++} \left(V^{++} dU^{++} - U^{++} dV^{++} \right) \\
&+ \frac{\Delta_\theta a^2 \sin^2\theta (r + r_{++})^2}{4\kappa_{++}^2 \rho^2 \Xi^2} \left(V^{++} dU^{++} - U^{++} dV^{++} \right)^2,
\end{aligned} \tag{2.64}$$

where $\rho_{++}^2 = r_{++}^2 + a^2\cos^2\theta$.

The above expression extends analytically to all of $(\mathbb{R}_{U^{++}} \times \mathbb{R}_{V^{++}}) \times S^2$ and it is straightforward to verify that it is non-degenerate at points of $\{(0,0)\} \times S^2$. This concludes the construction of $\mathscr{D}(r_{++})$ which is defined as $(\mathbb{R}_{U^{++}} \times \mathbb{R}_{V^{++}}) \times S^2$ equipped with the metric (2.64). Similar expressions for the metric can be obtained on the other Kruskal domains. We can now check that these extra points really do enable the extension of incomplete principal null geodesics contained in the horizons by welding together those from the different Boyer-Lindquist blocks. Recall from equation (2.63) the geodesic parametrisation of a generic integral curve, expressed in KdS^* coordinates, contained in the horizon $\mathscr{H}_i$ and coming from $KdS^{*\prime}$ (see figure 2.4):

$$\tilde{\gamma}(\lambda) = \left((r_{++}^2 + a^2) k_{++}^{-1} \ln(-k_{++}\lambda), r_i, \theta_0, a k_{++}^{-1} \ln(-k_{++}\lambda) \right), \lambda > 0,$$

this curve is past-incomplete and its expression in Kruskal coordinates is:

$$\begin{cases} U^{++} = 0, \\ V^{++} = -k_{++}\lambda, \\ \theta = \theta_0, \\ \phi^{++} = -\lim_{r \to r_{++}} A(r) - \dfrac{a}{r_{++}^2 + a^2} T(r), \end{cases} \qquad \lambda \in \mathbb{R}_+^*, \tag{2.65}$$

from these expressions we see that when $\lambda \to 0$, γ approaches a point on the crossing-sphere ($U^{++} = V^{++} = 0$).

If we consider now a similar curve in the horizon coming from KdS^*, then its geodesic parametrisation in KdS^* coordinates is, from (2.62):

$$\tilde{\gamma}(\lambda) = \left((r_{++}^2 + a^2) k_{++}^{-1} \ln(k_{++}\lambda), r_{++}, \theta_0, a k_{++}^{-1} \ln(k_{++}\lambda) \right), \lambda < 0.$$

This curve is future incomplete; converting to Kruskal coordinates:

$$\begin{cases} U^{++} = 0, \\ V^{++} = -k_{++}\lambda, \\ \theta = \theta_0, \\ \phi^{++} = -\lim_{r \to r_{++}} A(r) - \dfrac{a}{r_{++}^2 + a^2} T(r), \end{cases} \qquad \lambda \in \mathbb{R}_-^*. \tag{2.66}$$

The curves clearly analytically extend one another to form a complete geodesic. Through this example, we see that the role of the crossing-sphere ($U^{++} = V^{++} = 0$) really is to join together the two "vertical" horizons in figure 2.4 to form a single null hypersurface of equation $U^{++} = 0$. The results are similar when considering the principal null geodesics in the "horizontal" horizons of figure 2.4.

Building maximal slow Kerr-de Sitter KdS_s

We will now describe how to combine the Kruskal domains of section 2.5.4 to build the maximal slow Kerr-de Sitter spacetime KdS_s; the gluing pattern is illustrated in figure 2.5.

To realise the gluing, begin with the two manifolds K_1, K_2 defined by:

— K_1 is the manifold obtained by considering two sequences $(D_i^+)_{i\in\mathbb{Z}}, (D_j^-)_{j\in\mathbb{Z}}$ of isometric copies of $\mathscr{D}(r_+)$ and $\mathscr{D}(r_-)$ respectively. Define: $X = \coprod_i D_i^+$, $Y = \coprod_j D_j^-$. We

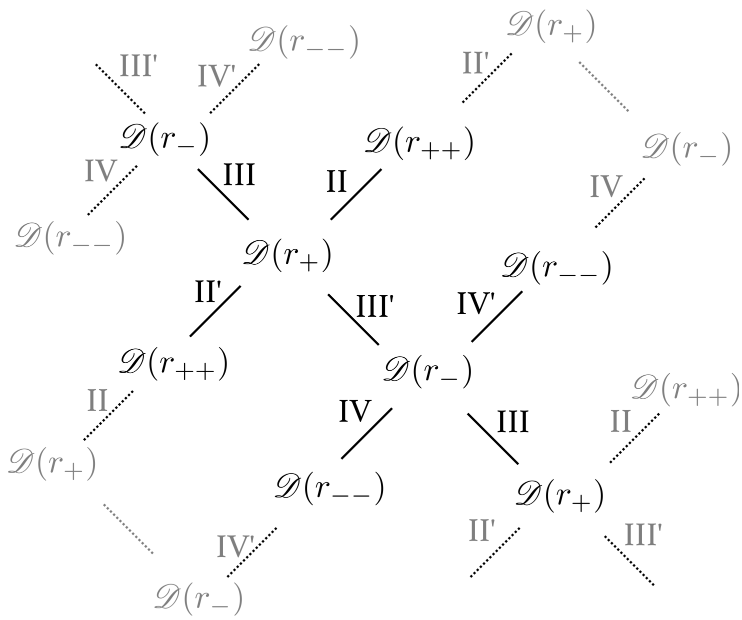


Figure 2.5 – Gluing pattern to construct KdS_s; the roman numeral labels indicate which Boyer-Lindquist block is used for the gluing

introduce some notations useful in the sequel:

— For each $k \in \mathbb{Z}$ denote by $i_+^k : D_k^+ \simeq \mathscr{D}(r_+) \to X$ and $i_-^k : D_k^- \simeq \mathscr{D}(r_-) \to Y$ the canonical injections.

— For any Boyer-Lindquist block $\mathcal{B} \subset \mathscr{D}(r_\pm)$, $\mathcal{B}_i^\pm$ will denote the image of that block by the isometry $\mathscr{D}(r_\pm) \simeq D_i^\pm$.

— $\boldsymbol{\mathcal{B}}_i^\pm = i_\pm^i(\mathcal{B}_i^\pm)$.

Define now [13]: $K_1 = X \coprod_\phi Y$ where $\phi : \coprod_i III_i \cup III'_i \to Y$ is constructed using the universal property of coproducts from the maps:

$$\phi_i : III_i \cup III'_i \subset D_i^+ \longrightarrow \boldsymbol{III_i} \cup \boldsymbol{III'_{i-1}} \subset Y,$$

which, when restricted to III_i (resp. III'_i) and expressed in Boyer-Lindquist coordinates, is simply the identity map.

— $K_2 = (\coprod_i D_i^{++}) \coprod (\coprod_j D_j^{--})$ is the disjoint union of the sequences $(D_i^{++})_{i\in\mathbb{Z}}, (D_j^{--})_{j\in\mathbb{Z}}$ of isometric copies of $D_i^{++} \simeq \mathscr{D}(r_{++})$ and $D_j^{--} \simeq \mathscr{D}(r_{--})$.

As illustrated in section 2.5, KdS_s can be built from K_1 and K_2 by gluing infinitely

13. see appendix A.5

many copies of these manifolds along blocks with the same label. More precisely, consider two sequences $(M_i)_{i\in\mathbb{Z}}$ and $(N_j)_{j\in\mathbb{Z}}$ of manifolds. This time, for each $i \in \mathbb{Z}$, M_i (resp. N_i) is an isometric copy of K_1 (resp. K_2). Define $\tilde{X} = \coprod_i M_i, \tilde{Y} = \coprod_j M_j$ and denote by $I_i : M_i \to \tilde{X}$ and $J_i : N_i :\to \tilde{Y}$ the canonical injections. KdS_s will then be $\tilde{X} \coprod_\psi \tilde{Y}$ for a well chosen isometry ψ.

ψ can be specified in several stages from maps $(\psi_k^{\pm\, i})_{(i,k)\in\mathbb{Z}^2}$:

$$\psi_k^{+\, i} : II_k \cup II'_k \subset D_k^+ \longrightarrow \boldsymbol{II}^{++}_{\boldsymbol{(i,k)}} \cup \boldsymbol{II'}^{++}_{\boldsymbol{(i-1,k)}} \subset \tilde{Y},$$
$$\psi_k^{-,\, i} : IV'_k \cup IV'_k \subset D_k^- \longrightarrow \boldsymbol{IV'}^{++}_{\boldsymbol{(i,k)}} \cup \boldsymbol{IV}^{--}_{\boldsymbol{(i-1,k)}} \subset \tilde{Y},$$

where, $\boldsymbol{II}^{++}_{\boldsymbol{(i,k)}} = J_i \circ i^{++}_{(i,k)}(II)$ and $i^{++}_{(i,k)}$ is the canonical injection of D_k^{++} into N_i; the other sets are defined similarly. Again, when restricted to a given Boyer-Lindquist block and expressed in Boyer-Lindquist coordinates, these are just the identity maps. Using a natural generalisation of point 3 of proposition A.5.1 in appendix A.5, for every $i \in \mathbb{N}$ this specifies a map:

$$\psi^i : \bigcup_{k\ \in\mathbb{Z}} \bar{i}_+^{(i,k)}(II_k \cup II'_k) \cup \bar{i}_-^{(i,k)}(IV_k \cup IV'_k) \subset M_i \to \tilde{Y}.$$

These maps, using the universal property of coproducts, define together an isometry:

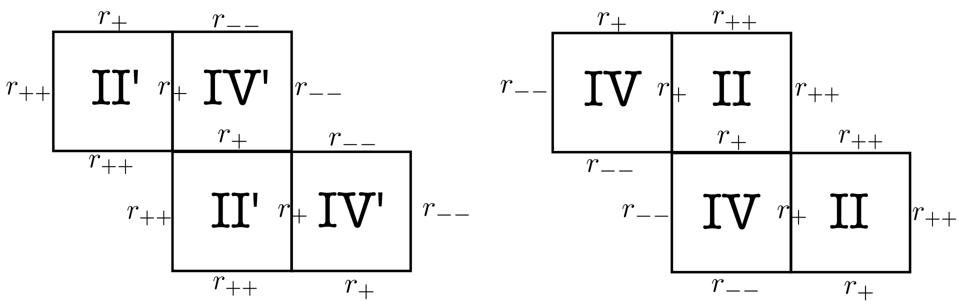
$$\psi : \coprod_{i\in\mathbb{Z}} \bigcup_{k\ \in\mathbb{Z}} \bar{i}_+^{(i,k)}(II_k \cup II'_k) \cup \bar{i}_-^{(i,k)}(IV_k \cup IV'_k) \subset M_i \to \tilde{Y}.$$

2.5.5 Maximal extreme and fast KdS spacetimes

Straightforward adaptations of the techniques of the previous section enable us to construct the maximal extreme and fast KdS spacetimes. For the extreme spacetimes, as discussed in Section 2.4, there are three cases: $r_+ = r_-$, $r_{++} = r_+$ or $r_{++} = r_+ = r_-$.

$KdS_e^1 : r_+ = r_-$

We begin with the case where the two black hole horizons coincide and in which the Boyer-Lindquist block III disappears. The Kruskal domains $\mathscr{D}(r_{--})$ and $\mathscr{D}(r_{++})$ are unchanged, but the domains $\mathscr{D}(r_+)$ and $\mathscr{D}(r_-)$ are to be replaced by the domains I_1 and I_2 given in figure 2.6. The form of these domains can be understood from the fact that the horizon $\mathscr{H}_+$ now arises from a double root and the principal null geodesics trapped in it are complete; in particular there are no crossing spheres on the double horizons. The



(a) I_1 (b) I_2

Figure 2.6 – Kruskal domains

slightly simpler gluing pattern is illustrated in figure 2.7. As before, the roman numeral labels indicate the blocks that are identified.

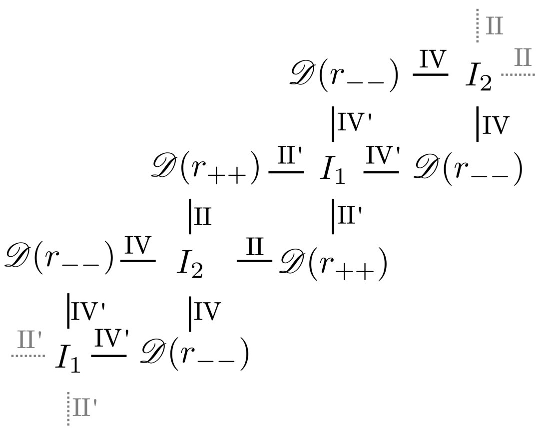


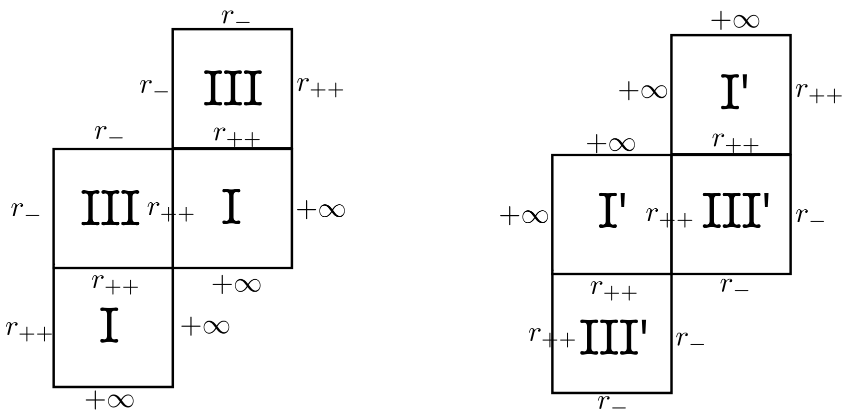
Figure 2.7 – Gluing pattern for KdS_e
$r_+ = r_-$

$KdS_e^2 : r_+ = r_{++}$

The second case is when the cosmological horizon r_{++} coincides with the outer black hole horizon r_+. Here the Kruskal domains $\mathscr{D}(r_{--})$ and $\mathscr{D}(r_-)$ are unchanged and the remaining blocks are replaced by the domains illustrated in figure 2.8. The stranger gluing pattern is illustrated in figure 2.9.

$KdS_e^3 : r_{++} = r_+ = r_- = x$

When Δ_r has a triple root x, we saw previously that all the horizons in the region $r > 0$ coincide; Boyer-Lindquist blocks II and III consequently vanish. Contrary to the



(a) I_1 (b) I_2

Figure 2.8 – Kruskal domains

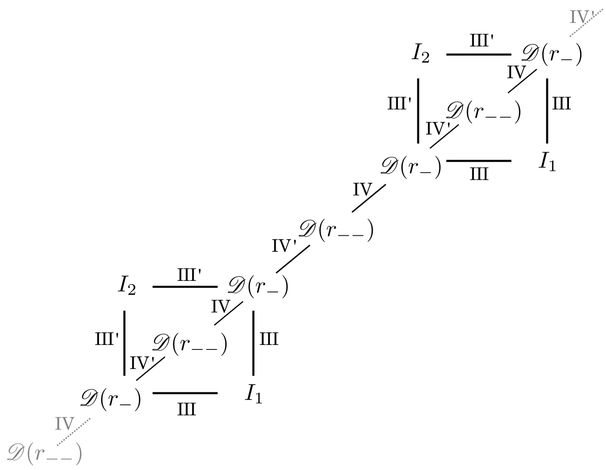


Figure 2.9 – Gluing pattern for KdS_e^2
$r_{++} = r_+$

other cases, only two Kruskal domains are required to construct a maximal extension: the domain $\mathscr{D}(r_{--})$, as illustrated in 2.3, and the domain $\mathscr{D}_0(x) \equiv \mathscr{D}(r_{++})$ illustrated in figure 2.10.

Diagram 2.10 has a striking ressemblance to that of $\mathscr{D}(r_{++})$ in figure 2.3, but is profoundly different due to the absence of the crossing sphere. Hence, whilst correctly depicting the assembly process leading to $\mathscr{D}_0(x)$, it is misleading for the interpretation of the geometry. In particular, like for the double horizons, Kruskal coordinates do not have analytic extensions to the whole domain.

As expected, the gluing pattern for KdS_e^3, illustrated in figure 2.11, is much simpler than in the other cases due to the fewer number of horizons and Boyer-Lindquist blocks.

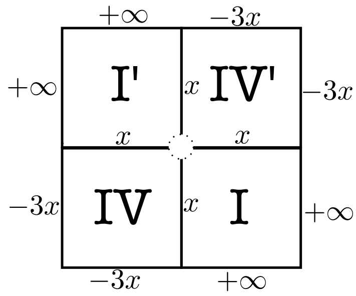


Figure 2.10 – $\mathscr{D}_0(x) \equiv \mathscr{D}(r_{++})$

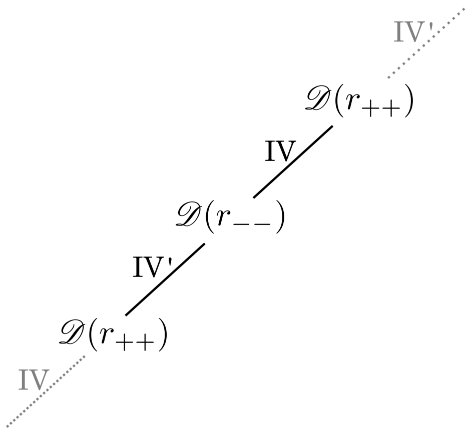


Figure 2.11 – Gluing pattern for KdS_e^3
$r_{++} = r_+ = r_- = x \quad r_{--} = 3x$

Maximal Fast KdS spacetimes

This final case, where Δ_r has only two simple real roots r_{--} and r_{++}, is in all points analogous to slow Kerr-spacetime as presented in [ONe14]; the main qualitative difference is that time orientation is reversed. There are only two Kruskal domains, $\mathscr{D}(r_{++})$ and $\mathscr{D}(r_{--})$ as illustrated in figure 2.3, with the exception that, due to the absence of blocks II and III , labels II and II' in figure 2.3 should be replaced by IV and IV' respectively. The gluing pattern is identical to that in figure 2.11.

2.6 Conclusion

The aim of this rather technical note was to give a detailed mathematical discussion regarding the construction of maximal analytical extensions to the Kerr-de Sitter solution to Einstein's equation with cosmological constant, as well as a review of the basic geometric properties of these spacetimes. The latter discussion can be found in Section 2.3. To the best of the author's knowledge, in existing literature, the construction is only briefly commented upon and is not carried out explicitly as in Section 2.5.

Section 2.4 is devoted to the study of the roots of the polynomial Δ_r in terms of the parameters (a, l, M), and hence, the horizon structure of the blackhole. The referees brought to the attention of the author that similar discussions, although less mathematical, are present in earlier publications, namely [SS04] for Kerr-de Sitter and [SH00] for the more general situation of Kerr-Newmann black holes on a background with non-zero cosmological constant.

Acknowledgements

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CHAPTER 3

AN ANALYTICAL SCATTERING THEORY FOR MASSIVE DIRAC FIELDS IN EXTREME KERR-DE SITTER SPACETIME

3.1 Preamble

In this chapter, we will study the Dirac equation near an *extreme* Kerr-de Sitter blackhole. More specifically, we are interested in the case where the two blackhole horizons coincide (see [Bor18, Proposition 4]) to form what we will call a double horizon. In physics, the Dirac equation describes free spin-1/2 particles, like the electron. Although physicists are more interested in the second quantised version, we study here the *classical* equation. We also consider the equation on a fixed geometric background, therefore ignoring the retroaction of the particle on the gravitational field; this is known as the linear approximation. Our fields will evolve in a region B situated between the double horizon and the cosmological horizon. Both will be treated as asymptotic regions: no boundary conditions will be set there. The choice of this particular region B is motivated by a number of interesting properties that lead us to suspect that it was possible to adapt and generalise the methods in [NH04; Dau10] in order to construct a scattering theory. On one hand, there is a global Killing field $\frac{\partial}{\partial t}$ on B associated with a function t whose level hypersurfaces are spacelike and isometric. This means that we can assimilate B to the direct product $\mathbb{R} \times \Sigma$ where Σ is a fixed Riemannian 3-fold, and reformulate the Dirac equation as an evolution problem on Σ. An advantage of Dirac fields, as opposed to Klein-Gordon fields, is that despite superradiance due to the rotation, there is still a conserved current leading to a natural norm (and inner product) on space-like slices and giving the necessary framework for spectral methods.

The setting is hence as follows: we have a Hilbert space $\mathscr{H}$ and a self-adjoint operator H on $\mathscr{H}$ with dense domain $D(H)$. Near the horizons, H intuitively approaches a simpler operator H_0; our aim is to understand to what extent this comparison is meaningful. More precisely, we seek to show the existence of strong limits of the following type:

$$s - \lim_{t \to +\infty} e^{iH_0 t} e^{-iHt} \equiv \Omega_+. \tag{3.1}$$

Ω_+ is known as a wave operator; when it exists, it satisfies for any $\phi_0 \in \mathscr{H}$:

$$\left\| e^{-iH_0 t} \Omega_+ \phi_0 - e^{-iHt} \phi_0 \right\| \underset{t \to +\infty}{\longrightarrow} 0.$$

Therefore, as long as we replace the initial data ϕ_0 by $\Omega_+ \phi_0$, we can use H_0 to describe the solution in the limit $t \to +\infty$. In general, Ω_+ will not be defined on all of $\mathscr{H}$. At best, it will be defined on the subspace $\mathscr{H}_{ac}$ composed of all vectors $x \in \mathscr{H}$ whose spectral measure $\mu_x(S) = (\mathbb{E}(S)x, x)$ is absolutely continuous with respect to the Lebesgue measure. In our particular case, the Dirac equation is fully separable and the existence of the cosmological horizon excludes the possibility for any discrete spectrum [BC09; BC10]. Moreover, via Mourre theory [Mou81] [1], we will show that there is no singular continuous spectrum. These facts imply that: $\mathscr{H} = \mathscr{H}_{ac}$.

Nonetheless, we will not be able to prove the existence of (3.1) directly, and will need to adjust the definition of Ω_+ to take into account some long-range effects. This is because the usual elementary methods to prove the existence of the wave operators rely on the assumption that $H - H_0$ is short-range in a neighbourhood of the asymptotic regions, which is not satisfied near the double horizon. Furthermore, the rotation of the blackhole adds a certain amount of anisotropy to the picture. We will show that all of these difficulties can be overcome through several intermediate comparisons and an *ad hoc* decomposition of the Hilbert space, which, surprisingly, enables us to reduce our problem to a spherically symmetric one. Mourre theory will play an important role in this part of the proof, since it will enable us to establish so-called « asymptotic velocity estimates » [SS88; GF98] which are essential in the proof of the existence of some of our intermediate wave operators. Last of all, at the double horizon, we will need to slightly modify H_0 to compensate for the fact that $H - H_0$ is long range there. To this end, we will perform a Dollard type modification [DV66] which consists in incorporating to e^{itH_0} the time evolution of the Coulomb potential terms that obstruct the existence of the classical wave operators.

An important prerequisite underlying the techniques used in this part of the work

1. see also Section 3.5

is the ability to construct a good functional calculus. Formulae like that of Helffer-Sjöstrand [HS87] are key, for instance, in showing that some of the operators we study are compact. This perspective was particularly elucidating whilst studying [AMG96]; a key reference for the techniques in the main text.

The following text is in a large part taken from a prepublication [Bor20] submitted for publication in the *Annales de l'Institut Fourier*. I have additionally included in the preamble a short discussion on the geometry of Dirac fields.

3.2 A short note on spinors and spin structures

The Dirac equation, in itself, is a fascinating mathematical object. Before our presentation of the analytical scattering theory in extreme Kerr-de Sitter spacetime, it is worth inspecting some of the geometry behind it. The first feature we should point at is the nature of the unknown. The Dirac field, or spinor, is different in nature from the tensor fields that appear in most equations in Physics, indeed, they do not correspond to a representation of the proper Lorentz group $SO_+(1,3)$, but rather of a particular 2 leaf covering [2], $Sp(1,3)$, which, due to an accidental isomorphism in dimension 4 can be identified with $SL(2,\mathbb{C})$.

The exceptional isomorphism can be described as follows: first we identify $\mathbb{R}^4$ with the vector space of 2×2 Hermitian matrices via:

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix} = \underline{x},$$

The relation:

$$\underline{\Lambda(A)x} = A\underline{x}A^*, \quad A\in SL(2,\mathbb{C}), A^* = {}^t\bar{A}, \tag{3.2}$$

then defines a representation of $SL(2,\mathbb{C})$. One can show that Λ maps $SL(2,\mathbb{C})$ onto $SO_+(1,3)$ and that its kernel is $\{-1,1\}$.

The group $Sp(1,3)$ is defined as a subgroup of the group of invertible elements of the Clifford algebra $Cl_{1,3}(\mathbb{R})$. In brief, $Cl_{1,3}(\mathbb{R})$ is the « most general » real algebra with unit

2. which is simply connected so it is in fact the universal covering

containing $\mathbb{R}^4$ such that for any $v \in V$,

$$v^2 = \eta(v,v) \cdot 1, \quad \eta(v,v) = -v_0^2 + \sum_{i=1}^{3} v_i^2.$$

$Sp(1,3)$ is the identity component of the group generated by the set:

$$\{v_1 \dots v_{2p}, p \in \mathbb{N}, v_i \in \mathbb{R}^4, \eta(v_i) = \pm 1\}.$$

The geometric explanation to this definition is that if $v \in \mathbb{R}^4$ with $\eta(v,v) = \pm 1$, then v is invertible in the algebra and the inverse is given by $v^{-1} = \frac{v}{\eta(v,v)}$. Consequently the map $x \in \mathbb{R}^4 \mapsto -vxv^{-1}$ is an endomorphism of $\mathbb{R}^4$ and:

$$-vxv^{-1} = -vx\frac{v}{\eta(v,v)} = x - 2\frac{\eta(v,x)}{\eta(v,v)}v.$$

Hence, $x \mapsto -vxv^{-1}$ is the reflection about the plane orthogonal to v. Since such reflections generate $O(1,3)$, after correctly generalising $x \mapsto -vxv^{-1}$ to more general elements of $Cl_{1,3}(\mathbb{R})^{\times}$, we have a way of reproducing $O(1,3)$. Restricting to an even number of factors, we get $SO(1,3)$. The important point is that the algebra multiplication has a strict relationship with the metric η.

A spinor representation is a complex representation of $Sp(1,3)$ induced by a representation of the Clifford algebra, $Cl_{1,3}(\mathbb{R})$. The isomorphism $SL(2,\mathbb{C}) \cong Sp(1,3)$, means that [3] it is not necessary for us to delve deeper into this side of things, and we can just think of spinors as corresponding to complex representations of $SL(2,\mathbb{C})$. The link between $SL(2,\mathbb{C})$ and $SO_+(1,3)$ is what justifies the point of view that these spinor representations are in some sense representations of $SO_+(1,3)$. For our needs, the most important representations are:

- **2-spinors** : the fundamental representation ρ_f de $SL(2,\mathbb{C})$,
- **pointed 2-spinors** : the representation $\rho_{fc} : A \mapsto \overline{A}$,
- **Dirac spinors** : the direct sum of the dual representation to ρ_f and ρ_{fc},
- **complexified vectors** : the product representation $\rho_f \otimes \rho_{fc}$.

In order to speak of spinor fields on an oriented Lorentzian manifold (M,g), we need to construct vector bundles corresponding to these spinor representations. This should be

3. much to my dismay

done in such a way that the Clifford algebra behind the spin representation, corresponds to the fibrewise metric of each tangent space. The solution involves a « reduction »[4] of the positive orthonormal frame bundle $\mathscr{F}_0(M)$ that lifts in each fibre $SO_+(1,3)$ to $SL(2,\mathbb{C})$. The precise definition, in 4 dimensions, is as follows:

Definition 3.2.1. Let M be a smooth manifold of dimension 4, a spin structure is an $SL(2,\mathbb{C})$-principal fibre bundle $(S(M), \pi_s)$ over M and a fibre bundle morphism $\lambda : S(M) \longrightarrow \mathscr{F}_0(M)$ into a reduction of $L(TM)$ to a $SO_+(1,3)$-principal fibre bundle over M, $(F_0(M), \pi)$, such that for any $s \in S(M)$, and $g \in SL(2,\mathbb{C})$:

$$\pi(\lambda(s)) = \pi_S(s), \tag{3.3}$$

$$\lambda(sg) = \lambda(s)\Lambda(g), \tag{3.4}$$

where the morphism $\Lambda : SL(2,\mathbb{C}) \longrightarrow SO_+(1,3)$ is defined by Equation (3.2).

The definition can be generalised to arbitrary dimension and signature (p,q) but this requires a little more theory than we can afford to explore here. We refer the interested reader to [Fri00; LM89]. We have also chosen to present the metric structure (hidden in the bundle $F_0(M)$) as *derived* from the spin structure. In practice, however, the metric structure, and by extension, the bundle $\mathscr{F}_0(M)$, is already given in advance and we seek a spin structure compatible with it. This problem can always be solved locally, for instance, in a local bundle chart of the bundle $\mathscr{F}_0(M)$, however, there may be an obstruction to a global solution [Ger68; Ger70]. For our needs, the following result of Geroch [Ger68; Ger70] will be sufficient:

Theorem 3.2.1. *An orientable and globally hyperbolic* 4*-dimensional spacetime has a spin structure.*

Unfortunately, a spin structure is generally not unique, see for example the n-spheres [Tra93]. The importance of this choice is however a global question and will not concern us.

We should mention some consequences of Definition 3.2.1. First of all, the Lie group homomorphism Λ induces a Lie algebra isomorphism between $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{so}_{1,3}(\mathbb{R})$, hence, a connection on either one of the bundles $S(M)$ or $\mathscr{F}_0(M)$ automatically determines one on the other: $S(M)$ will be systematically assumed to be equipped with the connection induced by the Levi-Civita connection on $\mathscr{F}_0(M)$.

4. The terminology in English is unfortunate, the French word « élargissement », describes better the situation we consider here...

If a 4-dimensional Lorentzian manifold M has a spin-structure, « spinor » bundles are defined as associated vector bundles to $(S(M), \lambda)$. We will write $\mathbb{S}$ and $\mathbb{S}'$ for the bundles corresponding respectively to the 2-spinor and pointed 2-spinor representations. The fibre bundle morphism λ guarantees the compatibility of $S(M)$ with the Lorentzian structure of the base M by inducing a vector bundle isomorphism:

$$\mathbb{S} \otimes \mathbb{S}' \cong \mathbb{C} \otimes TM.$$

When using the abstract index notation (cf. Paragraph 1.1.1), sections k and χ of $\mathbb{S}$ and $\mathbb{S}'$ respectively will be written k^A, and $\chi^{A'}$. There is also an anti-linear isomorphism between $\mathbb{S}$ and $\mathbb{S}'$, obtained by factorising complex conjugation. We will write:

$$\overline{k^A} = \overline{k}^{A'}.$$

Somewhat abusively, the inverse map will be written in the same way: $\overline{\chi^{A'}} = \overline{\chi}^A$.

Last of all, there is a canonical linear map:

$$\varepsilon : \Gamma(\mathbb{S}) \wedge \Gamma(\mathbb{S}) \to C^\infty(M),$$

written ε_{AB} and satisfying $g_{ab} = \varepsilon_{AB}\overline{\varepsilon}_{A'B'}$, where g is the metric tensor on M. In each of the fibres of $\mathbb{S}$ it restricts to a symplectic form. Just like a metric, ε_{AB} can be used to identify $\mathbb{S}$ to its dual:

$$k^A \mapsto k^A\varepsilon_{AB} \equiv k_B.$$

Due to the fact that it is *anti-symmetric*, we need to be a little more careful when raising and lowering indices, for example:

$$\begin{aligned} k^A &= \varepsilon^{AB}k_B = -k_B\varepsilon^{BA}, \\ k_B &= k^A\varepsilon_{AB} = -\varepsilon_{BA}k^A, \\ \delta_C^B &= \varepsilon_C{}^B = \varepsilon_{AC}\varepsilon^{AB} = -\varepsilon_{AC}\varepsilon^{BA} = -\varepsilon^B{}_C. \end{aligned}$$

The above properties of spin structures, as well as the close relationship with the bundle $\mathscr{F}_0(M)$ encourage us to think of $S(M)$ as analogous to a « square root » of $\mathscr{F}_0(M)$. In fact, in [PR84], Penrose argues that the spin structure is in some sense more fundamental than $\mathscr{F}_0(M)$. Elementary particles like electrons or quarks seem to be spinorial, so, if General Relativity has any part to play in a quantum theory of gravity, a spin structure is a pre-requisite. Spin structures are also topologically more restrictive than metrics, although Geroch's result points out that the restriction is transparent for many physically reasonable metrics. Nevertheless, we could imagine starting out with $SL(2, \mathbb{C}) \cong Sp(1, 3)$ principal fibre bundle $S(M)$ over a manifold M and seeking fibre bundle morphisms λ from $S(M)$ to the frame bundle $L(TM)$ over the base. If we find such a λ, we can then identify the image as a positive orthonormal frame bundle $\mathscr{F}_0(M)$, then deduce the metric *and* orientation from it; which indicates that all the essential information of the spacetime is encoded in the spin structure.

Scattering theory for Dirac fields near an Extreme Kerr-de Sitter black hole

The following is a modified version of the author's final preprint of the eponymous article submitted to Annales de l'Institut Fourier in May 2020

3.3 Introduction

Over the past two decades or so there has been quite a bit of mathematical interest in scattering theories for particles in black-hole type geometries. This is useful for the understanding of these geometries and the detection of black holes but also in the study of Quantum Field Theory on curved spacetimes, see for example [DMP11; GHW20].

For rotating black holes, due to super-radiance, it is well known that the usual energy functional of integer spin particle fields, described for instance by the wave or Klein-Gordon equation, is no longer positive-definite, this leads to obvious technical difficulties that have nevertheless been overcome in a handful of situations, such as the Klein-Gordon equation on (De Sitter) Kerr spacetimes [GGH17; Häf03], the wave equation on Kerr spacetime [DRS14] or the Maxwell equation on the Reissner-Nordström de-Sitter spacetime [Mok16].

On the other hand, for Dirac fields, there is still a conserved current which leads to a natural Hilbert space framework adapted to a spectral theory approach. Scattering theories for massive or massless Dirac fields have been constructed in this manner in the exterior region of Reissner-Nordström, slow Kerr and Kerr-Newman black holes [Dau04; NH04]. More recently, there has been interest in non-asymptotically flat backgrounds such as Schwarzschild-de Sitter [Ide16], slow Kerr Newman-de Sitter [DN16] and slow Kerr-Newman-AdS [BC10] black holes.

In this paper we study the case of an *extreme* Kerr-de Sitter black hole in a region situated between what we will refer to as a "double" horizon and a usual "simple" one (the cosmological horizon). The "double" horizon is the hypersurface resulting from the coincidence of the two inner black hole horizons[5], and differs quite significantly from the exterior horizon of, for instance, Kerr spacetime. The extreme case is of particular interest for the understanding of mechanisms behind stability/instability of black hole

5. which occurs for special choices of the parameters of the family

type spacetimes as it presents features of both types. This is analysed thoroughly in the case of an Extreme Reissner-Nordström black hole in [Are11a; Are11b], and complemented by the remarks in [BS07] on the asymptotic behavior to the wave equation. Regarding the Dirac equation, an integral representation of the Dirac progagator in the extreme Kerr metric is derived in [BF13]. Our main theorem, Theorem 3.7.1, formulated in Section **??**, is the asymptotic completeness of the Dirac operator in an extreme Kerr-de Sitter black hole, which can perhaps be interpreted in this context as a stability feature of these spacetimes.

Our global strategy follows closely that of [Dau04; Dau10; NH04]: we will adopt the point of view of a class of observers for which the two horizons are asymptotic and will show in Section 3.5 that a conjugate operator in the sense of Mourre theory [AMG96; Mou81] can be constructed in an analogous fashion to that in the exterior of a Kerr black hole as in [Dau04; NH04]. Furthermore, it has already been noted, for example in [BC10], that the presence of the simple horizon is enough to ensure that the usual proof of the absence of eigenvalues – via a Grönwall inequality exploiting the separability of the Dirac equation – follows through without modification. However, our results do not follow directly from these works due to long-range potentials at the extreme horizon and a significantly perturbed angular operator. In particular, the decomposition of the Hilbert space into spin harmonics, essential to the reduction to the spherically symmetric case treated in [Dau10] is no longer stable. A key ingredient to our analysis, carried out in Section 3.6.4 is constructing operators at both asymptotic ends with similar adapted decompositions and of which the full Dirac operator is a short-range perturbation. Furthermore, it is worth noting that since the mass terms do not survive at either of the horizons, despite constituting a long-range potential near the double one, some of the arguments in [Dau10] can be simplified.

3.3.1 The Kerr-de Sitter metric

Throughout this text, we will mainly use the usual Boyer-Lindquist like coordinates (t, r, θ, φ) in which the Kerr-de Sitter metric is known to be (signature $(+,-,-,-)$):

$$g = \frac{\Delta_r}{\Xi^2\rho^2}[\mathrm{d}t - a\sin^2\theta\mathrm{d}\varphi]^2 - \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 - \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 - \frac{\Delta_\theta\ \sin^2\theta}{\rho^2\Xi^2}[(r^2+a^2)\mathrm{d}\varphi - a\mathrm{d}t]^2, \quad (3.5)$$

where:

$$
\begin{aligned}
l^2 &= \frac{\Lambda}{3}, \qquad \Delta_r = r^2 - 2Mr + a^2 - l^2 r^2 (r^2 + a^2), \\
\Xi &= 1 + a^2 l^2, \quad \Delta_\theta = 1 + a^2 l^2 \cos^2\theta, \quad \rho^2 = r^2 + a^2 \cos^2\theta.
\end{aligned}
\tag{3.6}
$$

It depends on three parameters a, M, Λ, the angular momentum per unit mass of the black hole, the mass of the black hole and the cosmological constant, respectively. We will always assume $l > 0$.

The above expression is singular when $\Delta_r = 0$ or $\rho = 0$, however, the manifold can be analytically extended across the singularities $\{\Delta_r = 0\}$. In such an extension, the roots of Δ_r give rise to null hypersurfaces that we will refer to as horizons. They will be labelled by the root r_i to which they correspond as so: $\mathscr{H}_{r_i}$. If r_i is a double (resp. simple) root of Δ_r, $\mathscr{H}_{r_i}$ will be said to be a "double" (resp. "simple") horizon. In, for instance, [Bor18], it is shown that the roots of Δ_r can be labelled such that either:

1. $r_{--} < 0 < r_- < r_+ < r_{++}$
2. $r_{--} < 0 < r_- = r_+ < r_{++}$
3. $r_{--} < 0 < r_- < r_+ = r_{++}$
4. $r_{--} < 0 < r_- = r_+ - r_{++}$
5. $r_{--} < r_{++}, r_-, r_+ \in \mathbb{C} \setminus \mathbb{R}$.

We will refer to case (2) as extreme Kerr-de Sitter; a necessary and sufficient condition for this is:

$$
\begin{aligned}
&|a|l < 2 - \sqrt{3}, \\
&M^2 = \frac{(1 - a^2 l^2)(a^4 l^4 + 34 a^2 l^2 + 1) - \gamma^{\frac{3}{2}}}{54 l^2},
\end{aligned}
\tag{3.7}
$$

where $\gamma = (1 - a^2 l^2)^2 - 12 a^2 l^2$. In this situation the double root is given by:

$$
r_e{}^{6} = \frac{12 a^2 l^2 + (1 - a^2 l^2)(1 - a^2 l^2 - \sqrt{\gamma})}{18 M l^2}. \tag{3.8}
$$

For future reference, we quote the following useful properties of r_e :

$$
\begin{cases}
0 \leq r_e < \frac{4}{3}\frac{a^2}{M}, \\
l^2 r_e^4 + a^2 = M r_e.
\end{cases}
\tag{3.9}
$$

6. In [Bor18] it was denoted by x

Finally, we note that the other two roots r_{++} and r_{--} are equally those of the polynomial:

$$X^2 + 2r_e X - \frac{a^2}{l^2 r_e^2}. \tag{3.10}$$

To avoid unnecessarily complicated subscripts, we will now rename the roots of Δ_r as follows:

$$r_- < 0 < r_e < r_+.$$

The region, B, in which we will study the scattering of Dirac fields is defined in the coordinates (t, r, θ, φ) by $r_e < r < r_+$. In essence, $B = \mathbb{R}\times]r_e, r_+[\times S^2$, with the metric given by (3.5), that extends analytically to the poles. It is between two horizons, one double, one simple and it is the effect of the double horizon that we wish to understand.

The scattering problem will be considered from the point of view of a stationary observer with world-line:

$$r = r_0,\ \theta = \theta_0,\ \varphi = \omega t + \phi_0,\ \omega \in \mathbb{R}, r_0 \in]r_e, r_+[, \theta_0 \in]0, \pi[, \phi_0 \in]0, 2\pi[.$$

Proper time for such an observer differs from the coordinate function t only by a multiplicative constant depending on the parameters of the trajectory. For this family of observers photons travelling, say, along a principal null geodesic, which are in some sense the most direct trajectories for light to travel towards one of the horizons, will not reach it in finite time. For instance, the coordinate time t necessary for a photon, emitted from $r = r_0$ at $t = t_0$, to reach $\mathscr{H}_+$ travelling along such a curve is:

$$t - t_0 = \int_{r_0}^{r_+} \frac{\Xi(r^2 + a^2)}{\Delta_r} \mathrm{d}r = +\infty. \tag{3.11}$$

In fact, for our purposes, it will be appropriate to replace the coordinate r, by the Regge-Wheeler type coordinate $r^* = \int \frac{\Xi(r^2 + a^2)}{\Delta_r} \mathrm{d}r$ appearing in this computation. By definition:

$$\mathrm{d}r^* = \frac{\Xi(r^2 + a^2)}{\Delta_r} \mathrm{d}r. \tag{3.12}$$

It will be useful to calculate an explicit expression for r^* by a partial fraction decomposition of the integrand:

$$\frac{r^2 + a^2}{(r - r_-)(r - r_e)^2(r - r_+)} = \frac{\alpha}{r - r_-} + \frac{\beta}{r - r_+} + \frac{\gamma}{r - r_e} + \frac{\delta}{(r - r_e)^2}. \tag{3.13}$$

The coefficients $\alpha, \beta, \gamma, \delta$ are given by:

$$\alpha = -\frac{l}{2}\sqrt{\frac{r_e}{M}}\frac{r_-^2+a^2}{(r_e-r_-)^2} < 0, \quad \beta = \frac{l}{2}\sqrt{\frac{r_e}{M}}\frac{r_+^2+a^2}{(r_+-r_e)^2} > 0,$$
$$\delta = \frac{l^2 r_e^2(r_e^2+a^2)}{3Mr_e-4a^2} < 0, \quad \gamma = -\frac{2l^2 r_e^3(2r_e^2-7Mr_e+6a^2)}{(3Mr_e-4a^2)^2} < 0.$$

The sign of γ follows from the following relations:

$$\begin{cases} r_e^2 l^2(r_e^2+a^2) = r_e^2+a^2-2Mr_e, \\ 0 < 3Mr_e-4a^2-2r_e^2l^2(r_e^2+a^2) = 7Mr_e-6a^2-2r_e^2. \end{cases}$$

The expression of r^* is therefore:

$$\boxed{\begin{aligned} r^* = \frac{\Xi}{2l}\sqrt{\frac{r_e}{M}}\ln\left(\frac{|r-r_-|^{\eta_-}}{|r-r_+|^{\eta_+}}\right) + \frac{r_e^2(r_e^2+a^2)}{3Mr_e-4a^2}\frac{\Xi}{r-r_e} \\ + \frac{2r_e^3(2r_e^2-7Mr_e+6a^2)}{(3Mr_e-4a^2)^2}\Xi\ln|r-r_e| + R_0. \end{aligned}} \tag{3.14}$$

Above, R_0 is an arbitrary real constant and $\eta_\pm = \frac{r_\pm^2+a^2}{(r_e-r_\pm)^2}$.

From (3.14), one can deduce the following asymptotic equivalences:

Lemma 3.3.1.

$$r_+ - r \underset{r^*\to+\infty}{\sim} e^{-\frac{2l}{\Xi\eta_+}\sqrt{\frac{M}{r_e}}r^*}, \tag{3.15}$$

$$r - r_e \underset{r^*\to-\infty}{\sim} \frac{r_e^2(r_e^2+a^2)\Xi}{3Mr_e-4a^2}\frac{1}{r^*}. \tag{3.16}$$

(3.15) is true for a suitable choice of R_0: it is the usual behaviour that we have come to expect at a simple black hole horizon. The decay near the double horizon, however, is a lot slower and will be the source of technical difficulties when constructing a scattering theory.

3.3.2 The Dirac equation

Notations

On B, $\Delta_r > 0$ and the coordinate t is a "time function", providing a foliation $(\Sigma_t)_{t\in\mathbb{R}}$ of B into spacelike Cauchy hypersurfaces. B is therefore an orientable globally hyperbolic

4-manifold and as such, by a result due to R. Geroch [Ger68; Ger70], possesses a global spin structure.

The Dirac equation is most conveniently expressed with Penrose's abstract index notation [7] denoting by $\mathbb{S}^A$ the module of sections of the two-spinor bundle $\mathbb{S}$ and, $\mathbb{S}^{A'}$, that of the pointed two spinor bundle $\mathbb{S}'$; lowered indices are used for sections of the dual bundles. We recall that $\mathbb{S}^A$ is identified with $\mathbb{S}^{A'}$ via complex conjugation and to $\mathbb{S}_A$ via the canonical symplectic form ε_{AB} according to:

$$\begin{cases} \kappa_B = \kappa^A \varepsilon_{AB} = -\varepsilon_{BA}\kappa^A, \\ \kappa^{A'} = \overline{\kappa^A}, \end{cases} \qquad \kappa^A \in \mathbb{S}^A.$$

The bundle $\mathbb{S} \otimes \mathbb{S}'$ can be identified with the complexified tangent bundle $\mathbb{C} \otimes TB$ and finally:

$$\varepsilon_{AB}\varepsilon_{A'B'} = g_{ab}.$$

Following [Nic02], we will refer to elements of $\mathbb{S}_A \oplus \mathbb{S}^{A'}$ as *Dirac spinors*, the massive Dirac equation for a spin-$\frac{1}{2}$ Dirac spinor $(\phi_A, \chi^{A'})$ is then:

$$\begin{cases} \nabla^{AA'}\phi_A = \mu\chi^{A'}, \\ \nabla_{AA'}\chi^{A'} = -\mu\phi_A, \end{cases} \qquad \mu = \frac{m}{\sqrt{2}}\,. \tag{3.17}$$

As mentioned in the introduction, it is well known that the equation has a conserved current, namely:

$$j_{AA'} = \phi_A\bar{\phi}_{A'} + \chi_{A'}\bar{\chi}_A.$$

Thus the total charge:

$$Q = \int_{\Sigma_t} T^a j_a \omega_{g,\Sigma_t}, \tag{3.18}$$

is conserved. $\omega_{g,\Sigma_t} = \sqrt{\frac{\Delta_r}{\Delta_\theta}} \frac{\rho\sigma}{(r^2+a^2)\Xi^2} \mathrm{d}r^* \wedge (\sin\theta \mathrm{d}\theta \wedge \mathrm{d}\varphi)$ is the induced volume form on Σ_t [8] and T^a is colinear to $\nabla^a t$ and normalised, for convenience, such that $T^a T_a = 2$.

Q defines an inner product on spinors defined on any slice [9], Σ_t, $t \in \mathbb{R}$, and gives rise to a Hilbert space $\mathscr{H}_t$. Solving the Dirac equation can be thought of as finding a family

7. See again [PR84].
8. Oriented by $-\nabla t$.
9. These can be thought of as either sections of the pullback bundle of $\mathbb{S}$ via the canonical injection, or, sections of the spinor bundle on Σ_s; there is an identification between them since $\dim B = 4$.

of isometries $U(u,s) : \mathscr{H}_s \mapsto \mathscr{H}_u$ such that for any $u,s,w \in \mathbb{R}$:

$$U(s,s) = \mathrm{Id}, \quad U(u,s)U(s,w) = U(u,w).$$

The framework sketched here can nevertheless be significantly simplified since ∂_t is a global Killing field on B. All slices Σ_t are thus isometric, in particular, B is isometric to $\mathbb{R} \times \Sigma$ for some fixed Σ. Furthermore, the $\mathscr{H}_t$ can all be identified and so one can view the problem as an evolution problem on a fixed Hilbert space $\mathscr{H}$. For these reasons, we will seek expressly to write the Dirac equation as a Schrödinger type equation. Moreover, we will work directly with spinor densities [10] on Σ, i.e. the section of $(\mathbb{S} \oplus \mathbb{S}') \otimes \mathcal{E}(-\frac{n+1}{2})$ given by:

$$(\phi_A, \chi^{A'})|\omega_{g,\Sigma}|^{\frac{1}{2}}. \tag{3.19}$$

After a choice of spin-frame, this means that our Hilbert space $\mathscr{H}$ can be assimilated with $L^2(\Sigma) \otimes \mathbb{C}^4 = L^2(\mathbb{R}_{r^*} \times S^2) \otimes \mathbb{C}^4$ equipped with its natural inner product :

$$(\phi, \psi) = \int \langle \phi, \psi \rangle_{\mathbb{C}^4} \mathrm{d}r^* \mathrm{d}\Omega, \quad \mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi.$$

We refer to [Nic02] for a more detailed discussion on the framework outlined above.

To convert Equation (3.17) this into a system of four scalar equations we will use the local spin-connection forms $\alpha^{\boldsymbol{A}}{}_{\boldsymbol{B}a}$ of a local normalised spin frame $(\varepsilon^A_{\boldsymbol{A}})_{\boldsymbol{A}\in\{0,1\}}$ defined by:

$$\alpha^{\boldsymbol{A}}{}_{\boldsymbol{B}a} = \varepsilon^{\boldsymbol{A}}_B \nabla_a \varepsilon^B_{\boldsymbol{B}}.$$

Given any orthonormal frame $g^a_{\boldsymbol{a}}$ and a normalised spin frame $\varepsilon^A_{\boldsymbol{A}}$ such that the vector fields:

$$l^a = \varepsilon^A_0 \varepsilon^{A'}_{0'}; \quad n^a = \varepsilon^A_1 \varepsilon^{A'}_{1'}; \quad m^a = \varepsilon^A_0 \varepsilon^{A'}_{1'};$$

of the Newman-Penrose tetrad $(l^a, n^a, m^a, \bar{m}^a)$ satisfy:

$$\begin{cases} l^a = \frac{g^a_0 + g^a_1}{\sqrt{2}}, \\ n^a = \frac{g^a_0 - g^a_1}{\sqrt{2}}, \\ m^a = \frac{g^a_2 + i g^a_3}{\sqrt{2}}, \end{cases} \tag{3.20}$$

10. An orientation on Σ can be seen as a bundle morphisme between $\Lambda^n T^*\Sigma$ and the density bundle. See also Definition 1.4.1.

then the spin connection forms are given in terms of the local connection forms $\omega^{\boldsymbol{i}}{}_{\boldsymbol{j}}$ in the basis $g^a_{\boldsymbol{a}}$ by:

$$\alpha^0{}_0 = \frac{\omega^0{}_1 + i\omega^2{}_3}{2}, \ \alpha^1{}_0 = \frac{\omega^2{}_0 + \omega^2{}_1}{2} + i\frac{\omega^3{}_0 + \omega^3{}_1}{2}, \ \alpha^0{}_1 = \frac{\omega^2{}_0 - \omega^2{}_1}{2} - i\frac{\omega^3{}_0 - \omega^3{}_1}{2}. \tag{3.21}$$

A spin connection is a $\mathfrak{sl}(2,\mathbb{C})$-valued one-form, so necessarily:

$$\alpha^1{}_1 = -\alpha^0{}_0.$$

In terms of the covariant derivative, this is equivalent to the requirement that $\nabla_a \varepsilon_{AB} = 0$. The forms $\alpha^{\boldsymbol{A'}}{}_{\boldsymbol{B'}a} = \varepsilon^{\boldsymbol{A'}}_{B'}(\nabla_a \varepsilon^{B'}_{\boldsymbol{B'}})$ satisfy:

$$\alpha^{\boldsymbol{A'}}{}_{\boldsymbol{B'}a} = \overline{\alpha^{\boldsymbol{A}}{}_{\boldsymbol{B}a}} \tag{3.22}$$

Remark 3.3.1. It should be remarked that our conventions differ slightly from those in [PR84], namely, we identify $\mathbb{R}^4$ to $H(2,\mathbb{C})$ via the isomorphism :

$$\begin{array}{cccc} & \mathbb{R}^4 & \longrightarrow & H(2,\mathbb{C}) \\ \varphi : & \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} & \longmapsto & \begin{pmatrix} x_0 + x_1 & x_2 - ix_3 \\ x_2 + ix_3 & x_0 - x_1 \end{pmatrix} \end{array}$$

Remark 3.3.2. Consider the Lie group morphism $\Lambda : SL(2,\mathbb{C}) \to SO_+(1,3)$ defined by associating to any $A \in SL(2,\mathbb{C})$ the matrix $\Lambda(A)$ of the linear map u defined by $u(x) = \varphi^{-1}(A\varphi(x)A^*), x \in \mathbb{R}^4$ expressed in the canonical basis of $\mathbb{R}^4$. Then, viewing $\boldsymbol{\omega} = (\omega^{\boldsymbol{i}}{}_{\boldsymbol{j}})_{\boldsymbol{i},\boldsymbol{j} \in [\![0,3]\!]}$ and $\boldsymbol{\alpha} = (\alpha^{\boldsymbol{A}}{}_{\boldsymbol{B}})_{\boldsymbol{A},\boldsymbol{B} \in \{0,1\}}$ as matrix valued one-forms, it follows that for any $(p,v) \in TM$:

$$\boldsymbol{\alpha}_p(v) = \Lambda_*^{-1}(\boldsymbol{\omega}_p(v)),$$

where Λ_* is the Lie algebra isomorphism induced by Λ.

Once a choice of spin-frame has been made, Equation (3.17) can be written as four scalar equations in terms of the components $\phi^{\boldsymbol{A}}, \chi_{\boldsymbol{A'}}$ of the spinor fields. For instance, the equation:

$$\nabla_{AA'}\phi^A = -\mu\chi_{A'},$$

becomes,

$$\nabla_{\boldsymbol{AA'}}\phi^{\boldsymbol{A}} + \phi^{\boldsymbol{A}}\alpha^{\boldsymbol{B}}_{\ \boldsymbol{A}\,CC'}\varepsilon^{C}_{\boldsymbol{B}}\varepsilon^{C'}_{\boldsymbol{A'}} = -\mu\chi_{\boldsymbol{A'}}.$$

For $\boldsymbol{A} = 0'$, this translates to :

$$l^a\nabla_a\phi^0 + \bar{m}^a\nabla_a\phi^1 + \phi^0\left(\alpha^0_{\ 0a}l^a + \alpha^1_{\ 0a}\bar{m}^a\right) + \phi^1\left(\alpha^0_{\ 1a}l^a + \alpha^1_{\ 1a}\bar{m}^a\right) = -\mu\chi_{0'},$$

or, equivalently:

$$l^a\nabla_a\phi_1 - \bar{m}^a\nabla_a\phi_0 + \phi_1\left(\alpha^0_{\ 0a}l^a + \alpha^1_{\ 0a}\bar{m}^a\right) - \phi_0\left(\alpha^0_{\ 1a}l^a + \alpha^1_{\ 1a}\bar{m}^a\right) = \mu\chi^{1'}.$$

Overall, we obtain the following system of equations for the components:

$$\begin{cases} l^a\nabla_a\chi^{0'} + m^a\nabla_a\chi^{1'} + \chi^{0'}\overline{F} + \chi^{1'}\overline{G} = -\mu\phi_0, \\ \bar{m}^a\nabla_a\chi^{0'} + n^a\nabla_a\chi^{1'} + \chi^{0'}\overline{G_1} + \chi^{1'}\overline{F_1} = -\mu\phi_1, \\ m^a\nabla_a\phi_1 - n^a\nabla_a\phi_0 + \phi_1 G_1 - \phi_0 F_1 = -\mu\chi^{0'}, \\ l^a\nabla_a\phi_1 - \bar{m}^a\nabla_a\phi_0 + \phi_1 F - \phi_0 G = \mu\chi^{1'}, \end{cases} \tag{3.23}$$

where we have defined:

$$\begin{aligned} F &= \alpha^0_{\ 0a}l^a + \alpha^1_{\ 0a}\bar{m}^a, \quad G = \alpha^0_{\ 1a}l^a + \alpha^1_{\ 1a}\bar{m}^a, \\ F_1 &= \alpha^0_{\ 1a}m^a + \alpha^1_{\ 1a}n^a, \quad G_1 = \alpha^0_{\ 0a}m^a + \alpha^1_{\ 0a}n^a, \end{aligned}$$

and used the fact that, by Equation (3.22), for any complex vector fields u^a, v^a:

$$\alpha^{\boldsymbol{A'}}_{\ \boldsymbol{B'}a}\bar{u}^a + \alpha^{\boldsymbol{C'}}_{\ \boldsymbol{D'}a}\bar{v}^a = \overline{\alpha^{\boldsymbol{A}}_{\ \boldsymbol{B}a}u^a + \alpha^{\boldsymbol{C}}_{\ \boldsymbol{D}a}v^a}\,.$$

Dirac equation in the “Boyer-Lindquist" frame

We will first use the results in [Bor18] to write the Dirac equation in the frame:

$$\begin{aligned} g^a_0\frac{\partial}{\partial x^a} &= \frac{\Xi}{\rho\sqrt{\Delta_r}}\left((r^2+a^2)\partial_t + a\partial_\varphi\right), \quad g^a_1\frac{\partial}{\partial x^a} = \frac{\sqrt{\Delta_r}}{\rho}\partial_r, \\ g^a_2\frac{\partial}{\partial x^a} &= \frac{\sqrt{\Delta_\theta}}{\rho}\partial_\theta, \quad g^a_3\frac{\partial}{\partial x^a} = \frac{\Xi}{\sin\,\theta\sqrt{\Delta_\theta}\,\rho}\left(\partial_\varphi + a\sin^2\theta\partial_t\right). \end{aligned} \tag{3.24}$$

The expressions for F, G, F_1, G_1 are given by:

$$F = \frac{1}{2\sqrt{2}\sqrt{\Delta_r}\rho^3}\left(\frac{\Delta_r'}{2}\rho^2 + \Delta_r\tilde{r}\right), \quad F_1 = -F, \quad G_1 = G,$$
$$G = \frac{1}{2\sqrt{2}\sqrt{\Delta_\theta}\sin\theta\rho^3}\left(ia\Delta_\theta\sin^2\theta\tilde{r} + \cos\theta\rho^2(1 + a^2l^2\cos(2\theta))\right),$$

where $\Delta_r' = \frac{\partial\Delta_r}{\partial r}$ and $\tilde{r} = (r + ia\cos\theta)$. In matrix form, with $\boldsymbol{\psi} = {}^t\left(\phi_0, \phi_1, \chi^{0'}, \chi^{1'}\right)$, Equation (3.23) is then:

$$i(\gamma^\mu\partial_\mu + V)\boldsymbol{\psi} = m\boldsymbol{\psi}.$$

In the above:

$$V = \sqrt{2}\begin{pmatrix} 0 & 0 & i\bar{F} & i\bar{G} \\ 0 & 0 & i\bar{G} & -i\bar{F} \\ iF & iG & 0 & 0 \\ iG & -iF & 0 & 0 \end{pmatrix},$$

$$\gamma^t = \frac{\Xi(r^2 + a^2)}{\sqrt{\Delta_r\rho^2}}\begin{pmatrix} 0 & iI_2 \\ -iI_2 & 0 \end{pmatrix} - i\frac{a\sin\theta\Xi}{\sqrt{\Delta_\theta\rho^2}}\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix},$$
$$\gamma^r = i\sqrt{\frac{\Delta_r}{\rho^2}}\begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}, \quad \gamma^\theta = i\sqrt{\frac{\Delta_\theta}{\rho^2}}\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix},$$
$$\gamma^\varphi = \frac{a\Xi}{\sqrt{\Delta_r\rho^2}}\begin{pmatrix} 0 & iI_2 \\ -iI_2 & 0 \end{pmatrix} - i\frac{\Xi}{\sqrt{\Delta_\theta\rho^2}\sin\theta}\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}.$$

The γ^μ are the so-called "gamma matrices" that satisfy the Clifford algebra anti-commutation relations:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathrm{Id}_4.$$

$\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -i\sigma_x\sigma_y.$$

Change of spin-frame

Whilst adapted to the study of the algebraic structure of the curvature of the Kerr-de Sitter metric, the orthonormal frame $g^a_{\boldsymbol{a}}$ and its associated normalised spin-frame $\varepsilon^A_{\boldsymbol{A}}$ [11] are not well aligned with the foliation of B with the space-like level hypersurfaces of t, in the sense that g^a_0 is not parallel to $\nabla^a t$. Following [Dau04; NH04], we switch to a new frame in which the timelike vector is colinear to the future pointing vector field $\nabla^a t$. Since $\nabla^a t^\perp = \mathrm{span}(\partial_r, \partial_\theta, \partial_\varphi)$ we make the simplest choice:

$$g'^a_{\ 0} = \frac{\nabla^a t}{\sqrt{|\nabla^a t \nabla_a t|}}, \ g'^a_{\ 1}\frac{\partial}{\partial x^a} = \frac{1}{\sqrt{-g_{rr}}}\partial_r,$$
$$g'^a_{\ 2}\frac{\partial}{\partial x^a} = \frac{1}{\sqrt{-g_{\theta\theta}}}\partial_\theta, \ g'^a_{\ 3}\frac{\partial}{\partial x^a} = \frac{1}{\sqrt{-g_{\varphi\varphi}}}\partial_\varphi.$$

The matrix P of the Lorentz transformation L^b_a that sends $g^a_{\boldsymbol{a}}$ to $g'^a_{\boldsymbol{a}}$ is given by:

$$P = M_{g'^a_{\boldsymbol{a}}, g^b_{\boldsymbol{b}}}(Id) = \begin{pmatrix} \frac{\sqrt{\Delta_\theta}(r^2+a^2)}{\sigma} & 0 & 0 & -\frac{a\sin\theta\sqrt{\Delta_r}}{\sigma} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{a\sin\theta\sqrt{\Delta_r}}{\sigma} & 0 & 0 & \frac{\sqrt{\Delta_\theta}(r^2+a^2)}{\sigma} \end{pmatrix}, \tag{3.25}$$

where we have defined:

$$\sigma^2 = \Delta_\theta(r^2+a^2)^2 - \Delta_r a^2 \sin^2\theta. \tag{3.26}$$

Up to sign, the spin transformation $A \in SL(2;\mathbb{C})$ that corresponds to P is:

$$A = \begin{pmatrix} \sqrt{\frac{\sigma_+}{2\sigma}} & \frac{ia\sin\theta\sqrt{\Delta_r}}{\sqrt{2\sigma\sigma_+}} \\ -\frac{ia\sin\theta\sqrt{\Delta_r}}{\sqrt{2\sigma\sigma_+}} & \sqrt{\frac{\sigma_+}{2\sigma}} \end{pmatrix}, \tag{3.27}$$

in the above formula $\sigma_+ = \sigma + \sqrt{\Delta_\theta}(r^2+a^2)$. It is useful to note that σ_+ satisfies:

$$\sigma_+^2 - a^2\sin^2\theta\Delta_r = 2\sigma\sigma_+.$$

The appropriate change of basis matrix in $\mathbb{S}_{Ap} \oplus \mathbb{S}^{A'}_p$ at each point p of block II is given

11. determined up to sign

by:

$$\tilde{P} = \begin{pmatrix} {}^tA^{-1} & 0 \\ 0 & \bar{A} \end{pmatrix} = \sqrt{\frac{\sigma_+}{2\sigma}} I_4 + \frac{a\sin\theta\sqrt{\Delta_r}}{\sqrt{2\sigma\sigma_+}} \begin{pmatrix} -\sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}. \tag{3.28}$$

The equation satisfied by $\boldsymbol{\psi'} = \tilde{P}^{-1}\ \boldsymbol{\psi}$ is hence:

$$i\tilde{P}^{-1}(\gamma^\mu\partial_\mu + V)\tilde{P}\boldsymbol{\psi'} = \ m\boldsymbol{\psi'}. \tag{3.29}$$

The left-hand side is:

$$i\tilde{P}^{-1}(\gamma^\mu\partial_\mu + V)\tilde{P} = i\left(\tilde{\gamma}^\mu\partial_\mu + \tilde{V} + \tilde{P}^{-1}\gamma^\mu\frac{\partial\tilde{P}}{\partial x^\mu}\right),$$

where:

$$\gamma^r = \tilde{\gamma}^r, \quad \gamma^\theta = \tilde{\gamma}^\theta, \quad \tilde{V} = V, \quad \tilde{\gamma}^t = \frac{\Xi\sigma}{\sqrt{\Delta_r\Delta_\theta}\rho}\begin{pmatrix} 0 & iI_2 \\ -iI_2 & 0 \end{pmatrix},$$

$$\tilde{\gamma}^\varphi = \frac{a\Xi q^2\rho}{\sigma\sqrt{\Delta_r\Delta_\theta}}\begin{pmatrix} 0 & iI_2 \\ -iI_2 & 0 \end{pmatrix} - i\frac{\Xi\rho}{\sigma\sin\theta}\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}, \tag{3.30}$$

$$\tilde{P}^{-1}\gamma^r\frac{\partial\tilde{P}}{\partial r} = \frac{\sqrt{\Delta_r}}{\rho}f_r\begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix}, \quad \tilde{P}^{-1}\gamma^\theta\frac{\partial\tilde{P}}{\partial\theta} = \frac{\sqrt{\Delta_\theta}}{\rho}f_\theta\begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}.$$

In the above formulae, we have introduced the following notations:

$$q^2 = (\Delta_\theta(r^2+a^2) - \Delta_r)\rho^{-2},$$

$$f_r = \frac{a\sin\theta\sqrt{\Delta_\theta}}{2\sigma^2\sqrt{\Delta_r}}\left(-\frac{\Delta_r'}{2}(r^2+a^2) + 2r\Delta_r\right), f_\theta = -\frac{a\sqrt{\Delta_r}(r^2+a^2)\cos\theta\Xi}{2\sigma^2\sqrt{\Delta_\theta}}.$$

We conclude this section by writing the equation satisfied by the spinor density. In the trivialisation of the density bundle determined by $|\mathrm{d}r^*\wedge\mathrm{d}\Omega|^{\frac{1}{2}}$ the density can be written:

$$\Phi = \underbrace{\left(\frac{\Delta_r\rho^2\sigma^2}{\Delta_\theta(r^2+a^2)^2\Xi^4}\right)^{\frac{1}{4}}}_{\alpha(r,\theta)^{-1}}\boldsymbol{\psi'}. \tag{3.31}$$

Φ satisfies almost the same equation as $\boldsymbol{\psi}'$ except for two additional terms:

$$i\gamma^1\partial_r(\ln\alpha(r,\theta))\Phi + i\gamma^2\partial_\theta(\ln\alpha(r,\theta))\Phi.$$

Overall the equation becomes:

$$i\tilde{\gamma}^0\partial_t\Phi + i\tilde{\gamma}^1\partial_r\Phi + i\tilde{\gamma}^2\partial_\theta\Phi + i\tilde{\gamma}^3\partial_\varphi\Phi + iV_1\Phi = m\Phi, \tag{3.32}$$

with:

$$V_1 = \begin{pmatrix} 0 & 0 & i\bar{\tilde{F}} & i\bar{\tilde{G}} \\ 0 & 0 & i\bar{\tilde{G}} & -i\bar{\tilde{F}} \\ i\tilde{F} & i\tilde{G} & 0 & 0 \\ i\tilde{G} & -i\tilde{F} & 0 & 0 \end{pmatrix}, \tag{3.33}$$

$$\begin{aligned} \tilde{F} &= \sqrt{2}F + i\frac{\sqrt{\Delta_\theta}}{\rho}f_\theta + \frac{\sqrt{\Delta_r}}{\rho}\partial_r\ln\alpha(r,\theta), \\ \tilde{G} &= \sqrt{2}G - i\frac{\sqrt{\Delta_r}}{\rho}f_r + \frac{\sqrt{\Delta_\theta}}{\rho}\partial_\theta\ln\alpha(r,\theta). \end{aligned} \tag{3.34}$$

More explicitly:

$$\tilde{F} = \frac{i\sqrt{\Delta_r}a\cos\theta}{2\rho^3} - \frac{ia\sqrt{\Delta_r}(r^2+a^2)\cos\theta\Xi}{2\sigma^2\rho} + \frac{\sqrt{\Delta_r}a^2\sin^2\theta}{2\rho\sigma^2(r^2+a^2)}\left(\frac{\Delta_r'}{2}(r^2+a^2) - 2r\Delta_r\right),$$

$$\begin{aligned} \tilde{G} = \frac{ia\Delta_\theta\sin^2\theta r + \cos\theta\rho^2\Xi - 3a^2l^2\sin^2\theta\cos\theta\rho^2}{2\sqrt{\Delta_\theta}\sin\theta\rho^3} &+ \frac{\sqrt{\Delta_\theta}a^2\sin\theta\cos\theta}{2\rho\sigma^2}\left((r^2+a^2)\Xi - 2Mr\right) \\ &- \frac{ia\sin\theta\sqrt{\Delta_\theta}}{2\sigma^2\rho}\left(2r\Delta_r - \frac{\Delta_r'}{2}(r^2+a^2)\right), \end{aligned}$$

Rewriting Equation (3.32) as an evolution equation, and introducing $\mathfrak{D}_{S^2}$, the Dirac operator on the 2-sphere, we obtain the following form of the Dirac equation:

$$\begin{aligned} i\partial_t\Phi + i\frac{\Delta_r\sqrt{\Delta_\theta}}{\Xi\sigma}\Gamma^1\partial_r\Phi - \frac{\sqrt{\Delta_r}\Delta_\theta}{\Xi\sigma}\mathfrak{D}_{S^2}\Phi + \frac{iaq^2\rho^2}{\sigma^2}\partial_\varphi\Phi + \frac{i\sqrt{\Delta_r\Delta_\theta}}{\sigma\sin\theta}\left(\frac{\rho^2}{\sigma} - \frac{\sqrt{\Delta_\theta}}{\Xi}\right)\Gamma^3\partial_\varphi\Phi \\ + \frac{i\sqrt{\Delta_r\Delta_\theta}\rho}{\sigma\Xi}\tilde{V}_1\Phi = \frac{\sqrt{\Delta_r\Delta_\theta}}{\Xi\sigma}\rho m\Gamma^0\Phi. \end{aligned} \tag{3.35}$$

This can be written as a Schrödinger equation $i\dfrac{\partial \Phi}{\partial t} = H\Phi$ with H given by:

$$H = \frac{\Delta_r\sqrt{\Delta_\theta}}{\Xi\sigma}\Gamma^1 D_r + \frac{\sqrt{\Delta_r}\Delta_\theta}{\Xi\sigma}\mathfrak{D}_{S^2} + \frac{aq^2\rho^2}{\sigma^2}D_\varphi + \frac{\sqrt{\Delta_r\Delta_\theta}}{\sigma\sin\theta}\left(\frac{\rho^2}{\sigma} - \frac{\sqrt{\Delta_\theta}}{\Xi}\right)\Gamma^3 D_\varphi \\ - \frac{i\sqrt{\Delta_r\Delta_\theta}\rho}{\sigma\Xi}\tilde{V}_1 + \frac{\sqrt{\Delta_r\Delta_\theta}}{\Xi\sigma}\rho m\Gamma^0. \quad (3.36)$$

In the above, we have adopted similar notations to [Dau04]:

$$D_\varphi = -i\partial_\varphi, \quad D_r = -i\partial_r, \quad D_\theta = -i\partial_\theta,$$

$\mathfrak{D}_{S^2}$ is the Dirac operator on the 2-sphere:

$$\mathfrak{D}_{S^2} = \left(D_\theta - i\frac{\text{cotan}\theta}{2}\right)\Gamma^2 + \frac{D_\varphi}{\sin\theta}\Gamma^3,$$

the matrices Γ^i are defined by:

$$\Gamma^0 = i\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \Gamma^1 = \text{diag}(-1,1,1,-1), \Gamma^2 = \begin{pmatrix} -\sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}, \Gamma^3 = \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix}.$$

Defining an operator operation $c \boxtimes M$ with $c \in \mathbb{C}$ and $M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$ a block-diagonal matrix by:

$$c \boxtimes M = \begin{pmatrix} cM_1 & 0 \\ 0 & \bar{c}M_2 \end{pmatrix},$$

the potential $\tilde{V}_1$ can be written:

$$\tilde{V}_1 = \tilde{F} \boxtimes \Gamma^1 + \left(\tilde{G} - \frac{\text{cotan}\theta\sqrt{\Delta_\theta}}{2\rho}\right) \boxtimes \Gamma^2.$$

For computational purposes it is worth noting that the operation $\boxtimes$ enjoys the following properties:

1. $\boxtimes$ is distributive with respect to addition,
2. It is $\mathbb{C}$-homogenous in M and $\mathbb{R}$-homogenous in c,
3. $(c \boxtimes M)^* = \bar{c} \boxtimes M^*$,
4. If $c \in \mathbb{R}$, $c \boxtimes M = cM$,

5. If M is hermitian, $(-i(c \boxtimes M))^* = -i(c \boxtimes A) + 2i\Re(c)M$.

3.4 Analytic framework

3.4.1 Symbol spaces

In what follows we will attempt to treat the operator H defined by Equation (3.36) as a perturbation of another operator. In order to have a succinct language in which to distinguish the asymptotic behaviour of the coefficients of H, we introduce the following symbol spaces:

$$\Pi = \left\{ f \in C^\infty(\Sigma), \partial_r^{\alpha_1}\partial_\theta^{\alpha_2}\partial_\varphi^{\alpha_3} f \circ \psi^{-1} \in L^\infty(]r_e, r_+[\times S^2), \alpha_i \in \mathbb{N} \right\}.$$

For $(m,n) \in \mathbb{N}^2$:

$$\boldsymbol{S}^{m,n} = \left\{ f \in C^\infty(\Sigma), \partial_{r^*}^{\alpha_1}\partial_\theta^{\alpha_2}\partial_\varphi^{\alpha_3} f \circ \psi^{*-1} = \left\{ \begin{array}{l} \underset{r^*\to+\infty}{O}\left(e^{-m\kappa r^*}\right) \\ \underset{r^*\to-\infty}{O}\left(\frac{1}{r^{*n+\alpha_1}}\right) \end{array} \right. \quad \alpha_i \in \mathbb{N} \right\}.$$

ψ and ψ^* denote the coordinate charts (r,θ,φ) and (r^*,θ,φ) respectively and κ is defined by:

$$\kappa = \frac{l}{\Xi\eta_+}\sqrt{\frac{M}{r_e}}. \tag{3.37}$$

By extension, if $M \in C^\infty(\Sigma) \otimes M_4(\mathbb{C})$,we will also write $M \in \boldsymbol{S}^{m,n}$ (resp. $M \in \Pi$) if the operator norm of the matrix M, $||M||$, is an element of $\boldsymbol{S}^{m,n}$ (resp. Π); this is of course equivalent to the requirement that each of its components satisfies the appropriate condition. Finally, we define:

$$\boldsymbol{S}^{\infty,n} = \bigcap_m \boldsymbol{S}^{m,n}, \quad \boldsymbol{S}^{m,\infty} = \bigcap_n \boldsymbol{S}^{m,n}. \tag{3.38}$$

Many of the functions f at hand will be naturally expressed in the coordinate chart ψ, the following results will enable us to infer rapidly the asymptotic behaviour of the function when expressed in the chart ψ^*. The only missing information is the relationship between partial derivatives with respect to r and those with respect to r^*. From (3.12), one has:

$$\partial_{r^*} = \frac{\Delta_r}{\Xi(r^2+a^2)}\partial_r. \tag{3.39}$$

So the question is settled by:

Lemma 3.4.1. *Define the map α on Σ by its coordinate expression: $\alpha \circ \psi^{-1} = \frac{\Delta_r}{\Xi(r^2+a^2)}$, then $\alpha \in \boldsymbol{S}^{2,2}$.*

Proof. Remark first that, from equations (3.15) and (3.16), since r_e is a double root of the polynomial Δ_r, we have:

$$\begin{gathered} \Delta_r = \underset{r^* \to -\infty}{O}\left(\frac{1}{r^{*2}}\right), \quad \Delta_r = \underset{r^* \to +\infty}{O}\left(e^{-2\kappa r^*}\right), \\ \Delta'_r = \underset{r^* \to -\infty}{O}\left(\frac{1}{r^*}\right). \end{gathered} \tag{3.40}$$

Hence:

$$\begin{aligned} \alpha(r^*) &= \underset{r^* \to -\infty}{O}\left(\frac{1}{r^{*2}}\right), \quad \alpha(r^*) = \underset{r^* \to +\infty}{O}\left(e^{-2\kappa r^*}\right), \\ \partial_r\alpha(r^*) &= \underset{r^* \to -\infty}{O}\left(\frac{1}{r^*}\right), \quad \partial_r\alpha(r^*) = \underset{r^* \to +\infty}{O}(1). \end{aligned} \tag{3.41}$$

For any $n \geq 2$, it is easy to see that $\partial_r^n \alpha(r^*) = O(1)$. Now, $\partial_{r^*}\alpha(r^*) = \alpha(r^*)\partial_r\, \alpha(r^*)$, so we have the correct behaviour at infinity after the first derivative. We claim that for $n \geq 1$:

$$\partial_{r^*}^n \alpha(r^*) = \sum_{k=1}^{n} f_k(r^*)(\partial_r\alpha(r^*))^{\beta_k}(\alpha(r^*))^k, \tag{3.42}$$

where $\alpha_k \in \mathbb{N}$, $f_k \in \Pi$ and $\beta_k + 2k \geq n+2$ for each $k \in [\![1,n]\!]$.

This is obvious for $n = 1$ and if such a relationship is true for some $n \geq 1$, after differentiation one has:

$$\begin{aligned} \partial_{r^*}^{n+1}\alpha(r^*) = \sum_{k=1}^{n} \partial_r f_k(r^*)(\partial_r\alpha(r^*))^{\beta_k}(\alpha(r^*))^{k+1} + \beta_k f_k(r^*)\partial_r^2\alpha(r^*)(\partial_r\alpha(r^*))^{\alpha_k-1}(\alpha(r^*))^{k+1} \\ + \sum_{k=1}^{n} f_k(r^*)(\partial_r\alpha(r^*))^{\beta_k+1}(\alpha(r^*))^k. \end{aligned}$$

Therefore, $\partial_{r^*}^{n+1}\alpha(r^*)$ satisfies (3.42), with:

$$\begin{gathered} \tilde{\beta}_{n+1} = \max(0, \beta_n - 1), \\ \tilde{f}_{n+1} = \partial_r f_n(\partial_r\alpha)^{\beta_n - \tilde{\beta}_{n+1}} + \beta_n f_n \partial_r^2\alpha, \\ \tilde{f}_1 = f_1 = 1, \; \tilde{\beta}_1 = \beta_1 + 1 = n+1. \end{gathered}$$

and for $k \in [\![2, n]\!]$:

$$\tilde{\beta}_k = \min(\beta_k + 1, \max(0, \beta_{k-1} - 1)),$$
$$\tilde{f}_k = \partial_r f_{k-1}(\partial_r \alpha)^{\beta_{k-1}-\tilde{\beta}_k} + \beta_{k-1} f_{k-1} \partial_r^2 \alpha (\partial_r \alpha(r^*))^{\beta_{k-1}-1-\tilde{\beta}_k} + f_k(\partial_r \alpha)^{\beta_k+1-\tilde{\beta}_k}.$$

The $\tilde{f}_k$ clearly satisfy the required hypothesis; if $\tilde{\beta}_k \neq 0$, then, either $\tilde{\beta}_k = \beta_k + 1$ or $\tilde{\beta}_k = \beta_{k-1} - 1$. In the first case, then:

$$\tilde{\beta}_k + 2k \geq n + 4,$$

in the second case:

$$\tilde{\beta}_k + 2k \geq n + 2 + 2 - 1 = n + 3.$$

If $\tilde{\beta}_k = 0$, then necessarily this implies $\beta_{k-1} \leq 1$. By hypothesis, β_{k-1} satisfies: $\beta_{k-1} + 2k \geq n + 4$, so, $2k \geq n + 3$, and the hypothesis is equally satisfied. Hence, the result follows by induction. The asymptotics can now be read from (3.42), each term in the sum is $O(\alpha) = O(e^{-2\kappa r^*})$ at $r^* \to +\infty$ and every term in the sum is $O(r^{*-(n+2)})$ at $r^* \to -\infty$.

One can now use the Faà di Bruno formula [12] to show that:

$$f \in \Pi \Rightarrow f \in \boldsymbol{S}^{0,0},\ \partial_{r^*} f \in \boldsymbol{S}^{2,2}. \tag{3.43}$$

In particular, if $f \in \Pi$ and $f(r^*) = \underset{r^* \to -\infty}{O}(\frac{1}{r^*})$ then $f \in \boldsymbol{S}^{0,1}$.

3.4.2 φ-invariance

The metric on B does not depend on the coordinate φ; this invariance will be exploited in two ways in this paper. Firstly, diagonalising D_φ with anti-periodic boundary conditions, any $\phi \in \mathscr{H}$ can be represented as:

$$\phi(r, \theta, \varphi) = \sum_{p \in \mathbb{Z} + \frac{1}{2}} \phi_p(r, \theta) e^{ip\varphi}.$$

The subspaces of this Hilbert sum are stable under the action of H, and we could just consider the restriction of H to any such subspace; this would enable us to treat the terms with factor D_φ as potentials. However, some terms contain explicit coordinate

12. See, appendix B.2

singularities. To avoid technical difficulties due to this, it is more convenient to work with the operator H^p formally defined on $\mathscr{H}$ by:

$$H^p = \frac{\Delta_r\sqrt{\Delta_\theta}}{\Xi\sigma}\Gamma^1 D_r + \frac{\sqrt{\Delta_r}\Delta_\theta}{\Xi\sigma}\mathfrak{D}_{S^2} - \frac{i\sqrt{\Delta_r\Delta_\theta}\rho}{\sigma\Xi}\tilde{V}_1 + \frac{\sqrt{\Delta_r\Delta_\theta}}{\Xi\sigma}\rho m\Gamma^0 + \frac{aq^2\rho^2}{\sigma^2}p + \frac{\sqrt{\Delta_r\Delta_\theta}}{\sigma\sin\theta}\left(\frac{\rho^2}{\sigma} - \frac{\sqrt{\Delta_\theta}}{\Xi}\right)\Gamma^3 p. \quad (3.44)$$

The function $\frac{\sqrt{\Delta_r\Delta_\theta}}{\sigma\sin\theta}\left(\frac{\rho^2}{\sigma} - \frac{\sqrt{\Delta_\theta}}{\Xi}\right)$ is well-defined and bounded, because[13]:

$$\frac{\rho^2}{\sigma} - \frac{\sqrt{\Delta_\theta}}{\Xi} = \frac{1}{\sigma\Xi}\left(\frac{\Xi^2\rho^4 - \Delta_\theta\sigma^2}{\Xi\rho^2 + \sqrt{\Delta_\theta}\sigma}\right),$$

and:

$$\Xi^2\rho^4 - \Delta_\theta\sigma^2 = a^2\sin^2\theta\left(\Delta_\theta\Delta_r + 2\Xi(r^2+a^2)(l^2r^2-1) + a^2\sin^2\theta(\Xi^2 - l^4(r^2+a^2)^2)\right).$$

H^p coincides with H on the subspace corresponding to the eigenvalue $p \in \mathbb{Z} + \frac{1}{2}$ of D_φ and the coordinate singularity is absorbed into $\mathfrak{D}_{S^2}$ which is well-defined as an operator on the sphere.

In later analysis, it will also prove convenient to rotate the coordinate system so as to cancel some of the effects of rotation at the double horizon. Setting $c_0 = \frac{a}{r_e^2+a^2}$, the coordinate transformation is:

$$t' = t, \quad r^{*'} = r^*, \quad \theta' = \theta, \quad \varphi' = \varphi - c_0 t.$$

Naturally, φ and φ' are circular coordinates. Due to the φ-invariance of the metric, H^p transforms very little under this change of coordinates, in fact, we just have to perform the substitution:

$$H^p \to H^p - c_0 p.$$

From now on, unless otherwise stated, we will work in the *rotated coordinates*. For convenience however, we will continue to call φ the new circular coordinate $\varphi - c_0 t$. Thanks to the φ-invariance of our problem this should not cause any confusion.

13. σ is defined by Equation (3.26). One has: $\sigma^2 = \Xi(r^2+a^2)\rho + 2Mra^2\sin^2\theta \geq \Xi(r_e^2+a^2)r_e^2$

3.4.3 A comparison operator

Almost all the operators we will study in this paper are perturbations of a single operator H_0 given by:

$$H_0 = \Gamma_1 D_{r^*} + g(r^*)\mathfrak{D} + f(r^*). \tag{3.45}$$

The functions g and f satisfy:

$$g(r^*) = \frac{\sqrt{\Delta_r}}{\Xi(r^2+a^2)} \in \boldsymbol{S}^{1,1}, \quad f(r^*) = \frac{ap}{r^2+a^2} - \frac{ap}{r_e^2+a^2} \in \boldsymbol{S}^{0,1}, \tag{3.46}$$

whilst, the operator $\mathfrak{D}$ is defined by:

$$\mathfrak{D} = \Delta_\theta^{\frac{1}{4}} \mathfrak{D}_{S^2} \Delta_\theta^{\frac{1}{4}}. \tag{3.47}$$

The structure of this comparison operator is very similar to that of those used in [NH04; Dau04], except that, here, the angular part $\mathfrak{D}$ is a perturbation of the Dirac operator on the sphere $\mathfrak{D}_{S^2}$, rather than $\mathfrak{D}_{S^2}$ itself. The spectral properties of the latter, which are well-documented [14],were quite essential to the analysis in [NH04; Dau04], luckily, $\mathfrak{D}$ shares many of them.

Lemma 3.4.2. *Let S be the self-adjoint extension in $L^2(S^2) \otimes \mathbb{C}^2$ of the operator:*

$$(D_\theta - i\frac{\cot\theta}{2})\sigma_x - \frac{D_\varphi}{\sin\ \theta}\sigma_y,$$

defined on the subset of $[C^\infty(S^2)]^2$ with anti-periodic boundary conditions in φ. Denoting its domain $D(S)$, $\tilde{S} = \Delta_\theta^{\frac{1}{4}} S \Delta_\theta^{\frac{1}{4}}$ is self-adjoint on $D(S)$ and has compact resolvent.

Proof. S has a core consisting of smooth functions on which a simple calculation shows that:

$$\tilde{S} = \sqrt{\Delta_\theta} S - \frac{i}{2}\frac{a^2 l^2 \cos\theta \sin\theta}{\sqrt{\Delta_\theta}}\sigma_x.$$

14. see, for example [Abr02; CH96; Tra93]

The expression extends to all of $D(S)$ by continuity in the graph topology. The estimates:

$$0 \leq \sqrt{\Delta_\theta} - 1 \leq \frac{\Delta_\theta - 1}{\sqrt{\Delta_\theta} + 1} \leq \frac{a^2 l^2}{2},$$
$$\left\| i\sigma_x \frac{a^2 l^2 \cos\theta \sin\theta}{2\sqrt{\Delta_\theta}} u \right\|^2 \leq \frac{a^4 l^4}{4} ||u||^2, \quad u \in L^2(S^2, \mathbb{C}^2), \tag{3.48}$$

together imply for $u \in D(S)$:

$$\left\| (\sqrt{\Delta_\theta} - 1) Su - i\sigma_x \frac{a^2 l^2 \cos\theta \sin\theta}{2\sqrt{\Delta_\theta}} u \right\| \leq \frac{a^2 l^2}{2} \left(||Su|| + ||u|| \right). \tag{3.49}$$

It is easy to see from (3.7) that $\frac{a^2 l^2}{2} < 1$. Thus, by the Kato-Rellich Perturbation Theorem [Lax02; Kat80], $\tilde{S}$ is self-adjoint on $D(S)$. In order to show that $\tilde{S}$ has compact resolvent, it suffices to show that there is a $z \in \rho(\tilde{S})$ such that $R(\tilde{S}, z)$ is compact, for, by the resolvent identity, the property will follow for all $z \in \rho(\tilde{S})$. In fact, in this perturbation theory setup, it is sufficient to show that there is some $z \in \rho(S)$ such that the following inequality holds:

$$\frac{a^2 l^2}{2} ||R(z, S)|| \ + \frac{a^2 l^2}{2} ||SR(z, S)|| < 1, \tag{3.50}$$

where $R(z, S)$ denotes the resolvent of the operator S at z. Indeed, assuming (3.50), it follows from (3.49) that for any $u \in L^2(S^2, \mathbb{C}^2)$:

$$||(\tilde{S} - S)R(z, S)u|| \leq \frac{a^2 l^2}{2} ||SR(z, S)u|| + \frac{a^2 l^2}{2} ||R(z, S)u|| < ||u||.$$

$(\tilde{S} - S)R(z, S)$ is therefore a bounded linear operator and $I + (\tilde{S} - S)R(z, S)$ is invertible with bounded inverse. Moreover:

$$\tilde{S} - zI = S + \tilde{S} - S - zI = (I + (\tilde{S} - S)R(z, S))(S - zI).$$

Consequently, $\tilde{S} - zI$ has bounded inverse given by:

$$R(z, S)(I + (\tilde{S} - S)R(z, S))^{-1}.$$

$R(z, S)$ is compact because S has compact resolvent, so $(\tilde{S} - zI)^{-1} = R(\tilde{S}, z)$ is compact.

We now show there is $z \in \rho(S)$ such that (3.50) is satisfied. By self-adjointness, it suffices to seek z of the form $z = ic$. A classical resolvent estimate shows then that:

$||R(z, S)|| \leq \frac{1}{|c|}$ so that $||R(z, S)||$ is arbitrarily small for $|c|$ large enough. Furthermore, for any $z \in \rho(S)$ we have $||SR(z, S)|| \leq 1$, since $\frac{a^2 l^2}{2} < \frac{1}{2}$, (3.50) holds for any $|c| > 2$.

Lemma 3.4.3. *Let $\tilde{S}$ be as in Lemma 3.4.2, the following properties hold:*

— $-\sigma(\tilde{S}) = \sigma(\tilde{S})$,

— $\sigma(\tilde{S}) \cap]-1, 1[= \emptyset$.

In particular, the eigenvalues $(\lambda_k)_{k \in \mathbb{Z}^*}$ can be indexed by $\mathbb{Z}^*$, in such a way that $\lambda_{-k} = -\lambda_k$ for each $k \in \mathbb{Z}^*$. Furthermore, for each $k \in \mathbb{Z}^*$, there is a subset $J_k \subset \mathbb{Z} + \frac{1}{2}$, such that for each $n \in J_k$ one can find $\psi_{k,n}(\theta, \varphi) = \begin{pmatrix} \alpha_{k,n}(\theta) \\ \beta_{k,n}(\theta) \end{pmatrix} e^{in\varphi} \in L^2(S^2, \mathbb{C}^2), ||\psi_{k,n}|| = 1$, unique up to a complex phase, satisfying $\tilde{S}\psi_{k,n} = \lambda_k \psi_{k,n}$. Necessarily, these form a total orthonormal family of eigenvectors for $\tilde{S}$.

Proof. To prove that the spectrum of $\tilde{S}$ is disjoint from the open unit interval, it is sufficient to notice that, as a quadratic form, $\tilde{S}^2 \geq 1$. Indeed, for any $u \in D(S)$:

$$(\tilde{S}u, \tilde{S}u) = (\sqrt{\Delta_\theta} S \Delta_\theta^{\frac{1}{4}} u, S \Delta_\theta^{\frac{1}{4}} u)) \geq ||u||^2, \tag{3.51}$$

because $\Delta_\theta \geq 1$. The other points will be proved in a slightly more involved case in Section 3.6.4.

Due to the block diagonal form of $\mathfrak{D}$, the following is an immediate consequence of the above:

Corollary 3.4.1. *The family:*

$$\left\{ \psi_{k,n}^+ = \begin{pmatrix} \psi_{k,n} \\ 0 \end{pmatrix}, \psi_{k,n}^- = \begin{pmatrix} 0 \\ \psi_{k,n} \end{pmatrix}, k \in \mathbb{Z}^*, n \in J_k \right\},$$

is a total orthonormal family of eigenvectors of $\mathfrak{D}$.

These results are sufficient to construct a natural decomposition of $\mathscr{H}$ that can be used to obtain a convenient representation of the operator H_0. However, we begin by noting that the subspaces $L^2(\mathbb{R}) \otimes \operatorname{span}\{\psi_{k,n}^+, \psi_{k,n}^-\}, k \in \mathbb{Z}^*, n \in J_k$, are not stable under the action of Γ^1. Indeed, if ψ is an eigenvector with eigenvalue λ of $\mathfrak{D}$, then, since Γ^1 anti-commutes with Γ^2 and Γ^3, $\Gamma^1\psi$ is an eigenvector with eigenvalue $-\lambda$. In particular, the block diagonal form of Γ^1 implies that $\Gamma^1\psi_{k,n}^\pm$ and $\psi_{-k,n}^\pm$ must be colinear (because $\psi_{k,n}$ is unique up to scaling). In fact, Γ^1 being unitary and symmetric, one has $\Gamma^1\psi^{\pm}_{k,n} = \pm\psi^{\pm}_{-k,n}$. The family $\psi_{k,n}$ remains total and orthonormal if $\psi_{-k,n}$ is rescaled to absorb the sign, so one can assume that: $\Gamma^1\psi^{\pm}_{k,n} = \psi^{\pm}_{-k,n}$. The subspaces:

$$\mathscr{H}_{k,n} = L^2(\mathbb{R}) \otimes \operatorname{span}\left\{\psi^+_{k,n}, \psi^+_{-k,n}, \psi^-_{k,n}, \psi^-_{-k,n}\right\}, k \in \mathbb{N}^*, n \in J_k,$$

are then naturally stable under Γ^1 and therefore, under H_0, and $\mathscr{H} = \overset{\perp}{\bigoplus\limits_{k,n}}\mathscr{H}_{k,n}$. For each (k,n), $\mathscr{H}_{k,n}$ can be isometrically identified to $[L^2(\mathbb{R})]^4$ by the map:

$$\begin{array}{rccc} b_{k,n}: & \mathscr{H}_{k,n} & \longrightarrow & [L^2(\mathbb{R})]^4 \\ & \begin{array}{r} u_1\psi^+_{k,n} + u_2\psi^+_{-k,n} \\ +u_3\ \psi^-_{k,n} + u_4\psi^-_{-k,n} \end{array} & \longmapsto & \frac{1}{\sqrt{2}}\begin{pmatrix} u_1 - u_2 \\ u_1 + u_2 \\ u_3 + u_4 \\ u_3 - u_4 \end{pmatrix}. \end{array} \tag{3.52}$$

Through this identification the restriction, $H_0^{k,n}$, of H_0 to $\mathscr{H}_{k,n}$ can be written:

$$H_0^{k,n} = \Gamma^1 D_{r^*} - \lambda_{k,n} g(r^*)\Gamma^2 + f(r^*). \tag{3.53}$$

This is clearly a bounded perturbation of the self-adjoint operator $\Gamma^1 D_{r^*}$ with domain $[H^1(\mathbb{R})]^4$, hence it is self-adjoint on the same domain.

We are now ready to use the lemma below [15] to obtain a description of a domain where the formal expression for H_0 is self-adjoint.

Lemma 3.4.4. *Let X be a Hilbert space and $(X_n)_{n\in\mathbb{N}}$ a family of subspaces of X such that:*

$$X = \overset{\perp}{\bigoplus_{n\in\mathbb{N}}} X_n,$$

where the sum is topological. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of operators A_n on X_n, such that for each n, A_n is self-adjoint on its domain $D(A_n)$. Then the operator A defined by:

$$Ax = \sum_n A_n x_n,$$

15. see [NH04, Lemma 3.5]

if $x = \sum x_n, x_n \in X_n$ *for any* $n \in \mathbb{N}$ *is self-adjoint on:*

$$D(A) = \left\{ x = \sum_n x_n \in X, \sum_{n \in \mathbb{N}} ||A_n x_n||^2 < \infty \right\}.$$

Proof. It is clear that A is densely defined. In order to show that A is closed, denote by P_k the orthogonal projection onto X_k for each $k \in \mathbb{N}$ and suppose that $(x^m)_{m\in\mathbb{N}}$ is a sequence of points of $D(A)$ such that $x^m \to x$ and $Ax^m \to y$ in X. Then for any $k \in \mathbb{N}$, $P_k x^m \to P_k x$ and $P_k A x^m = A_k P_k x^m \to P_k y$ by definition, but since A_k is closed, it follows that $P_k x \in D(A_k)$ and $P_k y = A_k P_k x$. Thus, $\sum_k ||A_k P_k x||^2 = \sum_k ||P_k y||^2 < +\infty$ so $x \in D(A)$ and $Ax = y$.

To prove that A is self-adjoint we show that $A + z$ has dense range for any $z \in \mathbb{C} \setminus \mathbb{R}$. Let $y \in X$ be such that $(Ax + zx, y) = 0$ for any $x \in D(A)$. In particular, for each $k \in \mathbb{N}$, and every $x \in D(A_k)$, $(A_k x + zx, P_k y) = 0$, but then, since A_k is self-adjoint, $P_k y = 0$ for any $k \in \mathbb{N}$, i.e. $y = 0$.

The natural domain for H_0, which is always meaningful in the distributional sense, is certainly $\{u \in \mathscr{H}, H_0 u \in \mathscr{H}\}$, this, in fact, coincides with the domain of the operator given by the previous lemma:

$$D(H_0) = \{u = \sum_{k,n} u_{k,n} \in \mathscr{H}, \sum_{k,n} ||H_0^{k,n} u_{k,n}||^2 < \infty\}.$$

The proof is analogous to that of [NH04, Lemma 3.5].

Since for each $k \in \mathbb{N}^*, n \in J_k$, $D\left(H_0^{k,n}\right)$ is isometric to $[H^1(\mathbb{R})^4]$, and $\mathscr{S}(\mathbb{R})$ [16] is dense in $H^1(\mathbb{R})$, we deduce immediately a core for H_0, that we will simply denote by $\mathscr{S}$. This core will be convenient for many computations, in particular, it will justify the use of the Leibniz rule when computing commutators. More precisely:

Lemma 3.4.5. $\mathscr{S} = \overset{\perp}{\bigoplus_{k,n}} \mathscr{S}(\mathbb{R}) \otimes span\left\{\psi^+_{k,n}, \psi^+_{-k,n}, \psi^-_{k,n}, \psi^-_{-k,n}\right\}$ *is a core for* H_0.

Proof. For any k, n, $H_0^{k,n}$ is self-adjoint on $D(H_0^{k,n}) = b_{k,n}^{-1}([H^1(\mathbb{R})]^4)$ ($b_{k,n}$ is defined by Equation (3.52)) and $[\mathscr{S}(\mathbb{R})]^4$ is dense in $[H^1(\mathbb{R})]^4$. Denote by $P_{k,n}$ the orthogonal projection onto $\mathscr{H}_{k,n}$. Let $u \in D(H_0)$ and $\varepsilon \in \mathbb{R}^*_+$. For each k, n, one can find $\phi_{k,n} \in$

16. $\mathscr{S}(\mathbb{R})$ denotes the Schwartz space of rapidly decaying functions.

$[\mathscr{S}(\mathbb{R})]^4$ such that:

$$||b_{k,n}P_{k,n}\psi - \phi_{k,n}||_{[H^1(\mathbb{R})]^4} \leq \varepsilon \frac{2^{-\frac{k+n+2}{2}}}{C_k},$$

where $C_k = \lambda_k ||g||_\infty + ||f||_\infty + 1$, it follows that:

$$\sum_{k,n} ||P_{k,n}\psi - b_{k,n}^{-1}(\phi_{k,n})||^2 \leq \varepsilon^2, \sum_{k,n} ||H_0(P_{k,n}\psi - b_{k,n}^{-1}(\phi_{k,n}))||^2 \leq \varepsilon^2. \tag{3.54}$$

Therefore, $\sum_{k,n} P_{k,n}\psi - b_{k,n}^{-1}(\phi_{k,n})$ converges to some $y \in D(H_0)$. Set $\phi = \psi - y$, then $||\phi - \psi|| + ||H_0(\phi - \psi)|| \leq 2\varepsilon$, and for every k, n:

$$P_{k,n}\phi = P_{k,n}\psi - P_{k,n}y = b_{k,n}^{-1}(\phi_{k,n}),$$

i.e. $\phi \in \mathscr{S}$. ε being arbitrary this concludes the proof.

3.4.4 Short and long-range potentials

The construction of the wave operators, modified or not, will mainly be based on Cook's method [17] or minor variations thereof. Because of this, it will be interesting to investigate the integrability of the matrix-valued coefficients appearing in our differential operators. Amongst those, we will call "potentials", the parts of the order 0 component of its symbol that vanish on the horizons. For our purposes, they will be split into merely three groups. Namely a potential V is:

— short-range at $+\infty$ (resp. $-\infty$) if:

$$\sup_{r^* \geq 0, \vartheta \in S^2} ||\langle r^* \rangle^\alpha V|| < +\infty \quad (\text{resp.} \sup_{r^* \leq 0, \vartheta \in S^2} ||\langle r^* \rangle^\alpha V|| < +\infty) \tag{3.55}$$

for some $\alpha > 1$,

— long-range otherwise,

— of Coulomb-type at $+\infty$ (resp. $-\infty$) if V is long-range there and (3.55) holds with $\alpha = 1$.

The norm here is the operator norm on $M_4(\mathbb{C})$ and $\langle . \rangle$ denotes the Japanese bracket $\langle r \rangle = \sqrt{r^2 + 1}$. In relation with the symbol spaces we introduced previously, let $m, n \in \mathbb{Z}$ and suppose $V \in \boldsymbol{S}^{m,n}$, then:

17. See for example [Lax02, Chapter 37]

— $m \geq 1 \Rightarrow V$ short-range at $+\infty$,

— $n \geq 2 \Rightarrow V$ short-range at $-\infty$,

— $n = 1 \Rightarrow V$ of Coulomb type at $-\infty$.

3.4.5 Self-adjointness of H^p

It is now relatively easy to prove the self-adjointness of H^p, we first introduce the function:

$$h(r,\theta) = \Delta_\theta^{\frac{1}{4}} \sqrt{\frac{r^2+a^2}{\sigma}}, \tag{3.56}$$

it satisfies the following properties:

$$|h^2 - 1| \leq 1 - a^2 l^2 < 1, \tag{3.57}$$

$$\partial_\theta h = \Delta_r \frac{(r^2+a^2)a^2 \sin\theta\cos\theta\Xi}{2h\sqrt{\Delta_\theta}\sigma^3} \in \mathbf{S}^{2,2}. \tag{3.58}$$

Proof. The first property follows from the following chain of inequalities:

$$\begin{aligned} 0 \leq h^2 - 1 &= \frac{\Delta_r a^2 \sin^2\theta}{\sigma\left(\sigma + \sqrt{\Delta_\theta}(r^2+a^2)\right)} \\ &\leq \frac{\Delta_r a^2 \sin^2\theta}{\sigma^2} \leq \frac{a^2}{r^2} \leq \frac{a^2}{r_e^2} = \frac{6a^2l^2}{1 - a^2l^2 - \sqrt{(1-a^2l^2)^2 - 12a^2l^2}} \leq 1 - a^2l^2. \end{aligned}$$

By Equation (3.7), $1 - a^2l^2 < 1$, the conclusion follows.

The boundedness of $\partial_{r^*} h = \frac{\Xi\Delta_r}{r^2+a^2}\partial_r h$ and $\partial_\theta h$ shows that $h \in B(D(H_0))$. Indeed, $[H_0, h]$ is defined on $D(H_0)$ and:

$$[H_0, h]u = -i\Gamma^1 \partial_{r^*} h u - i\frac{\sqrt{\Delta_\theta \Delta_r}}{\Xi(r^2+a^2)}\Gamma^2 \partial_\theta h u, \quad u \in D(H_0).$$

Consequently, for any $u \in D(H_0)$:

$$||H_0 h u|| \leq ||h H_0 u|| + ||[H_0, h]u|| \leq C(||H_0 u|| + ||u||), \tag{3.59}$$

for some constant $C \in \mathbb{R}_+^*$. The following relationship between H_0 and H^p is therefore meaningful:

$$H^p = hH_0h + V_C + V_S, \tag{3.60}$$

with:

$$\begin{aligned} V_S = &-\frac{ap\sqrt{\Delta_\theta}}{\sigma} + \frac{a\Delta_\theta(r^2+a^2)p}{\sigma^2} - a\frac{\Delta_r p}{\sigma^2} + \frac{ap(h^2-1)}{r_e^2+a^2} \\ &+ i\left[\left(\frac{ia\Delta_\theta\sqrt{\Delta_r}}{2\sigma^3\Xi}\left(2r\Delta_r - \frac{\Delta_r'}{2}(r^2+a^2)\right)\right)\boxtimes\Gamma^2\right] \\ &- i\left[\left(\frac{i\Delta_r\sqrt{\Delta_\theta}a\cos\theta}{2\rho^2\sigma\Xi} - \frac{ia\Delta_r\sqrt{\Delta_\theta}(r^2+a^2)\cos\theta}{2\sigma^3}\right)\boxtimes\Gamma^1\right], \end{aligned} \tag{3.61}$$

$$V_C = \frac{\sqrt{\Delta_r\Delta_\theta}}{\sigma\sin\theta}\left(\frac{\rho^2}{\sigma} - \frac{\sqrt{\Delta_\theta}}{\Xi}\right)\Gamma^3 p + \frac{\sqrt{\Delta_r\Delta_\theta}}{\Xi\sigma}\rho m\Gamma^0 - i\left[\left(\frac{ia\sqrt{\Delta_r}\sin\theta r\Delta_\theta}{2\rho^2\sigma\Xi}\right)\boxtimes\Gamma^2\right]. \tag{3.62}$$

In the above, we have sorted the terms according to their asymptotic behaviour at $-\infty$, since at $+\infty$ all the potentials are short-range. More precisely, the terms in V_S are short-range at $-\infty$ and those of V_C are of Coulomb-type there. Equation (3.58) means that $h[H_0, h]$ is short-range at both infinities.

Using Equation (3.60), one shows that:

Lemma 3.4.6. *H^p is self-adjoint on $D(H_0)$, for any $p \in \mathbb{Z} + \frac{1}{2}$.*

Proof. It follows from Equation (3.60) that:

$$H^p = H_0 + (h^2-1)H_0 + h[H_0, h] + V_C + V_S,$$

since $[H_0, h]$ is bounded, H^p is H_0-bounded and, using the fact that :

$$|h^2 - 1| \leq 1 - a^2l^2 < 1,$$

the result follows from the Kato-Rellich Perturbation Theorem.

3.4.6 Further properties of H_0

Let us pursue the study of the simplified operator H_0; we aim to describe its domain as well as to generalise a useful criterion for proving compactness of functions of H_0.

Lemma 3.4.7. *As quadratic forms on $\mathscr{S}$, H_0^2 and $Q = D_{r^*}^2 + g^2(r^*)\mathfrak{D}^2$ are equivalent.*

Proof. On $\mathscr{S}$, the following equation makes sense:

$$H_0^2 = D_{r^*}^2 + g(r^*)^2\mathfrak{D}^2 + f(r^*)^2 + \frac{\Gamma^1}{i}g'(r^*)\mathfrak{D} + 2g(r^*)\mathfrak{D}f(r^*) + \{f(r^*), \Gamma^1 D_{r^*}\}\text{ [18]}.$$

Furthermore, for any $u \in \mathscr{S}$:

$$\begin{aligned} |(\{f(r^*), \Gamma^1 D_{r^*}\}u, u)| &\leq |(\Gamma^1 D_{r^*}u, f(r^*)u)| + |(f(r^*)u, \Gamma^1 D_{r^*}u)|, \\ &\leq 2||\Gamma^1 D_{r^*}u||||f(r^*)u||, \\ &\leq 2||f||_\infty||\Gamma^1 D_{r^*}u||||u||, \\ &\leq \frac{1}{2}||\Gamma^1 D_{r^*}u||^2 + 2||f||_\infty^2||u||^2. \end{aligned} \tag{3.63}$$

It follows that:

$$\frac{1}{2}D_{r^*}^2 + 2||f||_\infty^2 \geq \{f(r^*), \Gamma^1 D_{r^*}\} \geq -\frac{1}{2}D_{r^*}^2 - 2||f||_\infty^2.$$

Exploiting the fact that $|g'(r^*)| \leq C|g(r^*)|$ for some $C > 0$, one has:

$$\begin{aligned} |(\frac{\Gamma^1}{i}g'(r^*)\mathfrak{D}u, u)| &\leq ||g'(r^*)\mathfrak{D}u||||u||, \\ &\leq \frac{1}{4C^2}||g'(r^*)\mathfrak{D}u||^2 + C^2||u||^2, \\ &\leq \frac{1}{4}||g(r^*)\mathfrak{D}u||^2 + C^2||u||^2. \end{aligned} \tag{3.64}$$

We thus conclude that:

$$\frac{1}{4}g^2(r^*)\mathfrak{D}^2 + C^2 \geq \frac{\Gamma^1}{i}g'(r^*)\mathfrak{D} \geq -\frac{1}{4}g(r^*)^2\mathfrak{D}^2 - C^2.$$

In Equation (3.64), we have used the fact that:

$$g'^2(r^*)\mathfrak{D}^2 \leq C^2 g^2(r^*)\mathfrak{D}^2.$$

This follows from the functional calculus, since, if Z is an even function in the second

18. $\{A, B\}$ denotes the anti-commutator $AB + BA$ of two operators A and B, defined, if necessary, as a quadratic form.

variable:

$$(Z(r^*, \mathfrak{D})u, u) = \sum_{k,n} \int Z(r^*, \lambda_{k,n}) ||u_{k,n}||^2_{\mathbb{C}^4} \mathrm{d}r^*,$$
$$u = \sum_{k,n} b_{k,n}^{-1} u_{k,n}, u_{k,n} \in [L^2(\mathbb{R})]^4,$$

and so inequalities valid for Z pass to the operators, here:

$$Z(x, y) = g'(x)^2 y^2,$$

which clearly satisfies: $Z(x, y) \leq C^2 g(x)^2 y^2$. Finally:

$$\begin{aligned}|(2g(r^*)\mathfrak{D} f(r^*)u, u)| =& 2|(g(r^*)\mathfrak{D}u, f(r^*)u)|, \\ \leq& 2||f||_\infty ||g(r^*)\mathfrak{D}u|| ||u||, \\ \leq& \frac{1}{4}||g(r^*)\mathfrak{D}u||^2 + 4||f||^2_\infty ||u||^2.\end{aligned}$$

Thus:

$$\frac{1}{4} g(r^*)^2 \mathfrak{D}^2 + 4||f||^2_\infty \geq 2g(r^*)f(r^*)\mathfrak{D} \geq -\frac{1}{4} g(r^*)^2 \mathfrak{D}^2 - 4||f||^2_\infty,$$

and therefore:

$$H_0^2 \geq \frac{1}{2}(D_{r^*}^2 + g(r^*)^2 \mathfrak{D}^2) - C',$$

where $C' = 7||f||^2_\infty + C^2 > 0$. Overall :

$$\frac{1}{2} Q - C' \leq H_0^2 \leq 2Q + C',$$

which concludes the proof.

Lemma 3.4.7 has the following important consequences:

Corollary 3.4.2. *$D(H_0) \subset H^1_{loc}$ continuously and we have the following criterion for compactness*[19]*:*

$$\boxed{\text{If } f, \chi \in C_\infty(\mathbb{R}) \text{ then } f(r^*)\chi(H_0) \text{ is compact.}}$$

In the above corollary, $C_\infty(\mathbb{R})$ is the set of continuous functions that vanish at infinity.

Corollary 3.4.3. *$\Gamma^1 D_{r^*}$ and $g(r^*)\mathfrak{D}$ are elements of $B(D(H_0), \mathscr{H})$.*

19. The criterion is a consequence of the Rellich-Kondrachov theorem. See for example [Eva10].

The relationship between the operators Q and H_0^2 goes even further. Using similar arguments to those in [NH04], one can show that:

$$\boxed{D(H_0^2) = D(Q)} \tag{3.65}$$

3.5 Mourre theory

3.5.1 Brief overview

Mourre theory is a very powerful tool for constructing analytical scattering theories. It has been used in many different situations including the quantum N-particle problem [DG97] and for scattering of classical fields – with or without spin – in a range of black-hole type geometries [Häf03; Dau04; NH04]. The theory has been refined since E. Mourre's original article [Mou81] following, in particular, the theoretical developments in [AMG96]. There, it is discussed that one can substitute a certain regularity condition for some of the technical conditions in Mourre's original work. We present here a non-optimal "working" version of the theory. Mourre theory is a very powerful tool for constructing analytical scattering theories. It has been used in many different situations including the quantum N-particle problem [DG97] and for scattering of classical fields - with or without spin- in a range of black-hole type geometries [Häf03; Dau04; NH04]. The theory has been refined since E. Mourre's original article [Mou81] following, in particular, the theoretical developments in [AMG96]. There, it is discussed that one can substitute a certain regularity condition for some of the technical conditions in Mourre's original work. We present here a non-optimal "working" version of the theory.

We begin by making precise the aforementioned regularity condition :

Definition 3.5.1. Let A, H be two self-adjoint operators on a Hilbert space $\mathscr{H}$. We will say that $H \in C^1(A)$ if for any $u \in \mathscr{H}$ the map $s \mapsto e^{isA}(H-z)^{-1}e^{-isA}u$ is of class C^1 for a (and therefore all) $z \in \rho(H)$.

In other words, Definition 3.5.1 states that, in a certain sense, the resolvent of H evolves smoothly under the action of A [20]. An interesting technical consequence of this regularity

20. This interpretation fits nicely into the Heisenberg picture, where operators evolve instead of the wave function

is that (in the form sense) the following equation makes sense on $\mathscr{H}$.

$$[A,(H-z)^{-1}]=(H-z)^{-1}[H,A](H-z)^{-1},$$

we refer to [AMG96] for more details.

Definition 3.5.2. A pair (A,H) of self-adjoint operators on a Hilbert space $\mathscr{H}$ such that $H\in C^1(A)$ will be said to satisfy a Mourre estimate (with compact error) on some energy interval $I\subset\mathbb{R}$ if there is a compact operator K and a strictly positive constant μ such that:

$$\mathbf{1}_I(H)i[H,A]\mathbf{1}_I(H)\geq\mu\mathbf{1}_I(H)+K.$$

This will be written more briefly:

$$\mathbf{1}_I(H)i[H,A]\mathbf{1}_I(H)\underset{K}{\gtrsim}\mu\mathbf{1}_I(H). \tag{3.66}$$

The heart of Mourre theory is contained in the following theorem; the statement here differs from that in Mourre's original article [Mou81]; here we follow [Dau10; NH04].

Theorem 3.5.1 (Mourre). *Suppose that :*

1. *$i[H,A]$ defined as a quadratic form on $D(H)\cap D(A)$ extends to an element of $B(D(H),\mathscr{H})$,*
2. *$[A,[A,H]]$ defined as a quadratic form on $D(H)\cap D(A)$ extends to a bounded operator from $D(H)$ to $D(H)^*$.*
3. *(A,H) satisfy a Mourre estimate on $I\subset\mathbb{R}$.*

Then, H has no singular continuous spectrum in I, and H has at most a finite number of eigenvalues, counted with multiplicity, in I.

When a pair (A,H) satisfy the conditions of Theorem 3.5.1, A will be said to be a **conjugate operator** for H on I.

3.5.2 Our conjugate operators

We will now proceed to describe our choice of conjugate operators for H_0 and a class of perturbations of H_0 that will include $H^p, p\in\mathbb{Z}+\frac{1}{2}$. Mourre theory is very flexible in that the notion of conjugate operator is local in energy but also, using cut-off functions, in space-time; this is well-illustrated in [Dau04; NH04]. As a consequence, determining a candidate for the conjugate operator of a given operator H can be a very creative process, although in many examples from physics, the generator of dilatations, or minor variations thereof, is usually a good candidate. We will see that, despite the extreme blackhole geometry, our case is no exception. As in [Dau04], the full conjugate operator will be a combination of two operators A_+ and A_- tailored to deal with the distinct natures of the geometry at the two asymptotic ends. Throughout the sequel we separate the two infinities using smooth cut-off functions, j_+, j_-, j_1 satisfying:

$$\begin{cases} j_-(t) = 1 & \text{if } t \leq -2, \quad j_-(t) = 0 \quad \text{if } t \geq -\frac{3}{2}, \\ j_+(t) = 1 & \text{if } t \geq -\frac{1}{2}, \quad j_+(t) = 0 \quad \text{if } t \leq -1, \\ j_1(t) = 1 & \text{if } t \geq -1, \quad j_1(t) = 0 \quad \text{if } t \leq -\frac{3}{2}. \end{cases} \tag{3.67}$$

j_- and j_1 should be chosen such that their supports are disjoint.

At the simple horizon

Near $\mathscr{H}_{r_+}$, we will follow the treatment in [Dau04] and set:

$$A_+(S) = R_+(r^*, \mathfrak{D})\Gamma^1, \tag{3.68}$$

where:

$$R_+(r^*, \mathfrak{D}) = (r^* - \kappa^{-1} \ln |\mathfrak{D}|) j_+^2 \left(\frac{r^* - \kappa^{-1} \ln |\mathfrak{D}|}{S} \right). \tag{3.69}$$

Since $|\mathfrak{D}| \geq 1$, the same arguments in the proof of [Dau04, Lemma IV.4.4] can be used to show that:

Lemma 3.5.1. *For any $S \geq 1$, uniformly in λ_k, $k \in \mathbb{N}^*$:*

$$|R_+(r^*, \lambda_k)| \leq C \langle r^* \rangle. \tag{3.70}$$

In the above, C is a positive constant and $R_+(r^*, \lambda_k)$ denotes the restriction of $R_+(r^*, \mathfrak{D})$ to $\mathscr{H}_{k,n}$.

Despite the strange argument in the cut-off function, this choice is surprisingly simple and is essentially : $\Gamma^1 r^*$. This is motivated by the observation that, under the unitary transformation: $U = e^{-i\kappa^{-1}\ln(|\mathfrak{D}|)D_{r^*}}$, the toy model on $\mathbb{R}_+ \times S^2$ given by:

$$\not{H} = \Gamma^1 D_{r^*} + e^{-\kappa r^*}\mathfrak{D} + c,$$

transforms to :

$$\hat{\not{H}} = \Gamma^1 D_{r^*} + e^{-\kappa r^*}\frac{\mathfrak{D}}{|\mathfrak{D}|} + c.$$

The commutator with $\Gamma^1 r^*$ is then easily seen to be :

$$i[\hat{\not{H}}, \Gamma^1 r^*] = 1 + 2r^* e^{-\kappa r^*}\frac{\mathfrak{D}}{|\mathfrak{D}|}\Gamma^1.$$

Restricting to a compact energy interval using $\chi(H)$, $\chi \in C_0^\infty(\mathbb{R})$, the second term will lead to a compact error by Corollary 3.4.2. Note that without the unitary transformation U the commutator is :

$$i[\not{H}, \Gamma^1 r^*] = 1 + 2r^* e^{-\kappa r^*}\mathfrak{D}\Gamma^1.$$

Here the second term is problematic, as $r^* e^{-\kappa r^*}$ does not decay faster than $e^{-\kappa r^*}$ and hence we cannot control $||r^* e^{-\kappa r^*}\mathfrak{D}||$ with $||e^{-\kappa r^*}\mathfrak{D}||$.

Near the double horizon

Let us start our discussion at $\mathscr{H}_{r_e}$ by motivating the coordinate transformation we performed in Section 3.4.2.

At the double horizon ($r^* \to -\infty$), the function g appearing in the expression for H_0 decays as $O\left(\frac{1}{-r^*}\right)$. This is significantly slower than the exponential decay at a simple horizon, and is similar to the behaviour at space-like infinity in an asymptotically flat spacetime. In fact, when $r^* \to -\infty$ the principal symbol of H_0 formally ressembles:

$$\tilde{\not{H}} = \Gamma^1 D_{r^*} - \frac{C}{r^*}\mathfrak{D},$$

which *is* the massless Dirac operator (for the spinor density) for the asymptotically flat metric on $\mathbb{R}^*_- \times S^2$:

$$\eta = dt^2 - dr^{*2} - \left(\frac{r^*}{C}\right)^2 \frac{1}{\Delta_\theta}\mathrm{d}\sigma^2.$$

This suggests that we should try to treat the double horizon in a similar manner to space-like infinity, and in particular that $A = \frac{1}{2}\{D_{r^*}, r^*\}$ should be a reasonable candidate for a conjugate operator there; indeed,

$$i[\tilde{\not{H}}, A] = \tilde{\not{H}}. \tag{3.71}$$

However, had we used the original Boyer-Lindquist like coordinates (t, r, θ, φ), near $r^* \to -\infty$, we would have been lead to set:

$$\tilde{H}_0 = \Gamma^1 D_{r^*} + g(r^*)\mathfrak{D} + \tilde{f}(r^*),$$

where $\tilde{f} \in \boldsymbol{S}^{0,0}$ and $\lim_{r^* \to -\infty} \tilde{f}(r^*) = c_0 p = \dfrac{ap}{r_e^2 + a^2}$. The corresponding toy model would hence be : $\tilde{\not{H}} + c_0 p$. Since A commutes with constants, we need to modify it to generalise Equation (3.71). This can be achieved simply by appending $\Gamma_1 c_0 r^*$ to A. However, in doing so, we are immediately confronted to similar issues (that are carefully avoided by the unitary transformation U) described above at the simple horizon. The solution relies on the morphism properties of exp and the fact that $r^* e^{k_+ r^*} \underset{r^* \to -\infty}{=} o(1)$. In our situation, even if we can imagine trying to exploit the morphism properties of $t \mapsto \frac{1}{t}$, with a unitary transformation such as $\tilde{U} = e^{-\frac{i}{2} \ln |\mathfrak{D}| \{D_{r^*}, r^*\}}$, the error may not be compact simply because there is no decay left ! The coordinate change performed in Section 3.4.2 circumvents the problem entirely by shifting the potential to the simple horizon, where we know how to treat it. In the sequel we set:

$$A_-(S) = \frac{1}{2}\{R_-(r^*), D_{r^*}\}, \tag{3.72}$$

where,

$$R_-(r^*) = j_-^2(\frac{r^*}{S})r^*, \tag{3.73}$$

$\{\,\cdot\,,\,\cdot\,\}$ denotes the anti-commutator and $S \geq 1$ is a real parameter.

The conjugate operator A_I will vary depending on the energy interval I, in fact we will show that there is $S_I \in [1, +\infty)$ such that on I either:

$$\begin{aligned} &A_+(S_I) + A_-(S_I) \quad \text{if } I \subset (0, +\infty), \\ &A_+(S_I) - A_-(S_I) \quad \text{if } I \subset (-\infty, 0), \end{aligned} \tag{3.74}$$

is a conjugate operator on I.

3.5.3 The technical conditions

Despite being the key assumption in Mourre theory, the estimate (3.66) alone is not sufficient for the conclusion of Theorem 3.5.1. This section is devoted to the proof of the following results:

Proposition 3.5.1. *For any $S \geq 1$, $A_\pm(S)$ and $A_+(S) \pm A_-(S)$ are essentially self-adjoint on $\mathscr{S}$.*

Proposition 3.5.2. *Let H be an operator on $\mathscr{H}$ defined by :*

$$H = hH_0h + V, \tag{3.75}$$

where[21]*:*

— *V is a matrix-valued potential such that $V \in \boldsymbol{S}^{1,1}$*

— *$h \in C_b^\infty(\mathbb{R}_{r^*} \times]0, \pi[)$ such that $h > 0$, $|h^2 - 1| \leq c < 1$, $\partial_{r^*}h, \partial_\theta h$, $h^2 - 1 \in \boldsymbol{S}^{1,1}$.*

Any such operator is self-adjoint on $\mathscr{H}$ with domain $D(H_0)$ by the Kato-Rellich theorem. Furthermore for any $A \in \{A_\pm(S), A_+(S) \pm A_-(S)\}$:

1. *The quadratic forms $i[H, A]$ and $i[[H, A], A]$ on $D(H) \cap D(A)$ extend to elements of $\mathcal{B}(D(H), \mathscr{H})$,*
2. *$H \in C^2(A)$.*

We record here the following useful properties of the operators H defined in the previous Proposition:

Lemma 3.5.2. $D(H^2) = D(H_0^2)$.

For a proof we refer to [NH04, Lemma 4.6].

We also note that the functions h and $V = V_C + V_S$ of H^p satisfy slightly better conditions than those above :

Lemma 3.5.3.

— *The function h defined by Equation (3.56) satisfies :*

$$h^2 - 1, \partial_{r^*}h, \partial_\theta h \in \boldsymbol{S}^{2,2}.$$

21. These assumptions are not optimal

— *Let V_C and V_S be as in Equations (3.62) and (3.61) then :*

$$V_C \in \boldsymbol{S}^{1,1}, V_S \in \boldsymbol{S}^{1,2}.$$

We begin our presentation of the proof by remarking that the condition $H \in C^1(A)$ is quite difficult to check directly, despite the following characterisation:

Theorem 3.5.2 ([AMG96, Theorem 6.2.10]). *$H \in C^1(A)$ if and only if the following two conditions are satisfied:*

— *there is $c \in \mathbb{R}_+$ such that for all $u \in D(A) \cap D(H)$:*

$$|(Au, Hu) - (Hu, Au)| \leq c(||Hu||^2 + ||u||^2), \tag{3.76}$$

— *for some $z \in \rho(H)$ the set:*

$$\{u \in D(A), (H-z)^{-1}u \in D(A) \text{ and } (H-\bar{z})^{-1}u \in D(A)\},$$

is a core for A.

To overcome this, there is a useful scheme, based on Nelson's commutator theorem [RS75, Theorems X.36, X.37], that greatly simplifies the proof that $H \in C^1(A)$ in many cases. We first recall Nelson's theorem:

Theorem 3.5.3 (Nelson). *Let N be a self-adjoint operator with $N \geq 1$. Let A be a symmetric operator with domain D that is also a core for N. Suppose that:*

— *For some c and all $\psi \in D$,*

$$||A\psi|| \ \leq c||N\psi||. \tag{3.77}$$

— *For some d and all $\psi \in D$:*

$$|(A\psi, N\psi) - (N\psi, A\psi)| \leq d||N^{\frac{1}{2}}\psi||^2. \tag{3.78}$$

Then A is essentially self-adjoint on D and its closure is essentially self-adjoint on any other core for N.

Remark 3.5.1. Note that it follows that $D(N) \subset D(\bar{A})$ and A is essentially self-adjoint on $D(N)$.

The scheme is to find a third operator N – that we will refer to as the *comparison operator* – whose domain is a core for both H and A; which we establish using Nelson's lemma. We then seek to apply the following:

Theorem 3.5.4 ([GL02, Lemma 3.2.2]). *Let (H, H_0, N) be a triplet of self-adjoint operators on $\mathscr{H}$, with $N \geq 1$, A a symmetric operator on $D(N)$. Assume that:*

1. *$D(H) = D(H_0) \supset D(N)$,*
2. *$D(N)$ is stable under the action of $(H - z)^{-1}$,*
3. *H_0 and A satisfy (3.77) and (3.78),*
4. *for some $c > 0$ and any $u \in D(N)$, (3.76) is satisfied.*

Then:

— *$D(N)$ is dense in $D(A) \cap D(H)$ with norm $||Hu|| + ||Au|| + ||u||$,*

— *the quadratic form $i[H, A]$ defined on $D(A) \cap D(H)$ is the unique extension of $i[H, A]$ on $D(N)$,*

— *$H \in C^1(A)$.*

Our proof shall follow this outline.

3.5.4 The comparison operator N

Before identifying the comparison operator N, we begin with an important stability lemma:

Lemma 3.5.4. *For any $n \in \mathbb{N}^*, z \in \rho(H_0)$, the domain of $\langle r^* \rangle^n$ is stable under the resolvent $(H_0 - z)^{-1}$ and $\chi(H_0)$ for any $\chi \in C_0^\infty(\mathbb{R})$. The statement remains true if H_0 is replaced with H.*

The proof is identical to that of [Dau04, Proposition IV.3.2] and will not be repeated here. This lemma is very important for scattering purposes since it is an indication of how decay rates behave under the action of H, but it also serves to justify the use of the following comparison operator [22]:

$$N = D_{r^*}^2 + g(r^*)^2 \mathfrak{D}^2 + \langle r^* \rangle^2 = Q + \langle r^* \rangle^2. \tag{3.79}$$

22. That has an almost uncanny ressemblance to the harmonic oscillator...

Decomposing $\mathscr{H}$ as in Section 3.4.5, Lemma 3.4.4 and equation (3.65) imply that:

$$D(N) = D(Q) \cap D(\langle r^* \rangle^2) = D(H_0^2) \cap D(\langle r^* \rangle^2). \tag{3.80}$$

Finally (3.80) and Lemmata 3.5.4 and 3.5.2, together lead to:

$$\begin{aligned} \forall z \in \rho(H_0), \quad (H_0 - z)^{-1} D(N) \subset D(N), \\ \forall z \in \rho(H), \quad (H - z)^{-1} D(N) \subset D(N). \end{aligned} \tag{3.81}$$

Thus, the first two conditions of Theorem 3.5.4 are satisfied by the triplet (H, H_0, N).

3.5.5 Nelson's lemma

We will now check that H_0 and $A_\pm(S)$ satisfy the hypotheses of Theorem 3.5.3. To simplify notations, we will omit to specify the dependence on the real parameter S of the operator $A_\pm$ in this paragraph, as all the results discussed here hold for any $S \geq 1$. As a first step, we deduce immediately the following useful estimates from (3.79):

Lemma 3.5.5. *For any $u \in D(N)$:*

$$\begin{aligned} ||\Gamma^1 D_{r^*} u|| \leq ||N^{\frac{1}{2}} u||, \quad ||g(r^*)\mathfrak{D} u|| \leq ||N^{\frac{1}{2}} u||, \\ ||r^* u|| \leq ||N^{\frac{1}{2}} u||, \qquad\qquad ||u|| \leq ||N^{\frac{1}{2}} u||. \end{aligned} \tag{3.82}$$

Lemma 3.5.6. *With N as comparison operator, H_0 satisfies Equations (3.77) and (3.78).*

Proof. Fix $u \in D(N)$, from Lemma 3.5.5, we have:

$$\begin{aligned} ||H_0 u|| &\leq ||\Gamma^1 D_{r^*} u|| + ||g(r^*)\mathfrak{D} u|| + ||f(r^*) u||, \\ &\leq (2 + ||f||_\infty) ||N^{\frac{1}{2}} u||, \\ &\leq (2 + ||f||_\infty) ||N u||, \end{aligned} \tag{3.83}$$

this proves (3.77).

Moreover:

$$\begin{aligned}|([N,H_0]u,u)| \leq & 2|(\Gamma^1 r^* u,u)| + 2|(\Gamma^1 g'(r^*)g(r^*)\mathfrak{D}^2 u,u)| \\ & + 2||f'||_\infty ||D_{r^*}u|| ||u|| + 2||D_{r^*}u|| ||g'(r^*)\mathfrak{D}u||, \\ \leq & 2\Big(||r^*u|| ||u|| + C||g(r^*)\mathfrak{D}u||^2 \\ & + \ ||f'||_\infty ||D_{r^*}u|| ||u|| + C||D_{r^*}u|| ||g(r^*)\mathfrak{D}u||\Big), \\ \leq & 2(1+||f'||_\infty + 2C)||N^{\frac{1}{2}}u||^2.\end{aligned} \tag{3.84}$$

In (3.84), we have used the fact that there is $C \in \mathbb{R}_+^*$ such that:

$$|g'(r^*)| \leq C|g(r^*)|,$$

and the functional calculus as in the proof of Lemma 3.4.7.

In order to establish analogous estimates for A_-, we will also need the following estimates:

Lemma 3.5.7. *For any $u \in D(N)$,*

$$\begin{aligned}||r^{*2}u||^2 \leq ||Nu||^2 + ||u||^2, \\ ||Qu||^2 \leq ||Nu||^2 + ||u||^2.\end{aligned} \tag{3.85}$$

Proof. As usual, we will prove it for $u \in \mathscr{S}$. One has:

$$\begin{aligned}||Nu||^2 = & (N^2u,u) \\ = & ||Qu||^2 + ||r^{*2}u||^2 + ||u||^2 + (Qu, r^{*2}u) \\ & + (r^{*2}u, Qu) + 2(Qu,u) + 2||r^*u||^2.\end{aligned} \tag{3.86}$$

Since, for any $v \in D(Q)$, $(Qv,v) = ||\Gamma^1 D_{r^*}^2 v||^2 + ||g(r^*)\mathfrak{D}v||^2 \geq 0$, it follows that:

$$||Nu||^2 \geq ||Qu||^2 + \ ||r^{*2}u||^2 + ||u||^2 + (Qu, r^{*2}u) + (r^{*2}u, Qu). \tag{3.87}$$

Now,

$$(Qu, r^{*2}u) = (r^*Qu, r^*u) = (Qr^*u, r^*u) + (2iD_{r^*}u, r^*u), \tag{3.88}$$

and so, adding the hermitian conjugate $(r^{*2}u, Qu)$, one obtains:

$$\begin{aligned}(Qu, r^{*2}u) + (r^{*2}u, Qu) &= 2(Qr^*u, r^*u) + (2ir^*D_{r^*}u, u) - (2iD_{r^*}r^*u, u) \\ &= 2(Qr^*u, r^*u) - 2||u||^2 \geq -2||u||^2.\end{aligned}$$

Hence,

$$||Nu||^2 \geq ||Qu||^2 + ||r^{*2}u||^2 - ||u||^2. \tag{3.89}$$

Lemma 3.5.8. *There is a constant $d > 0$ such that for any $u \in D(Q) = D(H_0^2)$,*

$$||D_{r^*}^2 u||^2 \leq d(||Qu||^2 + ||u||^2). \tag{3.90}$$

Proof. As quadratic forms on $\mathscr{S}$:

$$\begin{aligned}Q^2 =& D_{r^*}^4 + (g^2(r^*)\mathfrak{D}^2)^2 + D_{r^*}^2 g^2(r^*)\mathfrak{D}^2 + g^2(r^*)\mathfrak{D}^2 D_{r^*}^2, \\ =& D_{r^*}^4 + (g^2(r^*)\mathfrak{D}^2)^2 + 2\mathfrak{D}g(r^*)D_{r^*}^2 g(r^*)\mathfrak{D} \\ &+ [D_{r^*}^2, g(r^*)]g(r^*)\mathfrak{D}^2 - g(r^*)\mathfrak{D}^2[D_{r^*}^2, g(r^*)], \\ \geq& D_{r^*}^4 + (g^2(r^*)\mathfrak{D}^2)^2 + [[D_{r^*}^2, g], g]\mathfrak{D}^2, \\ =& D_{r^*}^4 + (g^2(r^*)\mathfrak{D}^2)^2 - i[\{D_{r^*}, g'\}, g(r^*)]\mathfrak{D}^2, \\ =& D_{r^*}^4 + (g^2(r^*)\mathfrak{D}^2)^2 - 2(g'(r^*))^2\mathfrak{D}^2, \\ \geq& D_{r^*}^4 + (g^2(r^*)\mathfrak{D}^2)^2 - 2C^2 g(r^*)^2\mathfrak{D}^2, \\ \geq& D_{r^*}^4 + \frac{1}{2}(g^2(r^*)\mathfrak{D}^2)^2 - 2C^4, \\ \geq& D_{r^*}^4 - 2C^4.\end{aligned} \tag{3.91}$$

where we have used the fact that $|g'(r^*)| \leq C|g(r^*)|$.

Combining Lemmata 3.5.7 and 3.5.8 yields:

Corollary 3.5.1. *$r^{*2}, D_{r^*}^2 \in B(D(N), \mathscr{H})$.*

We are now ready to prove:

Lemma 3.5.9. *A_- satisfies (3.77) and (3.78).*

Proof. Until now we have not discussed the domain of A_- and will simply consider it as being defined for $u \in \mathscr{S}$, which is a core for N. Then, the following estimates hold:

$$\begin{aligned}||A_-u||^2 =&(R_-(r^*)D_{r^*}u, R_-(r^*)D_{r^*}u) + \frac{1}{4}||R'_-(r^*)u||^2 \\ &- \frac{1}{2}\left((R_-(r^*)D_{r^*}u, iR'_-(r^*)u) + (iR'_-(r^*)u, R_-(r^*)D_{r^*}u)\right), \\ \leq&(R_-(r^*)D_{r^*}u, R_-(r^*)D_{r^*}u) + \ ||R'_-(r^*)R_-(r^*)u||||D_{r^*}u|| + \frac{1}{4}||R'_-(r^*)u||^2.\end{aligned}$$

Since $R'_-(r^*)$ is a bounded operator, using Lemma 3.5.5 one can see that:

$$\begin{aligned}||R'_-(r^*)R_-(r^*)u||||D_{r^*}u|| + \frac{1}{4}||R'_-(r^*)u||^2 &\leq ||R'_-||_\infty||N^{\frac{1}{2}}u||^2 + \frac{1}{4}||R'_-||^2_\infty||u||^2 \\ &\leq ||R'_-||_\infty(1 + ||R'_-||_\infty)||Nu||^2.\end{aligned}$$

Moreover, by Lemmata 3.5.7 and 3.5.8:

$$\begin{aligned}|(R_-(r^*)D_{r^*}u, R_-(r^*)D_{r^*}u)| &= |(R^2_-(r^*)u, D^2_{r^*}u) + 2(iR'_-(r^*)R_-(r^*)u, D_{r^*}u)| \\ &\leq \sqrt{6d}||Nu||^2 + 2||R'_-||_\infty||Nu||^2.\end{aligned}$$

Combining the above gives (3.77). To prove (3.78) we start with the following estimates:

$$\begin{aligned}|([N, A_-]u, u)| =&\Big|(-\frac{i}{2}(R^{(3)}_-(r^*)u, u) - i(\{D^2_{r^*}, R'_-(r^*)\}u, u) \\ &+ 2i(r^{*2}j^2_-(\frac{r^*}{S})u, u) + (2ig'(r^*)g(r^*)R_-(r^*)\mathfrak{D}^2u, u)\Big|, \\ =&\Big| - i(\{D_{r^*}, R'_-(r^*)D_{r^*}\}u, u) - \frac{1}{2}(\{D_{r^*}, R''_-(r^*)\}u, u) \\ &+ 2i(r^{*2}j^2_-(\frac{r^*}{S})u, u) + (2ig'(r^*)g(r^*)R_-(r^*)\mathfrak{D}^2u, u)\Big|, \\ \leq&2||D_{r^*}u||\left(||R'||_\infty||D_{r^*}u|| + \frac{1}{2}||R''||_\infty||u||\right) \\ &+ 2||j_-(\frac{r^*}{S})r^*u||^2 + 2||g(r^*)\mathfrak{D}u||||g'(r^*)R_-(r^*)\mathfrak{D}u||.\end{aligned}$$

The only term that may pose problem is:

$$||R_-(r^*)g'(r^*)\mathfrak{D}u||. \tag{3.92}$$

However,

$$R_-(r^*)g'(r^*) = g(r^*)j_-^2(r^*)r^* \left(\frac{\frac{\Delta_r'}{2}}{\Xi(r^2+a^2)} - \frac{2r\Delta_r}{\Xi(r^2+a^2)^2} \right), \tag{3.93}$$

and the term between brackets is $\underset{r^*\to-\infty}{O}(\frac{1}{r^*})$ because when $r^* \to -\infty$, r approaches r_e, the double root of Δ_r, hence, both Δ_r and Δ_r' are at least $\underset{r\to x}{O}(r-r_e)$ and $r-r_e = \underset{r^*\to-\infty}{O}(\frac{1}{r^*})$ [23]. In conclusion, there is $C \in \mathbb{R}^*, |R_-(r^*)g'(r^*)| \leq C|g(r^*)|$ and thus, by the functional calculus:

$$||R_-(r^*)g'(r^*)\mathfrak{D}u|| \leq C||g(r^*)\mathfrak{D}u||. \tag{3.94}$$

Overall,

$$|([N, A_-]u, u)| \leq \left(||R_-''||_\infty + 2\left(||R_-'||_\infty + C + 1\right)\right)||N^{\frac{1}{2}}u||^2 \tag{3.95}$$

According to the above result, we can conclude that A_- is essentially self-adjoint on $D(N)$; the analogous result for A_+ is proved in [Dau04, Lemma IV.4.5], the arguments are identical. Theorem 3.5.3 also applies to $A = A_+ \pm A_-$. In all cases, we will consider the operators and their domains as being defined by the conclusion of Theorem 3.5.3.

3.5.6 Proof that $H_0, H \in C^1(A)$

In order to prove that $H, H_0 \in C^1(A)$, we require one more estimate that will be the object of this section. According to Theorem 3.5.4 it is sufficient to prove that for some $c > 0$ and any $u \in D(N)$ one has the estimate:

$$|(Hu, A_\pm u) - (A_\pm u, Hu)| \leq c(||Hu||^2 + ||u||^2). \tag{3.96}$$

As before, we will focus our attention on A_- and refer to [Dau04, Lemma IV.4.7] for A_+. In order to apply Mourre theory, we will additionally need to show that $i[H, A]$ extends to a bounded operator from $D(H) = D(H_0)$ [24] to $\mathscr{H}$. Both of these are covered by the following estimates, established, first, on the common core $\mathscr{S}$; we begin with H_0.

23. Note that in (3.93) $\Delta_r' = \frac{\partial \Delta_r}{\partial r}$
24. This equality is to be understood to imply that the graph norms are equivalent.

Let $u \in \mathscr{S}$, then:

$$\begin{aligned} ||i[H_0, A_-]u|| =& \left\| \Gamma^1 R_-'(r^*) D_{r^*} u - \frac{i}{2} \Gamma^1 R_-''(r^*) u - R_-(r^*) g'(r^*) \mathfrak{D} u - R_-(r^*) f'(r^*) u \right\|, \\ \leq & ||R_-'||_\infty ||D_{r^*} u|| + \frac{1}{2} ||R_-''||_\infty ||u|| + ||R_-(r^*) g'(r^*) \mathfrak{D} u|| + ||R_- f'||_\infty ||u||. \end{aligned}$$

Using (3.94) and Corollary 3.4.3, we thus conclude that for some $c > 0$ and any $u \in \mathscr{S}$:

$$||i[H_0, A_-]u|| \leq c(||H_0 u|| + ||u||). \tag{3.97}$$

Hence, $i[H_0, A_-]$ extends uniquely to an element of $B(D(H_0), \mathscr{H})$ and (3.96) holds. In order to establish the analogous result for H, we write:

$$[H, A_-] = h[H_0, A_-]h + i(hH_0 R_-(r^*)h' + R_-(r^*)h' H_0 h) + iR_-(r^*)V'.$$

Since $h, R_-'(r^*) \in B(D(H_0))$, $h[H_0, A_-]h$ and $R_-'(r^*)h' H_0 h$ extend to elements of $B(D(H_0), \mathscr{H})$. For similar reasons to h, $R_-(r^*)h' \in B(D(H_0))$ also, and, using Equation (3.43), $R_-(r^*)V' \in B(\mathscr{H})$. It follows then that $[H_0, A_-]$ extends to a bounded operator $D(H_0) \to \mathscr{H}$.

Assembling all the results above, we have thus shown that $H_0, H \in C^1(A)$ and that the first two assumptions of Theorem 3.5.1 are satisfied. It remains to verify the final assumption regarding the double commutator.

3.5.7 The double commutator assumption

Theorem 3.5.1 only requires that the double commutator extends to a bounded operator from $D(H)$ to $D(H)^*$, this section will be devoted to showing a slightly stronger result:

Lemma 3.5.10. *$[A, [A, H_0]]$ and $[A, [A, H]]$ extend to elements of $B(D(H), \mathscr{H})$.*

The consequence will be that H and H_0 are in fact $C^2(A)$ (see [AMG96, Chapter 5]), proving the final point of Propostion 3.5.2. Beginning with H_0, it is sufficient to prove this for the four double commutators $[A_\pm, [A_\pm, H_0]]$ separately; we will mainly concentrate on A_-, but it will also be informative to consider the mixed terms $[A_\pm, [A_\mp, H_0]]$.

(a) $[[H_0, A_-], A_-]$ A short calculation shows that:

$$\begin{aligned}(-i)[i[H_0, A_-], A_-] =&(-i)\Big(-\frac{1}{2}\Gamma^1 R'_-(r^*)R''_-(r^*) - i(R'_-(r^*))^2\Gamma^1 D_{r^*} \\ &+ iR_-(r^*)R''_-(r^*)\Gamma^1 D_{r^*} - \frac{i}{2}\Gamma^1 R_-(r^*)R'''(r^*) \\ &- iR_-(r^*)\left(\underline{(R_-(r^*)g'(r^*))'\mathfrak{D}} + (R_-(r^*)f'(r^*))'\right)\Big).\end{aligned} \tag{3.98}$$

Many of the terms in (3.98) extend clearly to elements of $B(D(H), \mathscr{H})$, either because they are bounded on $\mathscr{H}$ or using Corollary 3.4.3. The term that merits comment is underlined; it expands as follows:

$$R_-(r^*)g''(r^*)\mathfrak{D} + R'_-(r^*)g'(r^*)\mathfrak{D}. \tag{3.99}$$

We have already shown how to deal with the second term, and the first is treated very similarly as it is easily seen that $|g''(r^*)| \leq C|g(r^*)|$ for some $C \in \mathbb{R}^*_+$.

(b) $[i[H_0, A_-], A_+]$ This double commutator, as a quadratic form on $\mathscr{S}$, can be computed as:

$$(-i)[i[H_0, A_-], A_+] =(-i)\left([\Gamma^1 R'_-(r^*)D_{r^*}, A_+] - 2R_-(r^*)g'(r^*)R_+(r^*, \mathfrak{D})\Gamma^1\mathfrak{D}\right).$$

The first term vanishes, since on $\mathscr{S}$ it can be evaluated as:

$$[\Gamma^1 R'_-(r^*)D_{r^*}, A_+] = -R'_-(r^*)R'_+(r^*, \mathfrak{D}),$$

and j_+ and j_- have disjoint support (cf. (3.67)). The second term, which, on first glance, seems difficult to control, will equally vanish entirely due to our choice cut-off functions j_+, j_-, j_1. To see this, recall that:

$$R_+(r^*, \mathfrak{D}) = (r^* - \kappa^{-1}\ln|\mathfrak{D}|)j_+^2\left(\frac{r^* - \kappa^{-1}\ln|\mathfrak{D}|}{S}\right).$$

Hence, since j_1 satisfies $j_1(t) = 1, t \geq -1$, then:

$$R_+(r^*, \mathfrak{D}) = j_1^2(\frac{r^*}{S})R_+(r^*, \mathfrak{D}). \tag{3.100}$$

It follows that:

$$2R_-(r^*)g'(r^*)R_+(r^*,\mathfrak{D})\Gamma^1\mathfrak{D} = 2R_-(r^*)j_1^2(\frac{r^*}{S})g'(r^*)R_+(r^*,\mathfrak{D})\Gamma^1\mathfrak{D},$$

but, j_- and j_1 are chosen such that $\operatorname{supp} j_- \cap \operatorname{supp} j_1 = \emptyset$, therefore this term vanishes.

(c) $[i[H_0, A_+], A_-]$ Here, we start from [25]:

$$i[H_0, A_+] = R_+'(r^*,\mathfrak{D}) + 2ig(r^*)\mathfrak{D}R_+(r^*,\mathfrak{D})\Gamma^1,$$

this leads to:

$$[i[H_0, A_+], A_-] = R_+''(r^*,\mathfrak{D})R_-(r^*) + 2i\left(g(r^*)R_+(r^*,\mathfrak{D})\right)' R_-(r^*)\mathfrak{D}\Gamma^1.$$

Since (3.100) is equally true if $R_+(r^*,\mathfrak{D})$ is replaced by its first or second derivative with respect to r^*, one can argue as before and find that this double commutator vanishes entirely. We refer to [Dau04] for the appropriate treatment of $[[H_0, A_+], A_+]$.

This concludes the proof that (H_0, A) satisfies the first hypotheses of Mourre theory. To show that this is equally true of (H, A), we proceed as before using (3.60). For example:

$$\begin{aligned}[][[H, A_-], A_-] = h[[H_0, A_-], A_-]h + 2ih[H_0, A_-]R_-(r^*)h' \\ + 2iR_-(r^*)h'[H_0, A_-]h - 2h'R_-(r^*)H_0R_-(r^*)h' \\ - hH_0R(r^*)(R_-(r^*)h')' - R_-(r^*)(R_-(r^*)h')'H_0h - R_-(r^*)(R_-(r^*)V')'.\end{aligned}$$

This extends to an element of $B(D(H), \mathscr{H})$, thanks to the decay of h', V', etc. Similar computations show that this is equally true of the other double commutators. The reader may be concerned that a long-range potentiel may jeopardise our efforts in the mixed commutators, causing unbounded terms to appear. However, this is not the case since either commutation with A_- introduces the necessary decay through differentiation or terms vanish entirely due to the choice that j_1 and j_- have disjoint supports. For the first point, more precisely, if, for instance, $f \in \mathscr{S}(\mathbb{R})$, then :

$$[f(r^*), A_-] = iR_-(r^*)f'(r^*).$$

25. In this equation $R'(r^*,\mathfrak{D})$ denotes the operator obtained after differentiating with respect to r^*

In all cases encountered, f, when expressed as a function of r, has bounded derivative and therefore, at least, $[f(r^*), A_-] = O(\frac{1}{r^*})$.

3.5.8 Mourre estimates for H_0

We shall now move on to derive Mourre inequalities, naturally, we will treat $\mathscr{H}_{r_e}$ and $\mathscr{H}_{r_+}$ separately.

Near the double horizon

We begin with:

Lemma 3.5.11. *Let $\chi \in C_0^\infty(\mathbb{R})$ then for any $S \geq 1$;*

$$\chi(H_0) i[H_0, A_-(S)]\chi(H_0) \underset{K}{\sim} \chi(H_0) j_-(\frac{r^*}{S}) H_0 j_-(\frac{r^*}{S})\chi(H_0), \tag{3.101}$$

where $\underset{K}{\sim}$ is used to denote equality up to a compact error.

Proof. One has:

$$\begin{aligned} i[H_0, A_-(S)] =& \Gamma^1 R_-'(r^*) D_{r^*} - \frac{i}{2}\Gamma^1 R_-''(r^*) - R_-(r^*) g'(r^*)\mathfrak{D} - R_-(r^*) f'(r^*) \\ =& j_-(\frac{r^*}{S})\Gamma^1 D_{r^*} j_-(\frac{r^*}{S}) + 2r^* j_-(\frac{r^*}{S}) j_-'(\frac{r^*}{S})\Gamma^1 D_{r^*} \\ & \underline{-\frac{i}{S} j_-'(\frac{r^*}{S}) j_-(\frac{r^*}{S})} - \underline{\frac{ir^*}{S^2}(j_-'(\frac{r^*}{S}))^2} \\ & - \underline{\frac{ir^*}{S^2} j_-''(\frac{r^*}{S}) j_-(\frac{r^*}{S})} - R_-(r^*)\left(g'(r^*)\mathfrak{D} + f'(r^*)\right). \end{aligned} \tag{3.102}$$

Note that if $0 \leq \chi \leq 1$ is a smooth function with compact support in $\mathbb{R}$, since j' has compact support, Corollary 3.4.2 implies that the terms underlined above will only lead to compact terms in $\chi(H_0) i[H_0, A_-(S)]\chi(H_0)$, consequently:

$$\begin{aligned} \chi(H_0) i[H_0, A_-(S)]\chi(H_0) \underset{K}{\sim} \chi(H_0)\Bigg(& j_-(\frac{r^*}{S})\Gamma^1 D_{r^*} j_-(\frac{r^*}{S}) + 2r^* j_-(\frac{r^*}{S}) j_-'(\frac{r^*}{S})\Gamma^1 D_{r^*} \\ & - R_-(r^*)\left(g'(r^*)\mathfrak{D} + f'(r^*)\right)\Bigg)\chi(H_0). \end{aligned} \tag{3.103}$$

Using Corollary 3.4.3, one can show that $2r^* j_-(\frac{r^*}{S}) j_-'(\frac{r^*}{S})\Gamma^1 D_{r^*}\chi(H_0)$ is also compact.

Indeed, let $\gamma(r^*) = 2r^* j_-(\frac{r^*}{S}) j_-'(\frac{r^*}{S})$ and note that $\gamma \in C_0^\infty(\mathbb{R})$. For any $u \in \mathscr{H}$, one has:

$$\gamma(r^*)\Gamma^1 D_{r^*}\chi(H_0)u = \Gamma^1 D_{r^*}\gamma(r^*)\chi(H_0)u + i\Gamma^1\gamma'(r^*)\chi(H_0)u. \tag{3.104}$$

Corollary 3.4.3 implies that there is $C_1 > 0$ such that for any $u \in D(H_0)$.

$$||\Gamma^1 D_{r^*}u|| \leq C_1(||H_0 u|| + ||u||).$$

Hence:

$$\begin{aligned}
||\gamma(r^*)\Gamma^1 D_{r^*}\chi(H_0)u|| &\leq ||\Gamma^1 D_{r^*}\gamma(r^*)\chi(H_0)u|| + ||\gamma'(r^*)\chi(H_0)u||, \\
&\leq C_1||H_0\gamma(r^*)\chi(H_0)u|| + C_1||\gamma(r^*)\chi(H_0)u|| \\
&\qquad + ||\gamma'(r^*)\chi(H_0)u||, \\
&\leq C_1||\gamma(r^*)H_0\chi(H_0)u|| + C_1||\gamma(r^*)\chi(H_0)u|| \\
&\qquad + (1+C_1)||\gamma'(r^*)\chi(H_0)u||.
\end{aligned}$$

According to Corollary 3.4.2 the operators $\gamma(r^*)H_0\chi(H_0)$, $\gamma(r^*)\chi(H_0)$ and $\gamma'(r^*)\chi(H_0)$ are all compact and so it follows from a simple extraction argument that $\gamma(r^*)\Gamma^1 D_{r^*}\chi(H_0)$ must be too. Thus:

$$\begin{aligned}
\chi(H_0)i[H_0, A_-(S)]\chi(H_0) &\underset{K}{\sim} \chi(H_0)j_-(\frac{r^*}{S})\Gamma^1 D_{r^*} j_-(\frac{r^*}{S}) \\
&\qquad - R_-(r^*)\left(g'(r^*)\mathfrak{D} + f'(r^*)\right)\chi(H_0).
\end{aligned} \tag{3.105}$$

Now, (3.105) can be rewritten:

$$\begin{aligned}
\chi(H_0)i[H_0, A_-(S)]\chi(H_0) &\underset{K}{\sim} \chi(H_0)j_-(\frac{r^*}{S})H_0 j_-(\frac{r^*}{S})\chi(H_0) \\
&\quad - \chi(H_0)j_-^2(\frac{r^*}{S})\left(g(r^*) + r^*g'(r^*)\right)\mathfrak{D}\chi(H_0) \\
&\quad - \chi(H_0)j_-^2(\frac{r^*}{S})\left(f(r^*) + r^*f'(r^*)\right)\chi(H_0).
\end{aligned}$$

Since $f(r^*) + r^*f'(r^*) \to 0$ when $r^* \to -\infty$, it follows from Corollary 3.4.2 that the terms in the last line of the previous equation are compact. The compactness of those on the middle line is also a consequence of Corollary 3.4.2, because near the double horizon

$r_* \to -\infty$ $(r \to r_e)$ one has:

$$r^* g'(r^*) + g(r^*) = \left(1 + \frac{r^*}{\Xi(r_e^2+a^2)}\frac{\Delta_r'}{2} + O\left(\frac{1}{r^*}\right)\right) g(r^*),$$

and:

$$\begin{aligned}\Delta_r' &= 2l^2(r-r_e)(r_e-r_-)(r_+-r_e) + O((r-r_e)^2),\\ &= -2\frac{(3Mr_e-4a^2)(r-r_e)}{r_e^2} + O((r-r_e)^2).\end{aligned}$$

Using (3.16) we obtain that:

$$\Delta_r' = -2\frac{(r_e^2+a^2)\Xi}{r^*} + o(\frac{1}{r^*}).$$

From which it follows:

$$r^* g'(r^*) + g(r^*) = o(g(r^*)). \tag{3.106}$$

Therefore, there is a continuous function $\varepsilon \in C_\infty(\mathbb{R})$ such that:

$$\begin{aligned}||j_-^2(\frac{r^*}{S})(r^* g'(r^*) + g(r^*))\mathfrak{D}\chi(H_0)|| &= ||g(r^*)\mathfrak{D}\varepsilon(r^*)\chi(H_0)||,\\ &\le ||H_0\varepsilon(r^*)\chi(H_0)|| + ||\varepsilon(r^*)\chi(H_0)||.\end{aligned}$$

Compactness then follows with a similar argument as before.

We are now ready to prove:

Proposition 3.5.3. *Let χ be of a compact support contained in $(0,+\infty)$ and $\mu > 0$ be such that* $\operatorname{supp}\chi \subset [\mu,+\infty)$ *then for any $S \ge 1$:*

$$\chi(H_0)i[H_0, A_-(S)]\chi(H_0) \underset{K}{\gtrsim} \mu\chi(H_0)j_-^2(\frac{r^*}{S})\chi(H_0). \tag{3.107}$$

The result holds also if $\operatorname{supp}\chi \subset (-\infty, 0)$, if we replace $A_-(S)$ by $-A_-(S)$.

Proof. Using Lemma 3.5.11, it is sufficient to prove that:

$$\chi(H_0)j_-(\frac{r^*}{S})H_0 j_-(\frac{r^*}{S})\chi(H_0) \underset{K}{\gtrsim} \mu\chi(H_0)j_-^2(\frac{r^*}{S})\chi(H_0).$$

Our first step is to note that, although $\chi(H_0)$ and $j_-(\frac{r^*}{S})$ do not commute, their commutator is a compact operator. This can be seen using the Helffer-Sjöstrand formula [HS87, Proposition 7.2] [26], for one has:

$$\begin{aligned}\left[\chi(H_0), j_-(\frac{r^*}{S})\right] &= \frac{i}{2\pi}\int_{\mathbb{C}} \partial_{\bar z}\tilde\chi(z)[(H_0-z)^{-1}, j_-(\frac{r^*}{S})]\mathrm{d}z\wedge\mathrm{d}\bar z,\\ &= -\frac{i}{2\pi}\int \partial_{\bar z}\tilde\chi(z)(H_0-z)^{-1}[H_0, j_-(\frac{r^*}{S})](H_0-z)^{-1}\mathrm{d}z\wedge\mathrm{d}\bar z. \end{aligned} \tag{3.108}$$

The second equation makes sense since $j_-(\frac{r^*}{S})$ is bounded and $[H_0, j_-(\frac{r^*}{S})]$ extends to a bounded operator on $\mathscr{H}$. Furthermore, the integral exists in the norm topology, so the compactness of the commutator follows from that of the integrand which, again, is a consequence of Corollary 3.4.2 since:

$$[H_0, j_-(\frac{r^*}{S})] = -\frac{i}{S}\Gamma^1 j_-'(\frac{r^*}{S}).$$

Now $\chi(H_0)j_-(\frac{r^*}{S})H_0 j_-(\frac{r^*}{S})\chi(H_0)$ is equal to

$$\begin{gathered} j_-(\frac{r^*}{S})\chi(H_0)H_0\chi(H_0)j_-(\frac{r^*}{S})\\ +\underline{j_-(\frac{r^*}{S})\chi(H_0)H_0[j_-(\frac{r^*}{S}),\chi(H_0)]} + \underline{[\chi(H_0), j_-(\frac{r^*}{S})]H_0 j_-(\frac{r^*}{S})\chi(H_0)}. \end{gathered}$$

The underlined terms form a symmetric compact operator and denoting [27] $\mathbb{E}$ the operator-valued spectral measure, for any $u\in\mathscr{H}$:

$$\begin{aligned} (j_-(\frac{r^*}{S})\chi(H_0)H_0\chi(H_0)j_-(\frac{r^*}{S})u,u) &= (\chi(H_0)H_0\chi(H_0)j_-(\frac{r^*}{S})u, j_-(\frac{r^*}{S})u),\\ &= \int t\chi^2(t)(\mathbb{E}(\mathrm{d}t)j_-(\frac{r^*}{S})u, j_-(\frac{r^*}{S})u),\\ &\geq \mu(j_-(\frac{r^*}{S})\chi(H_0)^2 j_-(\frac{r^*}{S})u,u). \end{aligned}$$

In other words:

$$\begin{aligned} j_-(\frac{r^*}{S})\chi(H_0)H_0\chi(H_0)j_-(\frac{r^*}{S}) &\geq \mu j_-(\frac{r^*}{S})\chi(H_0)^2 j_-(\frac{r^*}{S}),\\ &\underset{K}{\gtrsim} \mu\chi(H_0)j_-^2(\frac{r^*}{S})\chi(H_0), \end{aligned} \tag{3.109}$$

26. See also Appendix B.1
27. following the notations of [Lax02].

where we have used once more the compactness of the commutator $[\chi(H_0), j_-(\frac{r^*}{S})]$. Similar arguments prove the final point.

At the simple horizon

The decomposition of the Hilbert space constructed in Section 3.4.3 and the results discussed there concerning the properties of the eigenvalues, mean that the proof of the Mourre estimate at the simple horizon in [Dau04], applies to our case without any essential modification. Hence we quote without proof:

Proposition 3.5.4 ([Dau04, Lemma IV.4.11]). *Let $\lambda_0 \in \mathbb{R}$, then there are $\chi \in C_0^\infty(\mathbb{R})$ such that $\lambda_0 \in \operatorname{supp}\chi$ and $\mu \in \mathbb{R}_+^*$ such that:*

$$\chi(H_0)i[H_0, A_+(S)]\chi(H_0) \underset{K}{\gtrsim} \mu\chi(H_0)j_1^2(\frac{r^*}{S})\chi(H_0), \tag{3.110}$$

for large enough $S \in \mathbb{R}_+^*$.

Remark 3.5.2. It is interesting to remark the difference in the formulation of Propositions 3.5.3 and 3.5.4. Only the latter truly restricts the size of the neighbourhood on which we have a Mourre estimate, Proposition 3.5.3 on the other hand, simply forbids a Mourre estimate on a neighbourhood of 0.

Combining the two previous results leads to:

Proposition 3.5.5. *Let $\lambda_0 \in \mathbb{R}^*$:*

— *If $\lambda_0 > 0$, then one can find an interval $I \subset (0, +\infty)$ containing λ_0 and $\mu > 0$ such that:*

$$\mathbf{1}_I(H_0)i[H_0, A_+(S) + A_-(S)]\mathbf{1}_I(H_0) \underset{K}{\gtrsim} \mu\mathbf{1}_I(H_0), \tag{3.111}$$

for large enough $S \in \mathbb{R}_+^*$.

— *If $\lambda_0 < 0$, then one can find an interval $I \subset (-\infty, 0)$ containing λ_0 and $\mu > 0$ such that:*

$$\mathbf{1}_I(H_0)i[H_0, A_+(S) - A_-(S)]\mathbf{1}_I(H_0) \underset{K}{\gtrsim} \mu\mathbf{1}_I(H_0), \tag{3.112}$$

for large enough $S \in \mathbb{R}_+^*$.

3.5.9 Mourre estimate for H

Now that we have at our disposition a Mourre estimate for H_0, we can deduce from it Mourre estimates for any operator H satisfying (3.75). Their spectral theory is closely related to that of H_0 as illustrated by the following lemma.

Lemma 3.5.12. *For any $\chi \in C_0^\infty(\mathbb{R})$, $(H_0 - i)^{-1} - (H - i)^{-1}$ and $\chi(H_0) - \chi(H)$ are compact. In particular, H_0 and H have the same essential spectrum.(Weyl's Theorem).*

Proof. One has for any $z \in \mathbb{C} \setminus \mathbb{R}$:

$$\begin{aligned}(H_0 - z)^{-1} - (H - z)^{-1} &= (H - z)^{-1}(H - H_0)(H_0 - z)^{-1}, \\ &= (H - z)^{-1}((h^2 - 1)H_0 + \tilde{V})(H_0 - z)^{-1},\end{aligned}$$

for some matrix $\tilde{V}$ whose coefficients are in $C_\infty(\mathbb{R})$. Compactness of $(H_0 - i)^{-1} - (H - i)^{-1}$ is, once more, a consequence of Corollary 3.4.2. That of $\chi(H_0) - \chi(H)$ follows from this since the Helffer-Sjöstrand formula [28] leads to:

$$\chi(H) - \chi(H_0) = \frac{i}{2\pi} \int \partial_{\bar{z}} \tilde{\chi}(z) \left((H - z)^{-1} - (H_0 - z)^{-1} \right) \mathrm{d}z \wedge \mathrm{d}\bar{z}, \tag{3.113}$$

the integral converges in norm so compactness of the integrand implies that of the integral.

An immediate consequence of Lemma 3.5.12 is that for any $\chi \in C_0^\infty(\mathbb{R})$:

$$\chi(H)[iH, A(S)]\chi(H) \underset{K}{\sim} \chi(H_0)[iH, A(S)]\chi(H_0). \tag{3.114}$$

Now, writing $H = H_0 + (h^2 - 1)H_0 + h[H_0, h] + V$, let us consider:

$$\chi(H_0)[(h^2 - 1)H_0 + h[H_0, h] + V, A_\pm(S)]\chi(H_0),$$

we will in fact find that it is compact, so that:

$$\chi(H)[iH, A]\chi(H) \underset{K}{\sim} \chi(H_0)[iH_0, A]\chi(H_0). \tag{3.115}$$

We recall our main tool:

Corollary 3.5.2. *Corollary 3.4.2, Section 3.4.*

28. see [HS87, Proposition 7.2]

$$\boxed{\textit{If } f, \chi \in C_\infty \textit{ then } f(r^*)\chi(H_0) \textit{ is compact.}}$$

To simplify notations we drop the dependence on S of the operator A_-. Consider first:

$$[(h^2-1)H_0, A_\pm] = (h^2-1)[H_0, A_\pm] - [A_\pm, h^2-1]H_0. \tag{3.116}$$

$(h^2-1) \in \boldsymbol{S}^{1,1}$ so, by Corollary 3.4.2, $(h^2-1)\chi(H_0)$ is compact. Therefore, so is: $\chi(H_0)(h^2-1) = ((h^2-1)\chi(H_0))^*$. Since $[H_0, A_\pm] \in B(D(H_0), \mathscr{H})$, we conclude that $\chi(H_0)(h^2-1)[H_0, A_\pm]\chi(H_0)$ is compact. Moreover:

$$[A_-, h^2-1] = -iR_-(r^*)2hh' \in \boldsymbol{S}^{\infty,1},$$

so $[A_-, h^2-1]\chi(H_0)$ is also compact.

Next we consider the term:

$$[A_+, h^2-1] = \Gamma^1(R_+(r^*, \mathfrak{D})(h^2-1) - ((R_+(r^*, \mathfrak{D})(h^2-1))^*).$$

Note that:

$$\begin{aligned} R_+(r^*, \mathfrak{D})(h^2-1) &= R_+(r^*, \mathfrak{D})\langle r^*\rangle^{-1}\langle r^*\rangle(h^2-1), \\ &= R_+(r^*, \mathfrak{D})\langle r^*\rangle^{-1} j_1^2(\frac{r^*}{S})\langle r^*\rangle(h^2-1). \end{aligned}$$

The last equality is a consequence of the choice of support of j_1 and j_+: recall that $j_1(t) = 1$ for $t \geq -1$ and $r^* \geq -S$ when $j_+(\frac{r^*-\kappa^{-1}\ln|\mathfrak{D}|}{S}) \neq 0$ so $j^2(\frac{r^*}{S}) = 1$ whenever the term is non-zero. $\langle r^*\rangle j_1^2(\frac{r^*}{S})(h^2-1)\chi(H_0)$ is therefore compact because $j_1^2(\frac{r^*}{S})(h^2-1) \in \boldsymbol{S}^{1,\infty}$.

Additionally, Lemma 3.5.1 implies that $R_+(r^*, \mathfrak{D})\langle r^*\rangle^{-1}$ extends to a bounded operator on $\mathscr{H}$. The compactness of $\chi(H_0)[(h^2-1)H_0, A_\pm]\chi(H_0)$ follows. The term:

$$[V, A_+] = [VR_+(r^*, \mathfrak{D})\Gamma^1 - \Gamma^1 R_+(r^*, \mathfrak{D})V],$$

is treated identically:

$$R_+(r^*, \mathfrak{D})V = R_+(r^*, \mathfrak{D})\langle r^*\rangle^{-1}\langle r^*\rangle j_1^2(\frac{r^*}{S})V,$$

and $j_1^2(\frac{r^*}{S})V \in \boldsymbol{S}^{1,\infty}$ so $R_+(r^*, \mathfrak{D})V\chi(H_0)$ is compact. Lastly using:

$$\begin{aligned}
[V, A_-] &= iR_-(r^*)V' \in \boldsymbol{S}^{\infty,1}, \\
h[H_0, h] &= -ih(\Gamma^1\partial_{r^*}h + \sqrt{\Delta_\theta}g(r^*)\Gamma^2\partial_\theta h), \\
[h[H_0, h], A_-] =& R_-(r^*)\Big[h'(\Gamma^1 h' + \Gamma^2\sqrt{\Delta_\theta}g(r^*)\partial_\theta h) + h\Gamma^1\partial_{r^*}^2 h \\
&+ h\Gamma^2\sqrt{\Delta_\theta}g(r^*)\partial_{r^*}\partial_\theta h + h\Gamma^2\sqrt{\Delta_\theta}g'(r^*)\partial_\theta h\Big] \in \boldsymbol{S}^{\infty,1}, \\
[h[H_0, h], A_+] &= -i\Gamma^1[h\partial_{r^*}h, R_+(r^*, \mathfrak{D})] - i[h\sqrt{\Delta_\theta}g(r^*)\Gamma^2\partial_\theta h, \Gamma^1 R_+(r^*, \mathfrak{D})].
\end{aligned}$$

and similar arguments as before, we conclude that the remaining terms are also compact. Therefore, we have proved the following:

Proposition 3.5.6. *Let H be an operator defined by (3.75), then the conclusion of Proposition 3.5.5 is true with H in place of H_0.*

3.5.10 Propagation estimates and other consequences of the Mourre estimate

On the spectrum of H_0 and H

The first important consequence of the estimate above is that Theorem 3.5.1 applies to H and H_0, on any interval disjoint from $\{0\}$. Hence, H and H_0 have no singular continuous spectrum and all eigenvalues, other than possibly 0, are of finite multiplicity. In fact, H_0 has no eigenvalue, as the following classical "Grönwall lemma" argument shows.

Proof that H_0 has no pure point spectrum. We only need to seek eigenvalues for H_0 on each of the subspaces $\mathscr{H}_{k,n}$, which, we recall, can be identified with $[L^2(\mathbb{R})]^4$. Let $\lambda \in \mathbb{R}$ and suppose that $u \in [L^2(\mathbb{R})]^4$ satisfies:

$$H_0^{k,n}u = \left(\lambda + \frac{ap}{r_+^2 + a^2} - \frac{ap}{r_e^2 + a^2}\right)u,$$

then $u \in [H^1(\mathbb{R})]^4$ and u vanishes at infinity. This is also true of the function $w : r^* \mapsto e^{-i\Gamma^1\lambda r^*}u(r^*)$. w additionally satisfies:

$$\begin{aligned}
w'(r^*) &= e^{-i\Gamma^1\lambda r^*}(-i\Gamma^1)(\lambda u(r^*) - \Gamma^1 D_{r^*}u(r^*)), \\
&= e^{-i\Gamma^1\lambda r^*}(-i\Gamma^1)I(r^*)e^{i\Gamma^1\lambda r^*}w(r^*),
\end{aligned}$$

where: $I(r^*) = \left(-\lambda_k g(r^*)\Gamma^2 + f(r^*) - \left(\frac{ap}{r_+^2+a^2} - \frac{ap}{r_e^2+a^2}\right)\right)$. From this, we deduce:

$$||w(r^*)|| \leq \int_{r^*}^{+\infty} \|I(r^*)\| \, ||w(r^*)|| \, \mathrm{d}r^*,$$

Because $||I||$ is integrable near $+\infty$, it follows from the integral form of Grönwall's lemma that $w = 0$ and therefore $u = 0$.

Using the separability of the Dirac equation in Kerr-de Sitter, a modified version of this argument shows that the full Dirac operator has no eigenvalues, we refer to [BC09]. We summarise these conclusions in the following lemma:

Lemma 3.5.13. *Let H be an operator defined by (3.75) then:*

— *H has no singular continuous spectrum,*

— $\sigma_{ess}(H) = \mathbb{R}$,

— *$\sigma_{pp}(H) \subset \{0\}$ and if 0 is an eigenvalue then it has infinite multiplicity.* [29]

Strict Mourre estimates

Let $H \in C^1(A)$, (H, A) is said to satisfy a *strict* Mourre estimate on some interval $I \subset \mathbb{R}$, if it satisfies a Mourre estimate with vanishing compact error. This slightly stricter condition will be required shortly for the important conclusion of Theorem 3.5.5. Nevertheless, if (H, A) satisfies a Mourre estimate on some open interval $I \subset \mathbb{R}$, then for any $\lambda \in I$ that is *not* an eigenvalue of H, one can find a small neighbourhood $J = (-\varepsilon+\lambda, \lambda-\varepsilon)$ of $\lambda \in I$ such that it satisfies a strict Mourre estimate on J. To see this we give a simplified version of the argument in the proof of [AMG96, Lemma 7.2.12]: let, for any n large enough such that $(-\frac{1}{n}+\lambda, \lambda+\frac{1}{n}) \subset I$, $E_n = \mathbb{E}((-\frac{1}{n}+\lambda, \lambda+\frac{1}{n}))$; where $\mathbb{E}$ is the spectral measure of H. Then:

$$\operatorname*{s-lim}_{n\to\infty} E_n = \mathbb{E}(\{\lambda\}) = 0,$$

as λ is not an eigenvalue. It follows that for any compact operator K:

$$\lim_{n\to\infty} E_n K E_n = 0.$$

29. $\sigma_{\mathrm{pp}}(H)$, the *pure-point* spectrum, is the set of all eigenvalues of H. It is not to be confused with the discrete spectrum, $\sigma_{\mathrm{disc}}(H) = \mathbb{R} \setminus \sigma_{\mathrm{ess}}(H)$, the set of all isolated eigenvalues with finite multiplicity.

Therefore, if $\varepsilon > 0$, one can find N, such that for any $n \geq N$:

$$|(E_n K E_n|x) \leq \varepsilon||x||^2,$$

so that for $n \geq N$:

$$E_n K E_n \geq -\varepsilon \Rightarrow E_n K E_n \geq -\varepsilon E_n.$$

Hence, if $\mathbf{1}_I(H)i[H,A]\mathbf{1}_I(H) \geq \mu + K$, then:

$$E_n i[H,A]E_n \geq (\mu - \varepsilon)E_n.$$

Consequently, on small enough intervals around any non-eigenvalue, one has a strict Mourre estimate for any $\nu \in (0,\mu)$.

In the case of H and H_0, the only possible eigenvalue is 0. All our estimates avoid this point, therefore they can all be upgraded to strict estimates on small enough intervals around any point of $\mathbb{R}^*$.

Minimal velocity estimate

One of the most powerful consequences of the hypotheses of Mourre theory, largely discussed and optimised in [AMG96], is that it leads to a (generalised) limiting absorption principle. In our case, thanks to Proposition 3.5.2, $H_0, H \in C^2(A)$, and we directly have access to an abstract propagation estimate due to Sigal-Soffer [SS88]:

Theorem 3.5.5. *Let (H,A) be a pair of self-adjoint operators on a Hilbert space $\mathscr{H}$. Suppose that A is a conjugate operator for H on $I \subset \mathbb{R}$ and that $H \in C^{1+\varepsilon}(A), (\varepsilon \in \mathbb{R}_+^*)$. Let $\mu \in \mathbb{R}_+^*$ be such that:*

$$\mathbf{1}_I(H)i[H,A]\mathbf{1}_I(H) \geq \mu\mathbf{1}_I(H).$$

Then, for any $b, \chi \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\chi \subset I$ and $\operatorname{supp} b \subset (-\infty,\mu)$ one has:

$$\begin{aligned} &\forall u \in \mathscr{H}, \int_1^{+\infty} \left\| b\left(\frac{A}{t}\right)\chi(H)e^{-iHt}u\right\|^2 \frac{dt}{t} \leq C||u||^2, \\ &\underset{t\to+\infty}{\mathrm{s-lim}}\, b\left(\frac{A}{t}\right)\chi(H)e^{-iHt} = 0. \end{aligned} \tag{3.117}$$

The importance of Theorem 3.5.5 is more obvious when the conjugate operator can be replaced by simpler operators that help to understand the propagation of fields. In [Dau10, Lemma IV.4.13], it is shown that in the case of the operators under consideration here, A can be replaced with $|r^*|$, and we obtain:

Proposition 3.5.7. *Let $\chi \in C_0^\infty(\mathbb{R})$ such that* $\operatorname{supp}\chi \cap \{0\} = \emptyset$*, then for any H defined by Equation (3.75), there are $\varepsilon_\chi, C \in \mathbb{R}_+^*$ such that for any $\psi \in \mathscr{H}$:*

$$\int_1^\infty \left\| \mathbf{1}_{[0,\varepsilon_\chi]}\left(\frac{|r^*|}{t}\right)\chi(H)e^{-itH}\psi\right\|^2 \frac{dt}{t} \leq C||\psi||^2. \tag{3.118}$$

Furthermore:

$$\operatorname*{s-lim}_{t\to+\infty} \mathbf{1}_{[0,\varepsilon_\chi]}\left(\frac{|r^*|}{t}\right)\chi(H)e^{-itH} = 0. \tag{3.119}$$

This "minimal velocity estimate" means that, given a certain energy interval, all fields with energy in that interval, must be outside of the "cone" $\{|r^*| < \varepsilon_I t\}$ at late times; it will be crucial to the construction of the wave operators.

Maximal velocity estimate

Independently of Mourre theory, one can show that we also have a natural "maximal velocity estimate", that is a consequence of the geometry:

Proposition 3.5.8. *Let $\delta \in (0,1), b \in C_0^\infty(\mathbb{R})$ be such that* $\operatorname{supp} b \cap [-1-\delta, 1+\delta] = \emptyset$*, then there is some constant $C \in \mathbb{R}^*$ such that for any $u \in \mathscr{H}$:*

$$\int_1^{+\infty} \left\| b(\frac{r^*}{t})e^{-itH}u\right\|^2 \frac{dt}{t} \leq C||u||^2. \tag{3.120}$$

Furthermore, for any $b \in C^\infty(\mathbb{R})$ such that $b \equiv 0$ on $[-1-\delta, 1+\delta]$ and $b = 1$ for $|r|$ large, then:

$$\operatorname*{s-lim}_{t\to\infty} b(\frac{r^*}{t})e^{-itH} = 0 \tag{3.121}$$

The proof is identical to that of [Dau04, Proposition IV.4.4].

What of $t \to -\infty$?

Up to now, we have only discussed estimates in the far future, and have said nothing of the far past. After thorough inspection, one can convince onself that all the results here hold for $-H$ (the conjugate operator should also be replaced by its opposite), but, there is a faster way to see this. The Kerr-de Sitter metric (3.5) is invariant under the simultaneous substitutions:

$$t \to -t \quad a \to -a.$$

This is intuitively reasonable because a time reversed black-hole will rotate in the opposite way. Consequently, all the results in the section have suitable analogs at $t \to -\infty$.

3.6 Intermediate wave operators

3.6.1 Overall strategy

In this section our goal is to show that, despite the long-range non-spherically symmetric potentials at the double horizon, it is still possible to reduce the scattering problem to a 1-dimensional one. To this end, we introduce the following operators:

$$H_1 = H_0 + h^{-1} V_C h^{-1}, \tag{3.122}$$

$$H_e = H_0 + g(r^*)\vartheta(\theta), \tag{3.123}$$

with:

$$\vartheta(\theta) = \frac{a^2 \sin\theta}{\sqrt{\Delta_\theta}} \left(\frac{l^2 r_e^2 - 1}{r_e^2 + a^2} \right) \Gamma^3 p + \rho_e m \Gamma^0 - \frac{a \sin\theta r_e}{2\rho_e^2} \sqrt{\Delta_\theta} \tilde{\gamma}, \tag{3.124}$$

$$\rho_e = r_e^2 + a^2 \cos^2\theta, \quad \tilde{\gamma} = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}. \tag{3.125}$$

Finally, V_C and V_S are defined by equations (3.61) and (3.62), their asymptotic behaviour is described in Lemma 3.5.3.

Both H_1 and H_e are of the prescribed form (3.75), hence the theory presented in Section 3.5 applies to them. We will show that we can compare the full operator $H \equiv H^p = hH_0h + V_S + V_C$ to simplified dynamics as so:

$$H \underset{r^* \to \pm\infty}{\longrightarrow} \begin{cases} H_1 \underset{r^* \to +\infty}{\longrightarrow} H_0 \\ H_1 \underset{r^* \to -\infty}{\longrightarrow} H_e \end{cases}$$

3.6.2 First comparison

The first step is to compare H to H_1. Here, there is no distinction between the behaviour at the different horizons because:

$$\begin{aligned} H - H_1 &= (h^2-1)H_0 + h[H_0,h] + V_S + (h^2-1)h^{-2}V_C \\ &\equiv \underbrace{(h^2-1)}_{\in S^{2,2}}H_1 + V_S + h[H_0,h], \end{aligned}$$

and $V_S + h[H_0,h] \equiv \tilde{V}_S$ is short-range. Proposition 3.5.7 is the key to prove:

Proposition 3.6.1. *The generalised wave-operators:*

$$\begin{aligned} \Omega^1_\pm &= \mathrm{s-}\lim_{t\to\pm\infty} e^{itH_1}e^{-itH}P_c(H), \\ \tilde{\Omega}^1_\pm &= \mathrm{s-}\lim_{t\to\pm\infty} e^{itH}e^{-itH_1}P_c(H_1), \end{aligned} \tag{3.126}$$

exist, where, for any self-adjoint operator B, $P_c(B)$ denotes the projection onto the absolutely continuous subspace of B.

Proof. We show the existence of the first limit at $t\to+\infty$ the other cases are similar. We begin by remarking that:

$$\overline{\bigcup_{\substack{\chi\in C_0^\infty(\mathbb{R})\\ \operatorname{supp}\chi\cap\{0\}=\emptyset}} \chi(H)\mathscr{H}} = P_c(H)\mathscr{H},$$

so it is sufficient to prove the existence of the limit:

$$\mathrm{s-}\lim_{t\to+\infty} e^{itH_1}\chi(H)e^{-itH},$$

for every $\chi\in C_0^\infty(\mathbb{R}), \operatorname{supp}\chi\cap\{0\}=\emptyset$. Consider then such a χ and let ε_χ be defined by Proposition 3.5.7. Choose $j_0\in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} j_0 \subset (-\varepsilon_\chi,\varepsilon_\chi)$ and $j\equiv 1$ on a neighbourhood of 0. Set $j = 1-j_0$. (3.119) implies that:

$$\mathrm{s-}\lim_{t\to\infty} e^{itH_1}j_0(\frac{r^*}{t})e^{-itH}\chi(H) = 0.$$

It remains to prove the existence of:

$$\mathrm{s-}\lim_{t\to\infty} e^{itH_1}j(\frac{r^*}{t})\chi(H)e^{-itH}.$$

For this, we apply the methods of Cook and Kato [30], who remarked that the convergence, for every u in a dense set of $\mathscr{H}$, of the integral:

$$\int_1^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{itH_1} j(\frac{r^*}{t})\chi(H)e^{-itH}u\right),$$

was a sufficient condition for the limit to exist. To prove the convergence of the integral, there are two model arguments that will both be illustrated on this simple example. To begin with, let $u \in D(H) = D(H_1)$ then $\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{itH_1} j(\frac{r^*}{t})\chi(H)e^{-itH}\right)u$ equates to:

$$\begin{aligned} &e^{itH_1}\left(iH_1 j(\frac{r^*}{t}) - \frac{r^*}{t^2} j'(\frac{r^*}{t}) - j(\frac{r^*}{t})iH\right)\chi(H)e^{-itH}u \\ &= e^{itH_1}\left(ij(\frac{r^*}{t})(H_1 - H) + \frac{1}{t}(\Gamma^1 - \frac{r^*}{t})j'(\frac{r^*}{t})\right)\chi(H)e^{-itH}. \end{aligned}$$

The treatment of the first term, illustrates the first type of argument. Consider first:

$$H_1 - H = (h^2 - 1)H_1 + \tilde{V}_S.$$

On $\operatorname{supp} j$, one must have $|r^*| \geq \varepsilon t$ for some $\varepsilon \in \mathbb{R}_+^*$, thus, $\frac{1}{|r^*|} \leq \frac{1}{\varepsilon t}$ on $\operatorname{supp} j$. Consequently, $j(\frac{r^*}{t})(h^2-1) = O(t^{-2})$ and $j(\frac{r^*}{t})\tilde{V}_S = O(t^{-2})$. Because $H_1\chi(H)$ is bounded [31], the term:

$$e^{itH_1} j(\frac{r^*}{t})(H_1 - H)\chi(H)e^{-itH}u,$$

is therefore integrable.

The final term, $e^{itH_1}\frac{1}{t}\left(\Gamma^1 - \frac{r^*}{t}\right)j'(\frac{r^*}{t})\chi(H)e^{-itH}u$, that is not clearly integrable in the sense of Lebesgue, requires a different treatment, which will serve as illustration for the second type of argument we use. Lebesgue integrability is in fact sufficient, but not necessary; the key to Cook's argument is simply that for any ε and any t_1, t_2 sufficiently large:

$$\left\| e^{it_2H_1} j(\frac{r^*}{t_2})\chi(H)e^{-it_2H} - e^{it_1H_1} j(\frac{r^*}{t_1})\chi(H)e^{-it_1H} \right\| < \varepsilon.$$

30. see for example [DG97; Lax02]
31. H_1 is continuous on $D(H)$

Moreover, by the Hahn-Banach theorem, there is $v \in \mathscr{H}, ||v|| \le 1$ such that:

$$\begin{aligned} &||e^{it_2H_1}j(\frac{r^*}{t_2})\chi(H)e^{-it_2H}u - e^{it_1H_1}j(\frac{r^*}{t_1})\chi(H)e^{-it_1H}u|| \\ &\qquad = (v, e^{it_2H_1}j(\frac{r^*}{t_2})\chi(H)e^{-it_2H}u - e^{it_1H_1}j(\frac{r^*}{t_1})\chi(H)e^{-it_1H}u), \\ &\qquad = \int_{t_1}^{t_2}\left(v, \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{itH_1}j(\frac{r^*}{t})\chi(H)e^{-itH}u\right)\right)\mathrm{d}t. \end{aligned}$$

So, one only needs to verify that for t_1, t_2 sufficiently large the integral:

$$\int_{t_1}^{t_2}\left(v, \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{itH_1}j(\frac{r^*}{t})\chi(H)e^{-itH}u\right)\right)\mathrm{d}t,$$

can be made arbitrarily small. Choose now $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\tilde{\chi} \cap \{0\} = \emptyset$ and $\tilde{\chi}\chi = \chi$, $\tilde{j} \in C_0^\infty(\mathbb{R})$, that vanishes on a neighbourhood of zero and satisfies $\tilde{j}j' = j'$. Notice then that:

$$\begin{aligned} \frac{1}{t}(\Gamma^1 - \frac{r^*}{t})j'(\frac{r^*}{t})\chi(H) = &\tilde{\chi}(H_1)\tilde{j}(\frac{r^*}{t})\frac{1}{t}(\Gamma^1 - \frac{r^*}{t})\tilde{j}(\frac{r^*}{t})j'(\frac{r^*}{t})\chi(H) \\ &+ \frac{1}{t}(\Gamma^1 - \frac{r^*}{t})\tilde{j}(\frac{r^*}{t})j'(\frac{r^*}{t})(\tilde{\chi}(H) - \tilde{\chi}(H_1))\chi(H) \\ &+ \frac{1}{t}[(\Gamma^1 - \frac{r^*}{t}), \tilde{\chi}(H_1)]j'(\frac{r^*}{t})\chi(H) \\ &+ \frac{1}{t}(\Gamma^1 - \frac{r^*}{t})[j'(\frac{r^*}{t}), \tilde{\chi}(H_1)]\chi(H). \end{aligned}$$

The last three terms are $O(t^{-2})$ so are integrable, this is not changed by multiplying to the left with e^{itH_1} and to the right with e^{-itH}. Now, for any $v \in \mathscr{H}$, one certainly has:

$$\begin{aligned} &|(v, e^{itH_1}\frac{1}{t}\tilde{\chi}(H_1)\tilde{j}(\frac{r^*}{t})(\Gamma^1 - \frac{r^*}{t})\tilde{j}(\frac{r^*}{t})j'(\frac{r^*}{t})\chi(H)e^{-itH}u)| \\ &= \left|\left(\frac{1}{\sqrt{t}}\tilde{j}(\frac{r^*}{t})(\Gamma^1 - \frac{r^*}{t})\tilde{j}(\frac{r^*}{t})\tilde{\chi}(H_1)e^{-itH_1}v, \frac{1}{\sqrt{t}}j'(\frac{r^*}{t})\chi(H)e^{-itH}u\right)\right|, \\ &\qquad \le K\left\|\frac{1}{\sqrt{t}}\tilde{j}(\frac{r^*}{t})\tilde{\chi}(H_1)e^{-itH_1}v\right\|\left\|\frac{1}{\sqrt{t}}j'(\frac{r^*}{t})\chi(H)e^{-itH}u\right\|, \end{aligned}$$

for some $K \in \mathbb{R}_+^*$. In the above we have used the fact that:

$$\tilde{j}(\frac{r^*}{t})(\Gamma^1 - \frac{r^*}{t}) \in B(\mathscr{H}).$$

Applying the Cauchy-Schwarz inequality, we get the following estimate:

$$\int_{t_1}^{t_2} \left|(v, e^{itH_1}\frac{1}{t}\Gamma^1\tilde{\chi}(H_1)\tilde{j}(\frac{r^*}{t})j'(\frac{r^*}{t})\chi(H)e^{-itH}u)\right| \mathrm{d}t$$
$$\leq K\left(\int_{t_1}^{t_2}\left\|\tilde{j}(\frac{r^*}{t})\tilde{\chi}(H_1)e^{-itH_1}v\right\|^2\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}\left(\int_{t_1}^{t_2}\left\|j'(\frac{r^*}{t})\chi(H)e^{-itH}u\right\|^2\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}.$$

However, it follows from Proposition 3.5.7 that there is some constant $C \in \mathbb{R}_+^*$ such that:

$$\left(\int_{t_1}^{t_2}\left\|\tilde{j}(\frac{r^*}{t})\tilde{\chi}(H_1)e^{-itH_1}v\right\|^2\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}\left(\int_{t_1}^{t_2}\left\|j'(\frac{r^*}{t})\chi(H)e^{-itH}u\right\|^2\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}$$
$$\leq C||v||\left(\int_{t_1}^{t_2}\left\|j'(\frac{r^*}{t})\chi(H)e^{-itH}u\right\|^2\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}},$$
$$\leq C\left(\int_{t_1}^{t_2}\left\|j'(\frac{r^*}{t})\chi(H)e^{-itH}u\right\|^2\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}.$$

In the last inequality we have specialised to the case where $||v|| \leq 1$. This quantity can be made arbitrarily small, for large enough t_1, t_2, again by Proposition 3.5.7. The existence of the limit then follows.

3.6.3 Second comparison

Our aim now is to show that asymptotically the dynamics of H_1 can again be simplified. However, the comparisons we will make in this section depend on the asymptotic region we consider. We will separate incoming and outgoing states using cut-off functions, $c_\pm$, that are assumed to satisfy: $c_\pm \in C^\infty(\mathbb{R})$, $c_\pm \equiv 1$ in a neighbourhood of $\pm\infty$ and that vanish in a neighbourhood of $\mp\infty$. We then seek to show that the following limits exist:

$$\begin{aligned}
\Omega^2_{\pm,\mathscr{H}_{r_+}} &= \mathrm{s}-\lim_{t\to\pm\infty} e^{iH_0t}c_+(r^*)e^{-iH_1t}P_c(H_1),\\
\tilde{\Omega}^2_{\pm,\mathscr{H}_{r_+}} &= \mathrm{s}-\lim_{t\to\pm\infty} e^{iH_1t}c_+(r^*)e^{-iH_0t},\\
\Omega^2_{\pm,\mathscr{H}_{r_e}} &= \mathrm{s}-\lim_{t\to\pm\infty} e^{iH_et}c_-(r^*)e^{-iH_1t}P_c(H_1),\\
\tilde{\Omega}^2_{\pm,\mathscr{H}_{r_e}} &= \mathrm{s}-\lim_{t\to\pm\infty} e^{iH_1t}c_-(r^*)e^{-iH_et}P_c(H_e).
\end{aligned} \tag{3.127}$$

This appears to introduce a certain arbitrariness into the construction, the following lemma shows that this is not the case:

Lemma 3.6.1. *If the limits (3.127) exist, then they are independent of the choice of cut-off functions $c_\pm$.*

Proof. The main point is that two such functions can differ on at most a compact set, i.e. their difference is an element of $C_0^\infty(\mathbb{R})$. So let us prove that if $c \in C_0^\infty(\mathbb{R})$, then, for instance:

$$\underset{t\to+\infty}{\mathrm{s}-\lim}\, e^{iH_0 t} c(r^*) e^{-iH_1 t} P_c(H_1) = 0,$$

the other cases will be similar. As before, by density, we only need to prove that:

$$\underset{t\to+\infty}{\mathrm{s}-\lim}\, e^{iH_0 t} c(r^*) \chi(H_1) e^{-iH_1 t} = 0,$$

for any $\chi \in C_0^\infty(\mathbb{R}), \operatorname{supp}\chi \cap \{0\} \ = \emptyset$.

Let χ be as so and let $M \in \mathbb{R}_+^*$ be such that $\operatorname{supp} c \subset [-M, M]$. Choose $j_0 \in C_0^\infty(\mathbb{R})$ with support contained in $(-\varepsilon_\chi, \varepsilon_\chi)$ such that, say, $j_0(s) = 1$ for any $s \in [-\frac{\varepsilon_\chi}{2}, \frac{\varepsilon_\chi}{2}]$. Then, for any $t \geq 1$, $j_0(\frac{r^*}{t}) = 1$ for any $|r^*| \leq \frac{\varepsilon_\chi}{2} t$. Hence, for $t \geq \frac{2M}{\varepsilon_\chi}$,

$$c(r^*) = c(r^*) j_0(\frac{r^*}{t}), \text{ for any } r^* \in \mathbb{R}.$$

It follows that:

$$\underset{t\to+\infty}{\mathrm{s}-\lim}\, e^{iH_0 t} c(r^*) \chi(H_1) e^{-iH_1 t} = \underset{t\to+\infty}{\mathrm{s}-\lim}\, e^{iH_0 t} c(r^*) j_0(\frac{r^*}{t}) \chi(H_1) e^{-iH_1 t},$$

which vanishes by Proposition 3.5.7.

We now argue that the limits (3.127) exist, with emphasis on:

$$\underset{t\to+\infty}{\mathrm{s}-\lim}\, e^{iH_e t} c_-(r^*) e^{-iH_1 t} P_c(H_1), \tag{3.128}$$

the other cases being similar.

Lemma 3.6.2. *$H_1 - H_e$ is short-range near the double horizon.*

Proof. Note that:

$$h^{-2} V_C = g \underbrace{\left(\frac{\Xi}{\sin\theta} \left(\frac{\rho^2}{\sigma} - \frac{\sqrt{\Delta_\theta}}{\Xi} \right) \Gamma^3 p + \rho m \Gamma^0 - \frac{a \sin\theta r}{2\rho^2} \sqrt{\Delta_\theta \tilde{\gamma}} \right)}_{\Theta(r,\theta)}, \tag{3.129}$$

and $\Theta(r_e,\theta)=\vartheta(\theta)$. Thus, $\Theta(r,\theta)-\vartheta(\theta)=\underset{r\to r_e}{o}(r-r_e)=\underset{r^*\to-\infty}{o}(r^{*-1})$, which leads to:

$$h^{-2}V_C-g\vartheta(\theta)=\underset{r^*\to\infty}{O}\left(\frac{1}{r^{*2}}\right).$$

Proof of the existence of (3.128). As before, we only need to prove the existence of:

$$\underset{t\to+\infty}{\mathrm{s-lim}}\, e^{iH_e t}c_-(r^*)\chi(H_1)e^{-iH_1 t},$$

for any $\chi\in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\chi\cap\{0\}=\emptyset$.

Let χ be as so, and j_0, j be as in the proof of Proposition 3.6.1, then:

$$\underset{t\to+\infty}{\mathrm{s-lim}}\, e^{iH_e t}c_-(r^*)j_0\left(\frac{r^*}{t}\right)\chi(H_1)e^{-iH_1 t}=0,$$

and we must prove the existence of $\underset{t\to+\infty}{\mathrm{s-lim}}\, e^{iH_e t}c_-(r^*)j\left(\frac{r^*}{t}\right)\chi(H_1)e^{-iH_1 t}$. To simplify notations, set $M(t)=e^{iH_e t}c_-(r^*)j(\frac{r^*}{t})\chi(H_1)e^{-iH_1 t}$, its derivative, $M'(t)$, is given by:

$$e^{iH_e t}\left(iH_e c_-(r^*)j\left(\frac{r^*}{t}\right)-\frac{r^*}{t^2}c_-(r^*)j'\left(\frac{r^*}{t}\right)-c_-(r^*)j\left(\frac{r^*}{t}\right)iH_1\right)\chi(H_1)e^{-iH_1 t}.$$

The term between parentheses is:

$$\begin{aligned}&c_-(r^*)j\left(\frac{r^*}{t}\right)i(H_e-H_1)+\Gamma^1\left(c_-(r^*)j\left(\frac{r^*}{t}\right)\right)'-\frac{r^*}{t^2}c_-(r^*)j'\left(\frac{r^*}{t}\right)\\ &=c_-(r^*)j\left(\frac{r^*}{t}\right)i(H_e-H_1)+\Gamma^1\left(c_-'(r^*)j\left(\frac{r^*}{t}\right)\right)+\frac{1}{t}c_-(r^*)\left(\Gamma^1-\frac{r^*}{t}\right)j'\left(\frac{r^*}{t}\right).\end{aligned}$$

The only new term compared with the proof of Proposition 3.6.1 is:

$$\Gamma^1\left(c_-'(r^*)j\left(\frac{r^*}{t}\right)\right),$$

however this vanishes when t is sufficiently large because c' has compact support and j vanishes on a neighbourhood of 0. Moreover, since H_e-H_1 is short-range near the double horizon and c_- vanishes on a neighbourhood of $+\infty$, the first two terms are $O(t^{-2})$ and hence integrable. The last term is treated as at the end of the proof of Proposition 3.6.1.

3.6.4 The operator H_e

The expression of H_e suggests that we seek to understand the precise spectral theory of the operator, defined on the sphere by:

$$\mathfrak{D}_e = \mathfrak{D} + \vartheta(\theta). \tag{3.130}$$

In particular, we would like to show that there is a Hilbert space decomposition of $L^2(S^2)\otimes\mathbb{C}^4$ which enables us to decompose the full Hilbert space $\mathscr{H}$ into an orthogonal sum of stable subspaces, that can be used to study H_e. Since $\vartheta(\theta)$ is a bounded operator it is an immediate consequence of the Kato-Rellich perturbation theorem that $\mathfrak{D}_e$ has compact resolvent. However, we require a slightly more thorough understanding of the structure of the spectral subspaces and in particular how Γ^1 acts on them.

Dimension of spectral subspaces

Decompose $L^2(S^2)\otimes\mathbb{C}^4$ in the usual manner by diagonalising D_φ with anti-periodic boundary conditions, and consider the restriction $\mathfrak{D}_e^n$ of $\mathfrak{D}_e$ to the subspace with eigenvalue $n\in\mathbb{Z}+\frac{1}{2}$. In the following E_λ will denote the spectral subspace of $\lambda\in\mathbb{R}$ for this restricted operator.

An element f in this subspace is an eigenvector with eigenvalue $\lambda\in\mathbb{R}$ of $\mathfrak{D}_e^n$ if and only if it is a solution to the first order ordinary differential equation:

$$\sqrt{\Delta_\theta}\Gamma^2 D_\theta f - \frac{i}{2}\left(\frac{\Delta_\theta'}{2\sqrt{\Delta_\theta}} + \cot\theta\right)\Gamma^2 f - ar_e\sin\theta\frac{\sqrt{\Delta_\theta}}{2\rho_e^2}\tilde{\gamma}f \\ + \left(\frac{\sqrt{\Delta_\theta}}{\sin\theta}n + \frac{a^2\sin\theta}{\sqrt{\Delta_\theta}}\frac{l^2r_e^2-1}{r_e^2+a^2}p\right)\Gamma^3 f + \rho_e m\Gamma^0 f - \lambda f = 0. \tag{3.131}$$

Note that since Γ^1 anti-commutes with $\Gamma^0,\Gamma^3,\Gamma^2$ and $\tilde{\gamma}$, if f is a solution to (3.131) then $\Gamma^1 f$ is a solution to the analogous equation for $-\lambda$, in fact, Γ^1 is an isometry between E_λ and $E_{-\lambda}$. The study of (3.131) is slightly easier after the substitution $z=\cos\,\theta$, after which we obtain:

$$a_1(z)\Gamma^2 D_z + a_2(z)\Gamma^2 f + a_3(z)\Gamma^3 f + a_5(z)\tilde{\gamma}f + a_0(z)\Gamma^0 f - \lambda f = 0, \tag{3.132}$$

where:

$$
\begin{aligned}
a_0(z) &= \rho_e m, \quad a_1(z) = -\sqrt{\Delta_\theta}\sqrt{1-z^2},\\
a_2(z) &= -\frac{i}{2}\left(-a^2l^2\frac{z\sqrt{1-z^2}}{\sqrt{\Delta_\theta}} + \frac{z}{\sqrt{1-z^2}}\right),\\
a_3(z) &= \left(\frac{\sqrt{\Delta_\theta}}{\sqrt{1-z^2}}n + \frac{a^2\sqrt{1-z^2}}{\sqrt{\Delta_\theta}}\frac{l^2r_e^2-1}{r_e^2+a^2}p\right),\\
a_5(z) &= -a\sqrt{1-z^2}\frac{\sqrt{\Delta_\theta}}{2\rho_e^2}.
\end{aligned}
\tag{3.133}
$$

Save the expressions $\sqrt{1-z^2}$, $\frac{1}{\sqrt{1-z^2}}$, all other functions appearing in the coefficients (3.133) of the equation can be extended to analytic functions on a disc centered in 0 and with radius $1+\varepsilon$ for some $\varepsilon > 0$, the reason for this is that the parameters satisfy: $|al| < 2-\sqrt{3} < 1$ and $r_e > |a|$. This suggests that (3.132) extends naturally to a differential equation expressed on an open subset of the 1-dimension complex manifold $\mathcal{S}$:

$$
\mathcal{S} = \{(z,w)\in\mathbb{C}^2, z\in B(0,1+\varepsilon), z^2+w^2=1\},
$$

where z is used as local coordinate - the implicit function theorem implies that this can be done in a neighbourhood of any point in $\mathcal{S}$ save $(1,0),(-1,0)$. The functions z, w are globally defined and holomorphic on $\mathcal{S}$ and (3.132) can be rewritten:

$$
\begin{aligned}
&-\sqrt{\Delta_\theta}w\Gamma^2D_zf - \frac{i}{2}\left(-a^2l^2\frac{zw}{\sqrt{\Delta_\theta}} + \frac{z}{w}\right)\Gamma^2 f\\
&\quad + \left(\frac{\sqrt{\Delta_\theta}}{w}n + \frac{a^2w}{\sqrt{\Delta_\theta}}\frac{l^2r_e^2-1}{r_e^2+a^2}p\right)\Gamma^3 f - aw\frac{\sqrt{\Delta_\theta}}{2\rho_e^2}\tilde{\gamma}f + \rho_e m\Gamma^0 f - \lambda f = 0.
\end{aligned}
\tag{3.134}
$$

By the Cauchy-Lipschitz theorem the set of solutions to Equation (3.134) on $\mathcal{S}\setminus\{(1,0),(-1,0)\}$ is a 4-dimensional vector space. The solutions to (3.132) will be the restrictions to $]-1,1[$, (i.e. $z\in]-1,1[, w>0$) of those of (3.134). Amongst these, we must pick out those in $L^2]-1,1[$. Since $\mathfrak{D}_e$ has compact resolvent we already know that they exist only for a countable number of values of λ. We will not seek the exact condition for this, but, a simple analysis of the behaviour of the solutions near a point where $w=0$ will enable us to see that the subspace of $L^2]-1,1[$ solutions is at most of dimension 2. To this end, we switch to local coordinates defined around such a point, say, $(-1,0)$. In fact, again using the Implicit Function Theorem, one can choose w as local coordinate on a neighbourhood of $(-1,0)$, the equation then becomes:

$$\sqrt{\Delta_\theta} z\Gamma^2 D_w f - \frac{i}{2}\left(-a^2l^2\frac{zw}{\sqrt{\Delta_\theta}} + \frac{z}{w}\right)\Gamma^2 f + \left(\frac{\sqrt{\Delta_\theta}}{w}n + \frac{a^2 w}{\sqrt{\Delta_\theta}}\frac{l^2r_e^2-1}{r_e^2+a^2}p\right)\Gamma^3 f \\ - aw\frac{\sqrt{\Delta_\theta}}{2\rho_e^2}\tilde{\gamma} f + \rho_e m\Gamma^0 f - \lambda f = 0. \quad (3.135)$$

(3.135) has a singular-regular point at $w = 0$ [32], hence, one can apply the Frobenius method, i.e. there are solutions of the form $f(w) = w^\alpha \sum_k a_k w^k$. Plugging this anstaz into (3.135) we find that a_0 must be in the null space of the map:

$$M(\alpha) = i(\alpha + \frac{1}{2})\Gamma^2 + n\Gamma^3. \quad (3.136)$$

The kernel is non-trivial only if α satisfies:

$$(\alpha - n + \frac{1}{2})^2(\alpha + n + \frac{1}{2})^2 = 0. \quad (3.137)$$

For each solution to (3.137), the kernel of $M(\alpha)$ is of dimension 2, and so one can generate two linearly independent solutions for each α [33]. Only $\alpha = |n| - \frac{1}{2}$ can yield square integrable solutions to (3.132), thus it follows that:

Lemma 3.6.3. *In the notations of this paragraph, if $n \in \mathbb{Z} + \frac{1}{2}$ and $\lambda \in \sigma(\mathfrak{D}_e^n)$, then* $\dim E_\lambda \leq 2$.

We now complete the proof of Lemma 3.4.3 ; the eigenequation $\tilde{S}\psi_{k,n} = \lambda_k \psi_{k,n}$ is the special case of (3.132), where $r_e = p = m = 0$. In this case, the equation has another symmetry that amounts to saying that Γ^2 and Γ^3 anti-commute with the matrix $P = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$. Hence, P, like Γ^1, is an isometry of E_λ onto $E_{-\lambda}$, however for any $u \in \mathbb{C}^4 \setminus \{0\}$, Pu and $\Gamma^1 u$ are linearly independent, so that we must have equality in Lemma 3.6.3. The form of the solutions follows from the block diagonal form of the equations.

32. see [Inc56]
33. Note that, since the roots of (3.137) differ by a positive integer, the anstaz will need to be modified to include possible logarithmic terms in the solution when $\alpha = -|n| - \frac{1}{2}$

A reduction of H_e

Denote now:

— $\sigma(\mathfrak{D}_e) \cup \{0\} = (\mu_k)_{k\in\mathbb{Z}}$, enumerated such that $\mu_{-k} = -\mu_k$, for each $k \in \mathbb{Z}$.

— For each $k \in \mathbb{Z}$, $J(k)$ the set of integers $q \in \mathbb{Z}$ such that μ_k is an eigenvalue for $\mathfrak{D}_e^{q+\frac{1}{2}}$; note that also $J(k) = J(-k)$.

— If $k \in \mathbb{Z}, q \in J(k)$, $E_{k,q}$ the spectral subspace of the eigenvalue μ_k of $\mathfrak{D}_e^{q+\frac{1}{2}}$. By convention, if $0 \notin \sigma(\mathfrak{D}_e)$, we set $J(0) = \{0\}$ and $E_{0,0} = \{0\}$.

— For each $k \in \mathbb{N}^*$ and fixed $q \in J(k)$, $\tilde{E}_{k,q} = L^2(\mathbb{R}) \otimes (E_{k,q} \overset{\perp}{\oplus} E_{-k,q})$.

— $\tilde{E}_{0,q} = L^2(\mathbb{R}) \otimes E_{0,q}, q \ \in J(0)$

The subspaces $\tilde{E}_{k,q}$ are, by construction, stable under the action of H_e and:

$$\mathscr{H} = \bigoplus_{k\in\mathbb{N}, q\in J(k)} \tilde{E}_{k,q}.$$

Now, let $k \in \mathbb{N}^*, q \in J(k)$, if $(e_i)_{i\in[\![1,\dim E_{k,q}]\!]}$ is an orthonormal basis for $E_{k,q}$, then $(\Gamma^1 e_i)_{i\in[\![1,\dim E_{k,q}]\!]}$ is an orthonormal basis of $E_{-k,q}$ and so, since $E_{k,q}$ and $E_{-k,q}$ are orthogonal, one can concatenate these two bases to obtain an orthonormal basis $E_{k,q} \oplus E_{-k,q}$. This enables us to identify, isometrically, $\tilde{E}_{k,q}$ with $[L^2(\mathbb{R})]^{2\dim E_{k,q}}$ via the natural isomorphism:

$$((u_i)_{i\in[\![1,\dim E_{k,q}]\!]}, (v_i)_{i\in[\![1,\dim E_{k,q}]\!]}) \longmapsto \sum_{i=1}^{\dim E_{k,q}} (u_i + v_i\Gamma^1)e_i.$$

Through this isomorphism, the restriction, $H_e^{q,n}$ of H_e to $\tilde{E}_{k,q}$ corresponds to the following operator:

$$\Gamma D_r^* + \mu_k g(r^*)\tilde{\Gamma} + f(r^*),$$

where $\Gamma = \begin{pmatrix} 0 & I_{\dim E_{k,q}} \\ I_{\dim E_{k,q}} & 0 \end{pmatrix}, \tilde{\Gamma} = \begin{pmatrix} I_{\dim E_{k,q}} & 0 \\ 0 & -I_{\dim E_{k,q}} \end{pmatrix}$ and satisfy the important property that $\{\Gamma, \tilde{\Gamma}\} = 0$. It is easily seen to be unitarily equivalent to:

$$\begin{aligned} \Gamma^1 D_r^* - \mu_k g(r^*)\Gamma^2 + f(r^*) \quad &\text{if } \dim E_{k,q} = 2, \\ -\sigma_z D_r^* + \mu_k g(r^*)\sigma_x + f(r^*) \quad &\text{if } \dim E_{k,q} = 1. \end{aligned} \tag{3.138}$$

If $0 \in \sigma(\mathfrak{D}_e)$ then, $\dim E_{0,q} \in \{1, 2\}$, for any $q \in J(0)$ and through the natural identification described above is of the form $\Gamma D_{r^*} + f(r^*)$ where Γ here is just some unitary matrix. This is in all points analogous to (3.53), and we will now be able to complete the scattering theory in a unified fashion. It also follows that H_e has no eigenvalues by the same Grönwall lemma argument that was used for H_0 in Section 3.5.10. In short we have:

Lemma 3.6.4. *$\sigma(H_e) = \sigma_{ac}(H_e)$, consequently, $P_c(H_e) = Id$.*

3.6.5 The spherically symmetric operators

The final step required in order to obtain the full scattering theory is to compare H_e and H_0 to their natural asymptotic profiles, $\Gamma^1 D_{r^*} + c_\pm$ at $r^* \to \pm\infty$ respectively; $c_+ = \frac{ap}{r_+^2+a^2} - \frac{ap}{r_e^2+a^2}$ and $c_- = 0$.

In the previous paragraph, we established that the Hilbert space $\mathscr{H}$ could be decomposed into an orthogonal sum of stable subspaces on which H_e reduces to a spherically symmetric operator ; this was also shown to be the case of H_0 in Section 3.4.3. Consequently, in order to construct wave operators, we only need to work on one of these subspaces. Additionally, the similarities between the reduced forms of H_e and H_0 imply that we, in fact, only need to know how to construct the wave operators for [34]:

$$\mathfrak{h} = \Gamma^1 D_{r^*} - \mu g(r^*)\Gamma^2 + f(r^*), \tag{3.139}$$

on $[L^2(\mathbb{R})]^4$, and under the assumption that we have minimal/maximal velocity estimates. This is manifestly the case for our operators because the estimates are stable under restriction to a stable subspace.

The important point is that the operator $\mathfrak{h}$ in (3.139) is formally similar to the restriction to a spherical harmonic of the (charged) Dirac operator of the Reissner-Nordström black hole given in [Dau10, Equation 3.6]. The extreme black hole horizon ($r^* \to \infty$) can be assimilated with spacelike infinity and the symbols f, g have the same asymptotic behaviour at both infinities as the corresponding ones in [Dau10, Equation 3.6]. It follows that we can apply the results of [Dau10] to our case. We note that, in fact, our operator is simpler than the one studied in [Dau04; Dau10] since there are no surviving mass terms.

Precisely, using [Dau10, Propositions 5.6 and 5.7] we find that:

Proposition 3.6.2 (Microlocal velocity estimate). *Let $\chi \in C_0^\infty(\mathbb{R})$ be such that* $\operatorname{supp}\chi \cap \{0\} = \emptyset$ *and choose $0 < \theta_1 < \theta_2$, then there is a constant $C > 0$ such that for any*

34. We choose to discuss the case where $\dim E_{k,q} = 2$, but the reasoning is independent of this choice.

$u \in [L^2(\mathbb{R})]^4$:

$$\int_1^{+\infty} \left\| \mathbf{1}_{[\theta_1,\theta_2]}(\frac{|r^*|}{t})(\Gamma^1 - \frac{r^*}{t})\chi(\mathfrak{h})e^{-it\mathfrak{h}}u \right\|^2 \frac{dt}{t} \leq C||u||^2. \tag{3.140}$$

Furthermore:

$$\mathrm{s-}\lim_{t\to+\infty} \mathbf{1}_{[\theta_1,\theta_2]}(\frac{|r^*|}{t})(\Gamma^1 - \frac{r^*}{t})\chi(\mathfrak{h})e^{-it\mathfrak{h}} = 0. \tag{3.141}$$

Analogous results can be established at $t \to -\infty$, but one must replace Γ^1 with $-\Gamma^1$.

For the specific treatment of our operators, due to the lack of mass terms, it is possible to simplify the proofs in [Dau10], avoiding in particular the use of pseudo-differential operators. We shall proceed to show this.

The proof of Proposition 3.6.2 will be split into two cases. First, we will restrict to the part of the field that is escaping towards the simple horizon, this part of the proof is in all point analogous to Daudé's:

Proof of Proposition 3.6.2, first case . Instead of (3.140), let us seek to estimate:

$$\int_1^{+\infty} \left\| F(\frac{r^*}{t})(\Gamma^1 - \frac{r^*}{t})\chi(\mathfrak{h})e^{-it\mathfrak{h}}u \right\|^2 \frac{dt}{t} \leq C||u||^2, \tag{3.142}$$

where $F \in C_0^\infty(\mathbb{R})$, $F \equiv 1$ on a neighbourhood of $[\theta_1, \theta_2]$ and χ, θ_1, θ_2 satisfy the hypotheses of Proposition 3.6.2; the conditions on F restrict to the region $r^* > 0$. It is enough to assume that $[\theta_1, \theta_2]$ is a neighbourhood of $[\varepsilon_\chi, 1]$, for this covers the region where we lack information. Now define for each $t \geq 1$:

$$\phi(t) = \chi(\mathfrak{h})F(\frac{r^*}{t})\left(R(\frac{r^*}{t}) + (\Gamma^1 - \frac{r^*}{t})R'(\frac{r^*}{t})\right)F(\frac{r^*}{t})\chi(\mathfrak{h}).$$

$R \in C_0^\infty(\mathbb{R})$ is assumed to satisfy $R' \equiv 0$ on a neighbourhood of 0 and $R(r^*) = \frac{r^{*2}}{2}$ on $\operatorname{supp} F$. ϕ is uniformly bounded in t and:

$$\begin{aligned}\phi'(t) = &-\frac{1}{t}\chi(\mathfrak{h})\frac{r^*}{t}F'(\frac{r^*}{t})\left(R(\frac{r^*}{t}) + (\Gamma^1 - \frac{r^*}{t})R'(\frac{r^*}{t})\right)F(\frac{r^*}{t})\chi(\mathfrak{h})\\ &-\frac{1}{t}\chi(\mathfrak{h})F(\frac{r^*}{t})\left(R(\frac{r^*}{t}) + (\Gamma^1 - \frac{r^*}{t})R'(\frac{r^*}{t})\right)\frac{r^*}{t}F'(\frac{r^*}{t})\chi(\mathfrak{h})\\ &-\frac{1}{t}\chi(\mathfrak{h})F(\frac{r^*}{t})\left((\Gamma^1 - \frac{r^*}{t})\frac{r^*}{t}R''(\frac{r^*}{t})\right)F(\frac{r^*}{t})\chi(\mathfrak{h}).\end{aligned} \tag{3.143}$$

Moreover:

$$\begin{aligned} i[\mathfrak{h},\phi(t)] =&\chi(\mathfrak{h})\left(F(\frac{r^*}{t})(-i\mu)g(r^*)[\Gamma^2,\Gamma^1]R'(\frac{r^*}{t})F(\frac{r^*}{t})\right)\chi(\mathfrak{h}) \\ &+\frac{1}{t}\chi(\mathfrak{h})\Gamma^1F'(\frac{r^*}{t})\left(R(\frac{r^*}{t})+(\Gamma^1-\frac{r^*}{t})R'(\frac{r^*}{t})\right)F(\frac{r^*}{t})\chi(\mathfrak{h}) \\ &+\frac{1}{t}\chi(\mathfrak{h})F(\frac{r^*}{t})\left(R(\frac{r^*}{t})+(\Gamma^1-\frac{r^*}{t})R'(\frac{r^*}{t})\right)\Gamma^1F'(\frac{r^*}{t})\chi(\mathfrak{h}) \\ &+\frac{1}{t}\chi(\mathfrak{h})F(\frac{r^*}{t})\Gamma^1(\Gamma^1-\frac{r^*}{t})R''(\frac{r^*}{t})F(\frac{r^*}{t})\chi(\mathfrak{h}). \end{aligned} \tag{3.144}$$

So the Heisenberg derivative of ϕ is :

$$\begin{aligned} D_{\mathfrak{h}}\phi(t) =&\frac{1}{t}\chi(\mathfrak{h})(\Gamma^1-\frac{r^*}{t})F'(\frac{r^*}{t})\left(R(\frac{r^*}{t})+(\Gamma^1-\frac{r^*}{t})R'(\frac{r^*}{t})\right)F(\frac{r^*}{t})\chi(\mathfrak{h}) \\ &+\frac{1}{t}\chi(\mathfrak{h})F(\frac{r^*}{t})\left(R(\frac{r^*}{t})+(\Gamma^1-\frac{r^*}{t})R'(\frac{r^*}{t})\right)F'(\frac{r^*}{t})(\Gamma^1-\frac{r^*}{t})\chi(\mathfrak{h}) \\ &+\chi(\mathfrak{h})F(\frac{r^*}{t})(-i\mu g(r^*))[\Gamma^2,\Gamma^1]R'(\frac{r^*}{t})F(\frac{r^*}{t})\chi(\mathfrak{h}) \\ &+\frac{1}{t}\chi(\mathfrak{h})F(\frac{r^*}{t})(\Gamma^1-\frac{r^*}{t})R''(\frac{r^*}{t})(\Gamma^1-\frac{r^*}{t})F(\frac{r^*}{t})\chi(\mathfrak{h}) \end{aligned} \tag{3.145}$$

Consider the first term, and let $\tilde{F}\in C_0^\infty(\mathbb{R})$ be such that $\operatorname{supp}\tilde{F}\subset]-\infty,\theta_1[\,\cup\,[\theta_2,+\infty[$ and $\tilde{F}F'=F'$ on $\operatorname{supp}F'$. It can now be written: $\frac{1}{t}\chi(h)\tilde{F}(\frac{r^*}{t})B(t)\tilde{F}(\frac{r^*}{t})\chi(\mathfrak{h})$, where $B(t)$ is uniformly bounded, so, there is a constant $M>0$ such that:

$$\begin{aligned} &\frac{1}{t}\chi(\mathfrak{h})(\Gamma^1-\frac{r^*}{t})F'(\frac{r^*}{t})\left(R(\frac{r^*}{t})+(\Gamma^1-\frac{r^*}{t})R'(\frac{r^*}{t})\right)F(\frac{r^*}{t})\chi(\mathfrak{h}) \\ &\qquad\geq-\frac{M}{t}\chi(\mathfrak{h})\tilde{F}^2(\frac{r^*}{t})\chi(\mathfrak{h}). \end{aligned}$$

Moreover, according to the minimal and maximal velocity estimates, there is $C>0$ such that for any $u\in[L^2(\mathbb{R})]^4$:

$$\int_1^{+\infty}\left\|\tilde{F}(\frac{r^*}{t})\chi(\mathfrak{h})e^{-i\mathfrak{h}t}u\right\|^2\frac{\mathrm{d}t}{t}\leq C||u||^2.$$

The same reasoning applies for the second term. The third term is treated in the following manner: $g\in\boldsymbol{S}^{1,1}$, which means in particular that: $g(r^*)=\underset{r^*\to+\infty}{O}(r^{*-2})$, thus:

$$\chi(\mathfrak{h})F(\frac{r^*}{t})\mu g(r^*)i[\Gamma^1,\Gamma^2]R'(\frac{r^*}{t})F(\frac{r^*}{t})\chi(\mathfrak{h})\geq-\frac{M_1}{t^2}\chi(\mathfrak{h})F(\frac{r^*}{t})^2\chi(\mathfrak{h}),$$

for some $M_1 > 0$, and one certainly has:

$$\int_1^{+\infty} \left\| F(\frac{r^*}{t})\chi(\mathfrak{h})e^{-i\mathfrak{h}t}u \right\|^2 \frac{\mathrm{d}t}{t^2} \leq C_1 ||u||^2,$$

for any $u \in [L^2(\mathbb{R})]^4$ and some constant $C_1 > 0$. The desired estimate follows because $R'' = 1$ on supp F and ϕ is uniformly bounded, we refer to [DG97, Lemma B.4.1] for the details.

The argument used to treat the third term in (3.145) will not go through at the double horizon, simply because the potential $g(r^*)\Gamma^2$ is of Coulomb type. This was, of course, already the case at spacelike infinity in the Reissner-Nordström case. The origin of this troublesome term is simply the matrix-valued coefficients of our operator and the simple fact that $[\Gamma^1, \Gamma^2]$ is non-zero. However, rather large spectral subspaces of $\mathfrak{h}_0$ sit in one of the spectral spaces of Γ^1 and, restricted to these subspaces, the commutator is zero. This will turn out to be sufficient to conclude, since the Coulomb decay is enough for the Helffer-Sjöstrand formula to enable a control of operators of the form $F(\frac{r^*}{t})(\chi(\mathfrak{h}_0) - \chi(\mathfrak{h}))$. This rough idea is made very precise thanks to the notion of *locally scalar operators* introduced in [GM01]. The definition is as follows:

Definition 3.6.1. Let E be a finite-dimensional complex Hilbert space and $l : \mathbb{R} \to B(E)$ a continuous function such that $l(p)$ is symmetric for any $p \in \mathbb{R}$. Define the operator $L_0 = l(D_x)$ on $L^2(\mathbb{R}) \otimes E$, then, L_0 is said to be scalar on an open subset $I \subset \mathbb{R}$ if there is a Borel function $\mu : \mathbb{R} \to \mathbb{R}$ such that:

$$L_0 \mathbf{1}_I(L_0) = \mu(D_x)\mathbf{1}_I(L_0). \tag{3.146}$$

If $\lambda \in \mathbb{R}$, L_0 is said to be scalar at λ, if the above holds on some open neighbourhood of λ. Finally, L_0 is locally scalar on an open set I if and only if it is scalar at every point in I.

The authors of [GM01] had Dirac operators in mind as the main application of their theory and so it is no surprise that our 1-dimensional Dirac operators $\Gamma^1 D_{r^*} + c_\pm$ satisfy the hypothesis of the definition. We will nevertheless work out the details and show that they are locally scalar on $\mathbb{R}\backslash\{c_\pm\}$; as it is a good illustration of the terms in the definition. The most direct [35] way to do this is to use the Fourier transform and work with the matrix-

35. and informative

valued multiplication operators $\Gamma^1 p + c_\pm$. For each p this is a diagonal hermitian matrix that has only two eigenvalues $|p| + c_\pm$ and $-|p| + c_\pm$.

Let $\lambda \in \mathbb{R} \setminus \{c_\pm\}$ and let $I \subset \mathbb{R} \setminus \{c_\pm\}$ be any open interval containing λ, then, since I is connected, $I \cap]c_\pm, +\infty[= \emptyset$ or $I \cap]-\infty, c_\pm[= \emptyset$. Suppose the latter, then $\mathbf{1}_I(\Gamma^1 p + c_\pm)$ acts on $u \in L^2(\mathbb{R}) \otimes E$ by projecting $u(p)$ onto the eigenspace of the eigenvalue $|p| + c_\pm$ of the matrix $\Gamma^1 p + c_\pm$ for each p. Therefore, $(\Gamma^1 p + c_\pm)\mathbf{1}_I(\Gamma^1 p + c_\pm) = (|p| + c_\pm)\mathbf{1}_I(\Gamma^1 p + c_\pm)$, which, after returning to the original representation, equates to:

$$(\Gamma^1 D_{r^*} + c_\pm)\mathbf{1}_I(\Gamma^1 D_{r^*} + c_\pm) = (|D_{r^*}| + c_\pm)\mathbf{1}_I(\Gamma^1 D_{r^*} + c_\pm).$$

This does not hold on any neighbourhood of $c_\pm$ for there would always be two distinct eigenvalues.

Now, let L_0 be scalar on some interval I and define:

$$\Omega_I = \{p \in \mathbb{R}, \sigma(l(p)) \cap I \neq \emptyset\}, \tag{3.147}$$

where, $\sigma(l(p))$ denotes the spectrum of the operator $l(p)$. Then, in fact, the function μ in (3.146) can be chosen arbitrarily on $\mathbb{R} \setminus \Omega_I$; this is clear in the Fourier transform representation: $\mathbf{1}_I(l(p))$ acts on $u \in L^2(\mathbb{R}) \otimes E$ according to:

$$(\mathbf{1}_I(l(p))u)(p) = \sum_{\lambda \in \sigma(l(p))} \mathbf{1}_I(\lambda) P_\lambda(l(p))u(p), p \in \mathbb{R},$$

where P_λ denotes projection onto the λ-eigenspace of the matrix $l(p)$. Consequently, if $p \in \mathbb{R} \setminus \Omega_I$ then $(\mathbf{1}(l(p))u)(p) = 0$.

To see how to exploit this remark, let us study Ω_I in the specific case of our Dirac type operators; where we have already seen that $\mu(p) = |p| + c_\pm$. To determine Ω_I choose $\lambda \in]c_\pm, +\infty[$ and $\varepsilon > 0$ such that $I =]\lambda - \varepsilon, \lambda + \varepsilon[\subset]c_\pm, +\infty[$, then :

$$\Omega(I) = \{p \in \mathbb{R}, ||p| - (\lambda - c_\pm)| < \varepsilon\}.$$

This is the union of two disjoint subsets on each side of 0, one can therefore assume that outside of Ω_I, $\mu(p)$ is extended to a function $\mu \in C_0^\infty(\mathbb{R})$ and in this case we will also have:

$$\Gamma^1 \mathbf{1}_I(\Gamma^1 D_{r^*} + c_\pm) = \mu'(D_x)\mathbf{1}_I(\Gamma^1 D_{r^*} + c_\pm)$$

Again μ' can be replaced with $\nu(p) = \frac{p}{|p|}\zeta(p)$ where $\zeta \in C^\infty(\mathbb{R}), \zeta(p) = 1$ outside a small neighbourhood of 0 and $\zeta(p) = 0$ on a neighbourhood of 0. On this subspace, the operator is reduced to a pseudo-differential operator with symbol $\nu(p)$.

We now have the tools necessary to complete the proof at the double horizon, although, we will not need to exploit the above remark to its full extent, contrary to [Dau10], since the mass terms do not survive at the double horizon. We therefore propose a simpler proof, slightly different in spirit, in which the aim is to pinpoint exactly at which moment the locally scalar properties of the operator intervene.

Let θ_1, θ_2 and χ be as before, and, this time choose, $F \in C_0^\infty(\mathbb{R})$ identically equal to 1 on $[-\theta_2, -\theta_1]$, to single out the double horizon. Without loss of generality, we can assume that $\operatorname{supp}\chi$ is a closed interval of $\mathbb{R}$. Choose now, a connected open neighbourhood I of $\operatorname{supp}\chi$ disjoint from 0, and suppose, say, that $I \subset]0, +\infty[$ (the other case is identical), then $\mathfrak{h}_0 = \Gamma^1 D_{r^*}$ is scalar on I. Finally, let $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\chi \subset \operatorname{supp}\tilde{\chi} \subset I$ and $\tilde{\chi} = 1$ on a neighbourhood of $\operatorname{supp}\chi$.

Proof of Proposition 3.6.2, second case . Now, the proof begins exactly as before, but we treat the term with $g(r^*)$ more carefully, recall its expression:

$$\chi(\mathfrak{h})F(\frac{r^*}{t})\mu g(r^*)i[\Gamma^1, \Gamma^2]R'(\frac{r^*}{t})F(\frac{r^*}{t})\chi(\mathfrak{h}) = -2i\chi(\mathfrak{h})F(\frac{r^*}{t})\mu g(r^*)R'(\frac{r^*}{t})F(\frac{r^*}{t})\Gamma^2\Gamma^1\chi(\mathfrak{h}).$$

It is straightforward to see that:

$$\begin{aligned}\chi(\mathfrak{h})F(\frac{r^*}{t})\mu g(r^*)R'(\frac{r^*}{t})F(\frac{r^*}{t})\Gamma^2\Gamma^1\chi(\mathfrak{h}) &= \chi(\mathfrak{h})F(\frac{r^*}{t})\mu g(r^*)R'(\frac{r^*}{t})F(\frac{r^*}{t})\Gamma^2\Gamma^1\tilde{\chi}(\mathfrak{h}_0)\chi(\mathfrak{h}) \\ &+ \chi(\mathfrak{h})F(\frac{r^*}{t})\mu g(r^*)R'(\frac{r^*}{t})F(\frac{r^*}{t})\Gamma^2\Gamma^1(\tilde{\chi}(\mathfrak{h}) - \tilde{\chi}(\mathfrak{h}_0))\chi(\mathfrak{h}).\end{aligned}$$

The second term is $O(t^{-2})$ because $R'(\frac{r^*}{t}) = \frac{r^*}{t}$ on $\operatorname{supp} F$, $r^* g(r^*) = O(1)$ and $F(\frac{r^*}{t})(\tilde{\chi}(\mathfrak{h}) - \tilde{\chi}(\mathfrak{h}_0))$ is $O(t^{-1})$. The first term can be decomposed further as follows:

$$\begin{aligned}&\chi(\mathfrak{h})F(\frac{r^*}{t})\mu g(r^*)R'(\frac{r^*}{t})F(\frac{r^*}{t})\Gamma^2\Gamma^1\tilde{\chi}(\mathfrak{h}_0)\chi(\mathfrak{h}) = \\ &\qquad \chi(\mathfrak{h})[\tilde{\chi}(\mathfrak{h}), F(\frac{r^*}{t})\mu g(r^*)R'(\frac{r^*}{t})F(\frac{r^*}{t})]\Gamma^2\Gamma^1\tilde{\chi}(\mathfrak{h}_0)\chi(\mathfrak{h}) \\ &\qquad + \chi(\mathfrak{h})F(\frac{r^*}{t})\mu g(r^*)R'(\frac{r^*}{t})F(\frac{r^*}{t})(\tilde{\chi}(\mathfrak{h}) - \tilde{\chi}(\mathfrak{h}_0))\Gamma^2\Gamma^1\tilde{\chi}(\mathfrak{h}_0)\chi(\mathfrak{h}) \\ &\qquad + \chi(\mathfrak{h})F(\frac{r^*}{t})\mu g(r^*)R'(\frac{r^*}{t})F(\frac{r^*}{t})\tilde{\chi}(\mathfrak{h}_0)\Gamma^2\Gamma^1\tilde{\chi}(\mathfrak{h}_0)\chi(\mathfrak{h}) \quad (3.148)\end{aligned}$$

The first and second terms are $O(t^{-2})$ using the Helffer-Sjöstrand Formula, the last term, on the other hand vanishes. To see this, let us study:

$$\tilde{\chi}(\mathfrak{h}_0)\Gamma^2\Gamma^1\tilde{\chi}(\mathfrak{h}_0) = \tilde{\chi}(\mathfrak{h}_0)\mathbf{1}_I(\mathfrak{h}_0)\Gamma^2\Gamma^1\mathbf{1}_I(\mathfrak{h}_0)\tilde{\chi}(\mathfrak{h}_0).$$

Via Fourier transform, $\mathbf{1}_I(\mathfrak{h}_0)\Gamma^2\Gamma^1\mathbf{1}_I(\mathfrak{h}_0)$ is unitarily equivalent to the matrix-valued multiplication operator: $\mathbf{1}_I(\Gamma^1 p)\Gamma^2\Gamma^1\mathbf{1}_I(\Gamma^1 p)$, however, for $p \in \Omega_I$:

$$\begin{aligned}\mathbf{1}_I(\Gamma^1 p)\Gamma^2\frac{p}{|p|}\mathbf{1}_I(\Gamma^1 p) &= \mathbf{1}_I(\Gamma^1 p)\Gamma^2\Gamma^1\mathbf{1}_I(\Gamma^1 p),\\ &= -\mathbf{1}_I(\Gamma^1 p)\Gamma^1\Gamma^2\mathbf{1}_I(\Gamma^1 p) = -\mathbf{1}_I(\Gamma^1 p)\Gamma^2\frac{p}{|p|}\mathbf{1}_I(\Gamma^1 p).\end{aligned}$$

Hence, all terms in the above equality vanish; Proposition 3.6.2 follows.

The first consequence of (3.140) is:

Lemma 3.6.5. *Let $\chi \in C_0^\infty(\mathbb{R})$, such that $0 \notin \operatorname{supp}\chi$, and let $0 < \theta_1 < \theta_2$, then*

$$\operatorname*{s-lim}_{t\to+\infty} \mathbf{1}_{[\theta_1,\theta_2]}\left(\frac{|r^*|}{t}\right)\left(\Gamma^1 - \frac{r^*}{t}\right)\chi(\mathfrak{h})e^{-it\mathfrak{h}} = 0. \tag{3.149}$$

Proof. As before, assume that $\theta_1 < \varepsilon_\chi$ et $\theta_2 > 1$. It is a direct consequence of the estimate (3.140) that, if the limit exist, it should be 0. Let $F \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} F \subset [-\theta_2, -\theta_1] \cup [\theta_1, \theta_2]$, let us show that for any $u \in D(\mathfrak{h})$ the following limit exists:

$$\lim_{t\to+\infty} ||F(\frac{r^*}{t})(\Gamma^1 - \frac{r^*}{t})\chi(\mathfrak{h})e^{-it\mathfrak{h}}u||^2.$$

The desired result follows immediately as this limit is necessarily 0. We only need to show that the Cauchy criterion is satisfied by the above by studying the integral :

$$\int_{t_1}^{t_2}(u, \frac{\mathrm{d}}{\mathrm{d}t}e^{it\mathfrak{h}}\chi(\mathfrak{h})F(\frac{r^*}{t})^2(\Gamma^1 - \frac{r^*}{t})^2\chi(\mathfrak{h})e^{-it\mathfrak{h}}u)\mathrm{d}t, u \in D(\mathfrak{h}).$$

The derivative evaluates to:

$$\begin{aligned}&(u, \chi(\mathfrak{h})e^{it\mathfrak{h}}\frac{1}{t}F'(\frac{r^*}{t})(\Gamma^1 - \frac{r^*}{t})^3F(\frac{r^*}{t})\chi(\mathfrak{h})e^{-it\mathfrak{h}}u)\\ &+(u, \chi(\mathfrak{h})e^{it\mathfrak{h}}\frac{1}{t}F(\frac{r^*}{t})(\Gamma^1 - \frac{r^*}{t})^3F'(\frac{r^*}{t})\chi(\mathfrak{h})e^{-it\mathfrak{h}}u)\\ &-(u, \frac{2}{t}e^{it\mathfrak{h}}\chi(\mathfrak{h})F(\frac{r^*}{t})(\Gamma^1 - \frac{r^*}{t})^2F(\frac{r^*}{t})\chi(\mathfrak{h})e^{-it\mathfrak{h}}u)\\ &+(u, 2e^{it\mathfrak{h}}\chi(\mathfrak{h})F(\frac{r^*}{t})\mu g(r^*)\frac{r^*}{t}i[\Gamma^2, \Gamma^1]F(\frac{r^*}{t})e^{-it\mathfrak{h}}u).\end{aligned} \tag{3.150}$$

In order to show that the integral is arbitrarily small as long as t_1, t_2 are large enough, the first three terms are treated as at the end of the proof of Proposition 3.6.1, more precisely, one exploits the different velocity estimates according to the supports of F and F'; the last term is dealt with as in the second part of the proof of Proposition 3.6.2, one again uses the locally scalar properties to reveal that it is in fact $O(t^{-2})$ and hence integrable on $[1, +\infty]$.

Proposition 3.6.2 is known as a microlocal velocity estimate. It completes the asymptotic information about the operator $\frac{r^*}{t}$ – which is itself to be thought of as an approximate velocity operator – provided by minimal and maximal velocity estimates. For instance, combining the three, we show that:

Corollary 3.6.1. *For any $J \in C_\infty(\mathbb{R})$:*

$$\operatorname*{s-lim}_{t\to+\infty} e^{it\mathfrak{h}}(J(\frac{r^*}{t}) - J(\Gamma^1))e^{-it\mathfrak{h}} = 0, \tag{3.151}$$

Proof. First, by density, it is sufficient to consider $J \in C_0^\infty(\mathbb{R})$. For such J, the Helffer-Sjöstrand formula can be used to show that the following holds for any $j_0 \in C_0^\infty(\mathbb{R})$:

$$\begin{aligned}(J(\frac{r^*}{t}) - J(\Gamma^1))j_0(\frac{r^*}{t}) &= \frac{i}{2\pi}\int \partial_{\bar{z}}\tilde{J}(z)(\Gamma^1 - z)^{-1}\left(\frac{r^*}{t} - z\right)^{-1}(\Gamma^1 - \frac{r^*}{t})j_0(\frac{r^*}{t})\mathrm{d}z\wedge\mathrm{d}\bar{z}\\ &= B(t)(\Gamma^1 - \frac{r^*}{t})j_0(\frac{r^*}{t}).\end{aligned}$$

The $B(t)$ are uniformly bounded in t. By a further density argument we only need to prove that for any $\chi \in C_0^\infty(\mathbb{R})$, $0 \notin \operatorname{supp}\chi$:

$$\operatorname*{s-lim}_{t\to+\infty} e^{it\mathfrak{h}}(J(\frac{r^*}{t}) - J(\Gamma^1))\chi(\mathfrak{h})e^{-it\mathfrak{h}} = 0.$$

Fix χ and introduce a smooth partition of unity, j_1, j_2, j_3 subordinate to the open cover:

$$U_1 = \{|x| < \varepsilon_\chi - \frac{\delta}{2}\}, U_2 = \{|x| > 1 + \frac{\delta}{2}\}, U_3 = \{\ \varepsilon_\chi - \delta < |x| < 1 + \delta\},$$

where ε_χ is given by Proposition 3.5.7 and $\delta \in (0, 2\varepsilon_\chi)$. Then:

$$e^{it\mathfrak{h}}(J(\frac{r^*}{t}) - J(\Gamma^1))\chi(\mathfrak{h})e^{-it\mathfrak{h}} = \sum_i e^{it\mathfrak{h}}B(t)(\Gamma^1 - \frac{r^*}{t})j_i(\frac{r^*}{t})\chi(\mathfrak{h})e^{-it\mathfrak{h}}$$

The result now follows from the minimal, maximal and microlocal velocity estimates.

3.6.6 Asymptotic velocity operators and wave operators for the spherically symmetric operators

The first application of the results in the previous section is the proof of the existence of asymptotic velocity operators; they are usually defined by:

$$J(P^{\pm}) = \underset{t\to\pm\infty}{\text{s}-\lim} e^{it\mathfrak{h}} J(\frac{r^*}{t}) e^{-it\mathfrak{h}}, J \in C_\infty(\mathbb{R}). \tag{3.152}$$

Provided that these limits exist, one can show [36] that there is a unique operator, $P^{\pm}$, possibly non-densely defined, satisfying the above, we write:

$$P^{\pm} = \underset{t\to\pm\infty}{\text{s}-C_\infty-\lim}\ e^{it\mathfrak{h}}(\frac{r^*}{t})e^{-it\mathfrak{h}}.$$

We prove the following lemma:

Lemma 3.6.6. *The following limits exist:*

$$\underset{t\to\pm\infty}{\text{s}-\lim} e^{it\mathfrak{h}}\Gamma^1 e^{-it\mathfrak{h}}.$$

Proof. As usual, we will only discuss the $t \to +\infty$ case. By density, one only needs to prove the existence of:

$$\underset{t\to\pm\infty}{\text{s}-\lim} e^{it\mathfrak{h}}\chi(\mathfrak{h})\Gamma^1\chi(\mathfrak{h})e^{it\mathfrak{h}},$$

for any $\chi \in C_0^\infty(\mathbb{R})$ such that $\{0\} \notin \text{supp}\ \chi$. Furthermore, as in the proof of Proposition 3.6.1, Proposition 3.5.7 implies that it is sufficient to prove the existence of:

$$\underset{t\to\pm\infty}{\text{s}-\lim} e^{it\mathfrak{h}}\chi(\mathfrak{h})j(\frac{r^*}{t})\Gamma^1\chi(\mathfrak{h})e^{it\mathfrak{h}},$$

where $j \in C^\infty(\mathbb{R})$ is a bounded function that vanishes on a neighbourhood of 0 and such that $\text{supp}\, j' \subset (-\varepsilon_\chi, \varepsilon_\chi)$ with ε_χ given by Proposition 3.5.7. We apply Cook's method and calculate the derivative on $D(\mathfrak{h})$, one finds:

$$e^{it\mathfrak{h}}\chi(\mathfrak{h})\left(\frac{1}{t}\Gamma^1(\Gamma^1 - \frac{r^*}{t})j'(\frac{r^*}{t}) + i[\Gamma^2, \Gamma^1]g(r^*)j(\frac{r^*}{t})\right)\chi(\mathfrak{h})e^{-it\mathfrak{h}}.$$

The first term can be treated again as in the proof of Proposition 3.6.1, the second requires a bit more effort, but the method is essentially that of Proposition 3.6.2. First, without

36. see for example the appendices in [DG97].

loss of generality assume that $\operatorname{supp}\chi$ is a closed interval contained in, say $(0,+\infty)$, and let $\tilde{\chi}\in C_0^\infty(\mathbb{R})$ satisfy $\operatorname{supp}\chi\subset\operatorname{supp}\tilde{\chi}\ \subset(0,+\infty)$, $\chi\tilde{\chi}=\chi$. Introduce equally a partition of unity j_0, j_1 such that $j_1(s)=1$ for $s>2$ and vanishes for $s<1$. Then:

$$i[\Gamma^2,\Gamma^1]g(r^*)j(\frac{r^*}{t})=i[\Gamma^2,\Gamma^1]g(r^*)j_0(\frac{r^*}{t})j(\frac{r^*}{t})+i[\Gamma^2,\Gamma^1]g(r^*)j_1(\frac{r^*}{t})j(\frac{r^*}{t}).$$

$g(r^*)j_1(\frac{r^*}{t})j(\frac{r^*}{t})=O(t^{-2})$ so the second term is integrable.
Now:

$$\begin{aligned}\chi(\mathfrak{h})2i\Gamma^2\Gamma^1g(r^*)j_0(\frac{r^*}{t})j(\frac{r^*}{t})\chi(\mathfrak{h})&=\chi(\mathfrak{h})\tilde{\chi}(\mathfrak{h}_0)2i\Gamma^2\Gamma^1g(r^*)j_0(\frac{r^*}{t})j(\frac{r^*}{t})\chi(\mathfrak{h})\\&+\chi(\mathfrak{h})(\tilde{\chi}(\mathfrak{h})-\tilde{\chi}(\mathfrak{h}_0))j_0(\frac{r^*}{t})j(\frac{r^*}{t})2i\Gamma^2\Gamma^1g(r^*)\chi(\mathfrak{h}),\end{aligned}$$

Again, the second term is $O(t^{-2})$ and the first has to be further decomposed:

$$\begin{aligned}\chi(\mathfrak{h})\tilde{\chi}(\mathfrak{h}_0)\Gamma^2\Gamma^1g(r^*)j_0(\frac{r^*}{t})j(\frac{r^*}{t})\chi(\mathfrak{h})=&\chi(\mathfrak{h})\tilde{\chi}(\mathfrak{h}_0)\Gamma^2\Gamma^1g(r^*)j_0(\frac{r^*}{t})j(\frac{r^*}{t})\tilde{\chi}(\mathfrak{h})\chi(\mathfrak{h}),\\=&\chi(\mathfrak{h})\tilde{\chi}(\mathfrak{h}_0)\Gamma^2\Gamma^1[g(r^*)j_0(\frac{r^*}{t})j(\frac{r^*}{t}),\tilde{\chi}(\mathfrak{h})]\chi(\mathfrak{h})\\&+\chi(\mathfrak{h})\tilde{\chi}(\mathfrak{h}_0)\Gamma^2\Gamma^1(\tilde{\chi}(\mathfrak{h})-\tilde{\chi}(\mathfrak{h}_0))g(r^*)j_0(\frac{r^*}{t})j(\frac{r^*}{t})\chi(\mathfrak{h})\\&+\chi(\mathfrak{h})\tilde{\chi}(\mathfrak{h}_0)\Gamma^2\Gamma^1\tilde{\chi}(\mathfrak{h}_0)g(r^*)j_0(\frac{r^*}{t})j(\frac{r^*}{t})\chi(\mathfrak{h}).\end{aligned}$$

The last term vanishes, and one can use the Helffer-Sjöstrand formula to show that the others are $O(t^{-2})$.

Corollary 3.6.1 can now be used to show the existence of asymptotic velocity operators which are defined as the limits[37]:

$$P^\pm=\mathrm{s}-C_\infty-\lim_{t\to\pm\infty}e^{it\mathfrak{h}}\frac{r^*}{t}e^{-it\mathfrak{h}}.$$

In Lemma 3.6.6, we showed the existence of: $\operatorname{s-lim}_{t\to\pm\infty}e^{it\mathfrak{h}}(\pm\Gamma^1)e^{-it\mathfrak{h}}$, consequently:

$$\begin{aligned}P^\pm&=\operatorname{s-lim}_{t\to\pm\infty}e^{it\mathfrak{h}}(\pm\Gamma^1)e^{-it\mathfrak{h}},\\\sigma(P^\pm)&=\{-1,1\}.\end{aligned}\tag{3.153}$$

37. See the appendices of [DG97]

3.6.7 Modified wave operators in the spherically symmetric case

The final stage of the construction is to prove the existence of the (modified) operators in the spherically symmetric case. Here, the operators $P^{\pm}$ can be used to distinguish between the incoming and outgoing states instead of cut-off functions. The simplicity of their spectrum means in particular that:

$$\mathscr{H} = \mathscr{H}_{\text{in}} \oplus \mathscr{H}_{\text{out}},$$

where: $\mathscr{H}_{\text{in}} = \mathbf{1}_{\{-1\}}(P^{\pm})$, $\mathscr{H}_{\text{out}} = \mathbf{1}_{\{1\}}(P^{\pm})$.

At the simple horizon, the asymptotic dynamics is given by $\mathfrak{h}_1 = \Gamma^1 D_{r^*} + \left(\frac{a}{r_+^2+a^2} - \frac{a}{r_e^2+a^2}\right) p$. The difference between this and the operator $\mathfrak{h}$ is short range when $r^* \to +\infty$. Hence, the existence of the wave operators on $\mathscr{H}_{\text{out}}$ can be shown in exactly the same manner as that of (3.128).

At the double horizon, it is necessary to modify slightly the comparison dynamics in order to take into account the long range potentials, as in [Dau10], we choose to use the Dollard [DV66] modification; in particular, the existence of the modified wave operator is contained in the results presented in [Dau10, Sections VII.B (Theorem 7.2), VII.C].

We briefly recall the main idea of the Dollard modification. We seek to compare $\mathfrak{h} = \Gamma^1 D_{r^*} - \mu g(r^*)\Gamma^2 + f(r^*)$ to $\mathfrak{h}_0 = \Gamma^1 D_{r^*}$ on $\mathscr{H}_{\text{in}}$. Several remarks are in order: both the potentials are long-range near the double horizon and $\{\Gamma^2, \mathfrak{h}_0\} = 0$. This anti-commutation property means that the corresponding term can be thought of as an "artifical" long-range term; it is no obstruction to the existence of wave operators. This is perhaps best understood by looking at $\mathfrak{h}^2$:

$$\begin{aligned}\mathfrak{h}^2 = D_{r^*}^2 + \mu^2 g(r^*)^2 + f(r^*)^2 + \Gamma_1\{D_{r^*}, f(r^*)\} - 2\mu f(r^*) g(r^*)\Gamma^2 \\ - \mu \underbrace{\{\Gamma^1, \Gamma^2\}}_{=0} g(r^*) D_{r^*} + i\mu g'(r^*)\Gamma^1\Gamma^2.\end{aligned}$$

We observe that there are no surviving long-range times containing g.

The main idea of the Dollard modification can be explained as follows: if the potential $f(r^*)$ commuted with $\mathfrak{h}_0$, one could expect on a purely formal level that:

$$\begin{aligned}e^{i\mathfrak{h}t} e^{-if(r^*)t} e^{-i\mathfrak{h}_0 t} &= e^{i(\mathfrak{h}_0 - \mu g(r^*)\Gamma^2)t} e^{+if(r^*)t} e^{-if(r^*)t} e^{-i\mathfrak{h}_0 t} \\ &= e^{i(\mathfrak{h}_0 - \mu g(r^*)\Gamma^2)t} e^{-i\mathfrak{h}_0 t}.\end{aligned}$$

Hence, modifying the asymptotic dynamics with $e^{itf(r^*)}$ would enable us to construct a wave operator. Now, of course f does not commute with $\mathfrak{h}_0$, but, Proposition 3.6.2 and Corollary 3.6.1 suggest that, in some sense, $r^* \approx \Gamma^1 t$ when $t \to +\infty$, therefore it could be a good idea to attempt to approximate $f(r^*)$ with $f(\Gamma^1 t)$, which *does* commute with $\mathfrak{h}_0$! We are therefore lead to try the above reasoning with the dynamics $U(t, t_0)$ generated by $f(t\Gamma^1)$. In fact, the comparison only interests us for $r^* < 0$, so we will consider the dynamics generated by $\tilde{f}(t\Gamma^1) = j(t\Gamma^1)f(t\Gamma^1)$ where $j \in C^\infty(\mathbb{R})$ is a smooth cut-off function satisfying $j(s) = 0$ if $s > 1$ and $j(s) = 1$ if $s < \frac{1}{2}$. Since $t \mapsto \tilde{f}(t\Gamma^1) = V(t)$ is uniformly bounded in t, $U(t, t_0)$ of this time-dependent operator is given by the Dyson series, or, time-ordered exponential:

$$\begin{aligned} U(t, t_0) &= \sum_{n=0}^{+\infty} \frac{(-i)^n}{n!} \int_{[t_0,t]^n} T(V(t_1)V(t_2)\dots V(t_n))\mathrm{d}t_n \dots \mathrm{d}t_1 \\ &= T\exp\left((-i)\int_{t_0}^{t} V(s)\mathrm{d}s\right). \end{aligned}$$

In the above, the operator T denotes time ordering of the operators which is defined as:

$$T(V(t_1)\dots V(t_n)) = \sum_{\sigma \in \mathfrak{S}_n} \mathbf{1}(t_{\sigma(1)} > t_{\sigma(2)} > \dots > t_{\sigma(n)})V(t_{\sigma(1)})\dots V(t_{\sigma(n)}).$$

The uniform-boundedness of the operators $V(t)$ implies that this expansion converges in norm. We quote its main properties, let $(t, s, t_0) \in \mathbb{R}^3$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}U(t, t_0) &= -iV(t)U(t, t_0), \quad U(t, t) = \mathrm{Id}, \\ \frac{\mathrm{d}}{\mathrm{ds}}U(t, s) &= iU(t, s)V(s), \quad U(t, t_0) = U(t, s)U(s, t_0). \end{aligned}$$

Set $U(t) = U(t, 0)$, then according to [Dau10, Section 7.2]:

Proposition 3.6.3. *The following limits exist:*

$$\begin{aligned} &\mathrm{s-}\lim_{t\to\pm\infty} e^{it\mathfrak{h}}e^{-it\mathfrak{h}_1}\mathbf{1}_{\{1\}}(\pm\Gamma^1), \\ &\mathrm{s-}\lim_{t\to\pm\infty} e^{it\mathfrak{h}_1}e^{-it\mathfrak{h}}\mathbf{1}_{\{1\}}(P^\pm), \\ &\mathrm{s-}\lim_{t\to\pm\infty} e^{it\mathfrak{h}}U(t)e^{-it\mathfrak{h}_0}\mathbf{1}_{\{-1\}}(\pm\Gamma^1), \\ &\mathrm{s-}\lim_{t\to\pm\infty} e^{it\mathfrak{h}_0}U(t)^*e^{it\mathfrak{h}}\mathbf{1}_{\{-1\}}(P^\pm). \end{aligned} \tag{3.154}$$

Once more, we note that the proof in [Dau10, Section 7.2] is complicated by the presence of a mass term absent from our operators. To illustrate this we shall prove the existence of:

$$s-\lim_{t\to\ +\infty} e^{it\mathfrak{h}}U(t)e^{-it\mathfrak{h}_0}\mathbf{1}_{\{-1\}}(\Gamma^1). \tag{3.155}$$

Proof of the existence of (3.155). The asymptotic velocity operator is simply Γ^1 for $\mathfrak{h}_0$ which is the reason why we use it to split incoming and outgoing states for $\mathfrak{h}_0$. The first step is to replace the projection with an operator that is more convenient to work with. First of all, for any $J \in C_0^\infty(\mathbb{R})$ such that, $\operatorname{supp} J \subset (-\infty, 0)$ and $J(-1) = 1$, $J(\Gamma^1) = \mathbf{1}_{\{-1\}}(\Gamma^1)$. Furthermore for each t, one has:

$$\begin{aligned} e^{it\mathfrak{h}}U(t)J(\frac{r^*}{t})e^{-it\mathfrak{h}_0} &= e^{it\mathfrak{h}}U(t)e^{-it\mathfrak{h}_0}(e^{it\mathfrak{h}_0}J(\frac{r^*}{t})e^{-it\mathfrak{h}_0} - J(\Gamma^1)) \\ &\quad + e^{it\mathfrak{h}}U(t)e^{-it\mathfrak{h}_0}J(\Gamma^1). \end{aligned} \tag{3.156}$$

Now, $e^{it\mathfrak{h}}U(t)e^{-it\mathfrak{h}_0}$ is uniformly bounded in t so applying [38] Corollary 3.6.1 to $\mathfrak{h}_0$, we find that the strong limit of the first term exists and is 0, so, using another classical density argument we only need to prove the existence of:

$$s-\lim_{t\to+\infty} e^{it\mathfrak{h}}U(t)J(\frac{r^*}{t})e^{-it\mathfrak{h}_0}\chi(\mathfrak{h}_0),$$

for any $\chi \in C_0^\infty(\mathbb{R}), 0 \notin \operatorname{supp}\chi$, this in particular implies that $\chi \equiv 0$ on a neighbourhood of 0. Once more, we use Cook's method and to that end we calculate the derivative; one finds:

$$\begin{aligned} &e^{it\mathfrak{h}}\left(iJ(\frac{r^*}{t})(-\mu)g(r^*)\Gamma^2 + \frac{1}{t}J'(\frac{r^*}{t})(\Gamma^1 - \frac{r^*}{t})\right)\chi(\mathfrak{h}_0)U(t)e^{-it\mathfrak{h}_0} \\ &\quad + e^{it\mathfrak{h}}\left(iJ(\frac{r^*}{t})f(r^*) - iJ(\frac{r^*}{t})\tilde{f}(t\Gamma^1)\right)\chi(\mathfrak{h}_0)U(t)e^{-it\mathfrak{h}_0}. \end{aligned}$$

The term involving J' can be treated by the second method explained in the proof of Proposition 3.6.1; we will not repeat the reasoning here.

Let us examine the first term:

$$T_1 = e^{it\mathfrak{h}}(iJ(\frac{r^*}{t})(-\mu)g(r^*)\Gamma^2 e^{-it\mathfrak{h}_0}U(t)\chi(\mathfrak{h}_0),$$

where we have used the fact that Γ^1 commutes with $\mathfrak{h}_0$, hence $U(t)$ commutes with

38. although it is simpler for $\mathfrak{h}_0$

$\chi(\mathfrak{h}_0)$ and $e^{-it\mathfrak{h}_0}$. Since Γ^2 anti-commutes with Γ^1, $\Gamma^2 U(t) = \tilde{U}(t)\Gamma^2$, where[39] $\tilde{U}(t) = T\exp(i\tilde{f}(-\Gamma^1 t))$, so one can rewrite T_1 as follows:

$$T_1 = e^{it\mathfrak{h}} iJ(\frac{r^*}{t})(-\mu)g(r^*)\tilde{U}(t)e^{-it\mathfrak{h}_0}e^{it\mathfrak{h}_0}\Gamma^2 e^{-it\mathfrak{h}_0}\chi(\mathfrak{h}_0).$$

Set $E(t) = \int_0^t e^{is\mathfrak{h}_0}\Gamma^2 e^{-is\mathfrak{h}_0}\chi(\mathfrak{h}_0)\mathrm{d}s$. Γ^2 anti-commutes with $\mathfrak{h}_0$, therefore:

$$E(t) = \Gamma^2 \int_0^t e^{-2is\mathfrak{h}_0}\chi(\mathfrak{h}_0)\mathrm{d}s.$$

However, it follows from the bounded functional calculus that:

$$\left\|\int_0^t e^{-2is\mathfrak{h}_0}\chi(\mathfrak{h}_0)\mathrm{d}s\right\| = \sup_{\lambda\in\mathbb{R}}\left|\int_0^t e^{-2is\lambda}\chi(\lambda)\mathrm{d}s\right|.$$

Since $\chi \equiv 0$ on a neighbourhood of 0, this is finite and bounded independently of t, so $E(t)$ is a uniformly bounded function of t. Now, for any $t_1, t_2 \geq 1$,

$$\begin{aligned}\int_{t_1}^{t_2} T_1(t)\mathrm{dt} = &\left[e^{it\mathfrak{h}} iJ(\frac{r^*}{t})(-\mu)g(r^*)\tilde{U}(t)e^{-it\mathfrak{h}_0}E(t)\ \right]_{t_1}^{t_2} \\ &- \int_{t_1}^{t_2} \partial_t \left(e^{it\mathfrak{h}}(iJ(\frac{r^*}{t})(-\mu)g(r^*)\tilde{U}(t)e^{-it\mathfrak{h}_0}\right)E(t)\mathrm{dt}\end{aligned} \tag{3.157}$$

Since J vanishes on a neighbourhood of 0, and $E(t)$ is uniformly bounded, the term in the squared brackets vanishes as $t_1, t_2 \to +\infty$:

$$\begin{aligned}\left\|e^{it\mathfrak{h}} iJ(\frac{r^*}{t})(-\mu)g(r^*)\tilde{U}(t)e^{-it\mathfrak{h}_0}E(t)\ \right\| &= O\left(|g(r^*)|J(\frac{r^*}{t})\right) \\ &= O\left(\frac{1}{t}\right).\end{aligned} \tag{3.158}$$

Additionally, due to the further derivative, the integrand in the second term is $O(t^{-2})$ and hence integrable. It remains to treat the final terms:

$$T_2 = e^{it\mathfrak{h}}\left(iJ(\frac{r^*}{t})f(r^*) - iJ(\frac{r^*}{t})\tilde{f}(t\Gamma^1)\right)\chi(\mathfrak{h}_0)U(t)e^{-it\mathfrak{h}_0}.$$

39. The operators under consideration here are all bounded, the series defining $U(t)$ converges in norm and $\tilde{f}$ is continuous and bounded, so one only needs to check the anti-commutation property on polynomials.

Notice first that, $\operatorname{supp} J \subset (0, -\infty)$, so $J(\frac{r^*}{t}) = J(\frac{r^*}{t})j(r^*)$ and:

$$T_2 = e^{it\mathfrak{h}} iJ(\frac{r^*}{t})\left(\tilde{f}(r^*) - \tilde{f}(t\Gamma^1)\right)\chi(\mathfrak{h}_0)U(t)e^{-it\mathfrak{h}_0}.$$

It follows from (3.43) and the subsequent remarks that $\tilde{f} \in \boldsymbol{S}^{1,1}$, and one can use the Helffer-Sjöstrand formula to obtain an expression for $(\tilde{f}(r^*) - \tilde{f}(t\Gamma^1))J(\frac{r^*}{t})$ as in the proof of Lemma 3.6.1:

$$(\tilde{f}(r^*) - \tilde{f}(t\Gamma^1))J(\frac{r^*}{t}) = B(t)(\Gamma^1 - \frac{r^*}{t})J(\frac{r^*}{t}),$$

where B is a uniformly bounded operator in t. The desired integrability result is hence a consequence of the microlocal velocity estimate (3.140); the existence of (3.155) follows.

3.7 The full scattering theory

In the previous two sections, the original scattering problem was progressively reduced to a one-dimensional problem via two intermediate comparisons. We discussed the proof of the existence of a number of strong limits that are to be identified with intermediate waves operators. In this section, we assemble these results into the scattering theory we set out to construct; the whole construction was broken up into three comparisons as illustrated in Figure 3.1.

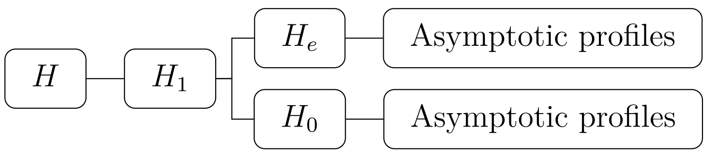

Figure 3.1 – Successive comparisons

3.7.1 Comparison I

The difference between H_1 and H being a short-range potential at both infinities, there was no obstruction to the existence of the classical wave operators (Proposition 3.6.1):

$$\begin{aligned}\Omega^1_\pm &= \operatorname*{s-lim}_{t\to\pm\infty} e^{itH_1}e^{-itH}P_c(H),\\ \tilde{\Omega}^1_\pm &= \operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-itH_1}P_c(H_1).\end{aligned} \tag{3.159}$$

The properties of these operators are well known [40], they satisfy:

$$\begin{aligned}\tilde{\Omega}^1_\pm = {\Omega^1_\pm}^*, \ \underbrace{\Omega^1_\pm H = H_1 \Omega^1_\pm}_{\text{Intertwining relation}}, \\ {\Omega^1_\pm}^* \Omega^1_\pm = P_c(H), \ \Omega^1_\pm {\Omega^1_\pm}^* = P_c(H_1),\end{aligned} \tag{3.160}$$

as such they are isometries between the absolutely continuous subspaces of H and H_1; the intertwining relation shows that H and H_1 are unitarily equivalent.

3.7.2 Comparison II

The second comparison was established in Section 3.6.3 and required to distinguish between states scattering to the double horizon $\mathscr{H}_{r_e}$ and those scattering to the simple horizon $\mathscr{H}_{r_+}$. This distinction was accomplished using smooth cut-off functions $c_\pm$, vanishing on a neighbourhood of $\mp\infty$ and equal to 1 on a neighbourhood of $\pm\infty$; we will denote by $\mathscr{C}_\pm$ the subset of smooth functions with these properties. We have shown the existence of the limits, for $c_\pm \in \mathscr{C}_\pm$:

$$\begin{aligned}\Omega^2_{\pm,\mathscr{H}_{r_+}} &= \operatorname*{s-lim}_{t\to\pm\infty} e^{iH_0t} c_+(r^*) e^{-iH_1t} P_c(H_1),\\ \tilde{\Omega}^2_{\pm,\mathscr{H}_{r_+}} &= \operatorname*{s-lim}_{t\to\pm\infty} e^{iH_1t} c_+(r^*) e^{-iH_0t},\\ \Omega^2_{\pm,\mathscr{H}_{r_e}} &= \operatorname*{s-lim}_{t\to\pm\infty} e^{iH_et} c_-(r^*) e^{-iH_1t} P_c(H_1),\\ \tilde{\Omega}^2_{\pm,\mathscr{H}_{r_e}} &= \operatorname*{s-lim}_{t\to\pm\infty} e^{iH_1t} c_-(r^*) e^{-iH_et}.\end{aligned} \tag{3.161}$$

The limits are independent of the choice of $c_\pm$; recall also that both H_e and H_0 only have absolutely continuous spectrum. [RS79, Proposition 4] shows that the ranges of both $\tilde{\Omega}^2_{\pm,\mathscr{H}_{r_e}}$ and $\tilde{\Omega}^2_{\pm,\mathscr{H}_{r_+}}$ are subsets of the absolutely continuous subspace of H_1, it follows then that:

$$\tilde{\Omega}^2_{\pm,\mathscr{H}_{r_e}} = {\Omega^2}^*_{\pm,\mathscr{H}_{r_e}}, \quad \tilde{\Omega}^2_{\pm,\mathscr{H}_{r_+}} = {\Omega^2}^*_{\pm,\mathscr{H}_{r_+}}. \tag{3.162}$$

One also has the intertwining relations on the absolutely continuous subspace of H_1:

$$H_0 \Omega^2_{\pm,\mathscr{H}_{r_+}} = \Omega^2_{\pm,\mathscr{H}_{r_+}} H_1, \tag{3.163}$$

$$H_e \Omega^2_{\pm,\mathscr{H}_{r_e}} = \Omega^2_{\pm,\mathscr{H}_{r_e}} H_1. \tag{3.164}$$

40. see, for example, [Lax02, Chapter 37]

Together, Equations (3.160), (3.163) and (3.164) give:

$$H_0\Omega^2_{\pm,\mathscr{H}_{r_+}}\Omega^1_\pm = \Omega^2_{\pm,\mathscr{H}_{r_+}}\Omega^1_\pm H, \quad H_e\Omega^2_{\pm,\mathscr{H}_{r_e}}\Omega^1_\pm = \Omega^2_{\pm,\mathscr{H}_{r_e}}\Omega^1_\pm H. \tag{3.165}$$

Now, since the limits are independent of the choice of $c_\pm \in \mathscr{C}_\pm$, one can always choose $c_\pm$ such that $c_+^2 + c_-^2 = 1$, consequently:

$$\Omega^{2*}_{\pm,\mathscr{H}_{r_e}}\Omega^2_{\pm,\mathscr{H}_{r_e}} + \Omega^{2*}_{\pm,\mathscr{H}_{r_+}}\Omega^2_{\pm,\mathscr{H}_{r_+}} = P_c(H_1). \tag{3.166}$$

One could have also chosen $c_\pm$ such that their supports were disjoint, therefore, we must also have:

$$\Omega^2_{\pm,\mathscr{H}_{r_e}}\Omega^{2*}_{\pm,\mathscr{H}_{r_+}} = \Omega^2_{\pm,\mathscr{H}_{r_+}}\Omega^{2*}_{\pm,\mathscr{H}_{r_e}} = 0. \tag{3.167}$$

In other words, relation (3.166) is an orthogonal sum decomposition of the absolutely continuous subspace of H_1 and the operators (3.161) are partial isometries. We therefore have a decomposition of $P_c(H_1)$ into incoming and outgoing states. In what follows, to simplify notations, we consider only the direct wave operators, analogous statements can be formulated for the reverse ones. Define:

$$X^{H_1}_{\text{in}} = (\ker\Omega^2_{+,\mathscr{H}_{r_e}})^\perp, \quad X^{H_1}_{\text{out}} = (\ker\Omega^2_{+,\mathscr{H}_{r_+}})^\perp.$$

In virtue of Equation (3.166), these subspaces have nice characterisations, indeed: $X^{H_1}_{\text{in}}$ is exactly $\ker\Omega^2_{+,\mathscr{H}_{r_+}} \cap P_c(H_1)\mathscr{H}$ and $\phi \in \ker\Omega^2_{+,\mathscr{H}_{r_+}} \cap P_c(H_1)\mathscr{H}$, if and only if :

$$\lim_{t\to+\infty} ||c_+(r^*)e^{-itH_1}\phi|| = 0,$$

for any $c_+ \in \mathscr{C}_+$. In other words, the states in $X^{H_1}_{\text{in}}$ are exactly those whose energy is concentrated on $\mathbb{R}_-$ at late times. Similarly, $\phi \in X^{H_1}_{\text{out}}$ if and only if:

$$\lim_{t\to+\infty} ||c_-(r^*)e^{-itH_1}\phi|| = 0,$$

for any $c_- \in \mathscr{C}_-$. An important point is that $\Omega^2_{+,\mathscr{H}_{r_e}}$ maps $X^{H_1}_{\text{in}}$ onto a similar subspace for H_e (and similarly at $\mathscr{H}_{r_+}$ for H_0). If ψ is in the range of $\Omega^2_{+,\mathscr{H}_{r_e}}$, then there is $\phi \in X^{H_1}_{\text{in}}$ such that:

$$\lim_{t\to+\infty} ||e^{-itH_e}\psi - c_-(r^*)e^{-itH_1}\phi|| = 0,$$

for any $c_- \in \mathscr{C}_-$. Hence for any $c_+ \in \mathscr{C}_+$, one can choose $c_- \in \mathscr{C}_-$ with support disjoint from that of c_+ so that:

$$0 = \lim_{t\to+\infty} ||c_+(r^*)e^{-itH_e}\psi - c_+(r^*)c_-(r^*)e^{-itH_1}\phi||,$$
$$= \lim_{t\to+\infty} ||c_+(r^*)e^{-itH_e}\psi||.$$

Conversely, all such states are mapped into $X_{\text{in}}^{H_1}$ by ${\Omega^2}^*_{+,\mathscr{H}_{r_e}}$.

Incoming and outgoing subspaces for H_e and H_0 were originally defined using the asymptotic velocity operators constructed in Section 3.6.6. These operators were constructed on each of the stable subspaces of the respective orthogonal sum decompositions associated to each of the operators, they are:

$$P_e^+ = \text{s}-\lim_{t\to+\infty} e^{itH_e}\Gamma^1 e^{-itH_e}, \quad P_0^+ = \text{s}-\lim_{t\to+\infty} e^{itH_0}\Gamma^1 e^{-itH_0},$$

and satisfy for any $J \in C_\infty(\mathbb{R})$:

$$\begin{aligned} J(P_e^+) &= \text{s}-\lim_{t\to+\infty} e^{itH_e} J(\frac{r^*}{t})e^{-itH_e}, \\ J(P_0^+) &= \text{s}-\lim_{t\to+\infty} e^{itH_0} J(\frac{r^*}{t})e^{-itH_0}. \end{aligned} \tag{3.168}$$

In terms of these operators, $X_{\text{in}}^{H_e} = \text{Ran}\mathbf{1}_{\mathbb{R}_-}(P_e^+) = \text{Ran}\mathbf{1}_{\{-1\}}(P_e^+)$. Using (3.168), one can show that $X_{\text{in}}^{H_e}$ as defined above coincides exactly with the image $\Omega^2_{+,\mathscr{H}_{r_e}} X_{\text{in}}^{H_1}$, for instance, for any $\phi \in \mathscr{H}$,

$$\mathbf{1}_{\{-1\}}(P_e^+)\phi = J(P_e^+)\phi = \lim_{t\to+\infty} e^{itH_e} J(\frac{r^*}{t})e^{-itH_e}\phi,$$

for any $J \in C_0^\infty(\mathbb{R})$ such that $\text{supp}\, J \subset (-\infty, 0), J(-1) = 1$. Hence, for any $c_+ \in \mathscr{C}_+$:

$$\lim_{t\to+\infty} c_+(r^*)e^{-itH_e}\mathbf{1}_{\{-1\}}(P_e^+)\phi = \lim_{t\to+\infty} c_+(r^*)J(\frac{r^*}{t})e^{-itH_e}\phi = 0.$$

The other inclusion is proved in a similar manner, one can show for example that:

$$\lim_{t\to+\infty} c_+(r^*)e^{-itH_e}\phi = 0, \text{for any } c_+ \in \mathscr{C}_+ \Rightarrow \phi \in \text{Ran}\mathbf{1}_{\{1\}}(P_e^+)^\perp. \tag{3.169}$$

Indeed, let ϕ satisfy the condition and let $\psi \in \text{Ran}\mathbf{1}_{\{1\}}(P_e^+)$. A similar argument to the one above shows that for any $c_- \in \mathscr{C}_-$:

$$\lim_{t \to +\infty} c_-(r^*) e^{-itH_e} \psi = 0.$$

Choose now $c_+ \in \mathscr{C}_+, c_- \in \mathscr{C}_-$ such that $c_+ + c_- = 1$, then for $t \in \mathbb{R}$:

$$\begin{aligned}(\phi, \psi) &= (e^{-itH_e}\phi, e^{-itH_e}\psi),\\ &= (c_+(r^*)e^{-itH_e}\phi, e^{-itH_e}\psi) + (e^{-itH_e}\phi, c_-(r^*)e^{-itH_e}\psi).\end{aligned} \tag{3.170}$$

By the Cauchy-Schwarz inequality, it follows that for any $t \in \mathbb{R}$:

$$|(\psi, \phi)| \leq ||\psi|| ||c_+(r^*)e^{-itH_e}\phi|| + ||\phi|| ||c_-(r^*)e^{-itH_e}\psi||,$$

The right-hand side approaches 0 as $t \to +\infty$ so that:

$$|(\psi, \phi)| = 0.$$

We can therefore define a global wave operator from the absolutely continuous subspace of H_1 onto the *external* direct sum $\mathrm{Ran}\mathbf{1}_{\{-1\}}(P_e^+) \oplus \mathrm{Ran}\mathbf{1}_{\{1\}}(P_0^+)$.

$$\begin{aligned}\Omega_+^2 : \; X_{\text{in}}^{H_1} \oplus X_{\text{out}}^{H_1} &\longrightarrow \mathrm{Ran}\mathbf{1}_{\{-1\}}(P_e^+) \oplus \mathrm{Ran}\mathbf{1}_{\{1\}}(P_0^+)\\ (\phi_1, \phi_2) &\longmapsto (\Omega^2_{+,\mathscr{H}_{r_e}}\phi_1, \Omega^2_{+,\mathscr{H}_{r_+}}\phi_2).\end{aligned} \tag{3.171}$$

3.7.3 Comparison III

Although the results in Section 3.6.5 can be used to construct a scattering theory for H_e and H_0 on the whole Hilbert space, the previous discussion shows that, for our needs, it only relevant to do this on $\mathrm{Ran}\mathbf{1}_{\{-1\}}(P_e^+)$ for H_e and on $\mathrm{Ran}\mathbf{1}_{\{1\}}(P_0^+)$ for H_0. The asymptotic profiles are given by:

$$\begin{aligned}H_{-\infty} &= \Gamma^1 D_{r^*},\\ H_{+\infty} &= \Gamma^1 D_{r^*} + \left(\frac{a}{r_+^2 + a^2} - \frac{a}{r_e^2 + a^2}\right) p.\end{aligned} \tag{3.172}$$

The outgoing and incoming states are identical for both of these operators and given by:

$$\mathscr{H}^+ = \mathrm{Ran}\mathbf{1}_{\{1\}}(\Gamma^1), \quad \mathscr{H}^- = \mathrm{Ran}\mathbf{1}_{\{-1\}}(\Gamma^1).$$

Due to the stability of the subspace under $\Gamma^1, H_e, H_{\pm\infty}$, the results in Section 3.6.5 prove that the following strong limits exist:

$$\begin{aligned}
\Omega^3_{+,\mathscr{H}_{r_+}} &= \underset{t\to+\infty}{\mathrm{s-lim}}\, e^{itH_{+\infty}} e^{-itH_0} \mathbf{1}_{\mathbb{R}_+}(P_0^+), \\
\Omega^3_{+,\mathscr{H}_{r_e}} &= \underset{t\to+\infty}{\mathrm{s-lim}}\, e^{itH_{-\infty}} \left(T \exp\left(-i\int_0^t \tilde{f}(\Gamma^1 s)\mathrm{d}s\right)\right)^* e^{-itH_e} \mathbf{1}_{\mathbb{R}_-}(P_e^+), \\
\tilde{\Omega}^3_{+,\mathscr{H}_{r_+}} &= \underset{t\to+\infty}{\mathrm{s-lim}}\, e^{itH_0} e^{-itH_{+\infty}} \mathbf{1}_{\mathbb{R}_+}(\Gamma^1), \\
\tilde{\Omega}^3_{+,\mathscr{H}_{r_e}} &= \underset{t\to+\infty}{\mathrm{s-lim}}\, e^{itH_e} T\exp\left(-i\int_0^t \tilde{f}(\Gamma^1 s)\mathrm{d}s\right) e^{-itH_{-\infty}} \mathbf{1}_{\mathbb{R}_-}(\Gamma^1),
\end{aligned}$$

One also has: $\tilde{\Omega}^3_{+,\mathscr{H}_{r_+}} = {\Omega^3}^*_{+,\mathscr{H}_{r_+}}$ and similarly for $\mathscr{H}_{r_e}$, this gives rise to a unitary map:

$$\begin{aligned}
\Omega^3_+ :\ \mathrm{Ran}\mathbf{1}_{\{-1\}}(P_e^+) \oplus \mathrm{Ran}\mathbf{1}_{\{1\}}(P_0^+) &\longrightarrow \quad \mathscr{H}^- \oplus \mathscr{H}^+ = \mathscr{H} \\
(\phi_1, \phi_2) &\longmapsto (\Omega^3_{+,\mathscr{H}_{r_e}}\phi_1, \Omega^3_{+,\mathscr{H}_{r_+}}\phi_2).
\end{aligned}$$

Finally, composition of $\Omega^1_+, \Omega^2_+, \Omega^3_+$ yields a unitary map W_+ between $P_c(H) = X^H_{\mathrm{in}} \oplus X^H_{\mathrm{out}}$ and $\mathscr{H}$, where:

$$X^H_{\mathrm{in}} = (\ker \Omega^2_{+,\mathscr{H}_{r_e}}\Omega^1_+)^\perp, \quad X^H_{\mathrm{out}} = (\ker \Omega^2_{+,\mathscr{H}_{r_+}}\Omega^1_+)^\perp,$$

given by:

$$\begin{aligned}
W_+ :\ X^H_{\mathrm{in}} \oplus X^H_{\mathrm{out}} &\longrightarrow \mathscr{H}^- \oplus \mathscr{H}^+ = \mathscr{H} \\
\phi_1 + \phi_2 &\longmapsto \Omega^3_{+,\mathscr{H}_{r_e}}\Omega^2_{+,\mathscr{H}_{r_e}}\Omega^1_+\phi_1 + \Omega^3_{+,\mathscr{H}_{r_+}}\Omega^2_{+,\mathscr{H}_{r_+}}\Omega^1_+\phi_2.
\end{aligned}$$

Remark 3.7.1. A simple application of the above result is to define the asymptotic velocity operator for the full dynamics. It is defined by the limits for $J \in C_\infty(\mathbb{R})$,

$$J(P^+) = \underset{t\to+\infty}{\mathrm{s-lim}}\, e^{iHt} J(\frac{r^*}{t}) e^{-iHt} = W_+^* J(\Gamma^1) W_+,$$

Using the results discussed in Section 3.6.6, it follows that: $P^+ = W_+^*\Gamma^1 W_+$.

3.7.4 Scattering for the Dirac operator

We now return to the notations we adopted prior to Section 3.6, where we dropped the explicit dependence of our operator H^p for notational convenience. We recall from Section 3.4.2 that H^p coïncides with the full Dirac operator on each of the subspaces associated with the eigenvalue $p \in \mathbb{Z} + \frac{1}{2}$ of D_ϕ. The global wave operators obtained in the previous section, although defined on all of $\mathscr{H}$, also depend on the parameter p. However the p-eigenspace is stable so that to obtain the scattering theory for the Dirac operator one only need to reassemble each of the harmonics. Since the Dirac operator has no pure point spectrum[41], there is no need to project onto the absolutely continuous subspace. Therefore, we state our final theorem:

Theorem 3.7.1. *For any* $\phi = \sum_{p \in \mathbb{Z}+\frac{1}{2}} \phi_p(r^*, \theta) e^{ip\phi} \in \mathscr{H}$ *set:*

$$\mathscr{P}^+ \phi = \sum_{p \in \mathbb{Z}+\frac{1}{2}} P_p^+ \phi_p e^{ip\phi}, \tag{3.173}$$

then $\mathscr{P}_+$ *is a bounded symmetric operator with spectrum* $\{-1, 1\}$*, and for any* $J \in C_\infty(\mathbb{R})$*,*

$$J(\mathscr{P}_+) = \mathrm{s}-\lim_{t \to +\infty} e^{iHt} J(\frac{r^*}{t}) e^{-iHt}.$$

Moreover, defining:

$$X_{in} = Ran\ \mathbf{1}_{\{-1\}}(\mathscr{P}^+), \quad X_{out} = Ran\ \mathbf{1}_{\{1\}}(\mathscr{P}^+),$$

then, $\mathscr{H} = X_{in} \oplus X_{out}$ *and the operator:*

$$\mathfrak{W}^+ \phi = \sum_{p \in \mathbb{Z}+\frac{1}{2}} W_+^p \phi_p e^{ip\phi}, \tag{3.174}$$

is a unitary operator such that:

$$\mathfrak{W}_+ X_{in} = \mathscr{H}_-, \quad \mathfrak{W}_+ X_{out} = \mathscr{H}_+,$$

41. see again [BC09]

and for the full Dirac operator $H + i\frac{a}{r_e^2+a^2}\partial_\phi$, *with* H *defined by Equation* (3.36)*:*

$$H_{-\infty}\mathfrak{W}_+\mathbf{1}_{\{-1\}}(\mathscr{P}^+) + H_{+\infty}\mathfrak{W}_+\mathbf{1}_{\{1\}}(\mathscr{P}^+) = \mathfrak{W}_+ H,$$

with:

$$H_{+\infty} = \Gamma^1 D_{r^*} + \left(\frac{a}{r_+^2 + a^2} - \frac{a}{r_e^2 + a^2}\right) D_\phi, \quad H_{-\infty} = \Gamma D_{r^*}$$

3.8 Conclusion

In this paper we have proposed an analytical construction for a scattering theory for particules in a region situated between a double and simple horizon of an extreme Kerr-de Sitter blackhole. The presence of the simple horizon alone simplified the problem considerably, being an obstruction to the existence of pure-point spectrum, and the existence of a conjugate operator in the sense of Mourre theory ruled out the possibility for any singular continuous spectrum. The setting was therefore ideal for an analytic scattering theory.

We found that, from an analytical point of view, the double horizon region was analogous to that of spacelike infinity in Kerr-Newmann spacetime. The theory is in fact slightly easier because the mass terms do no persist at the horizons, meaning that things appear to boil down to the massless case. As in this case, the reasoning hinges on the ability to obtain a minimal velocity estimate.

The main difference and novelty is that the double horizon exacerbates the effects of the rotation of the black hole by complicating the structure of the angular operator; the mass also plays a lesser role here. However, this did not prove to be an essential difficulty for the analytic methods used in this paper, which is another illustration of their robustness.

The methods used here do however have the clear disadvantage of not being very geometrical. In some sense, the study of the effects of the double horizon is reduced to the distinction between long and short-range potentials; it would be considerably more satisfying to seek a proof of the results in this paper with a clearer geometrical meaning.

Chapter 4

Projective differential geometry and asymptotic analysis

4.1 Introduction

« What is geometry ? » : It was long thought that the geometry we learn about in our early years at school, that is, in a naive sense, the study of figures in a plane or in 3D space, was the only sort of geometry. By this I mean that the popular sentiment amongst anyone interested in such questions was that the basic underlying axioms, upon which the rest of the theory is built, are so plain and clear that it would seemingly be nonsense to exclude any of them. Of course, there was always the famous « Parallel postulate », which, dealing with infinity, was arguably less obvious than the others, but there was a strong belief that, in fact, it was not an Axiom, but a Theorem that could be proved. In the XIXth century, it was established that the « Parallel postulate » is actually logically independent from the other axioms, and thus flourished a whole panoply of new examples of « geometries » in which it is false, such as projective, hyperbolic or spherical geometry. Naturally, this lead to the question of a possible common framework in which to think about all of these examples and it was F. Klein and his Erlangen program, that gave the answer which pervades the modern conception of geometry.

Post-Klein, geometry is the study of a (transitive) action on a set X, by a group of « symmetries » G. Since, for a transitive action, the set X is in one-to-one correspondence with the quotient set G/H where H is the stabiliser of any point of X, and the action is equivalent to the natural action of G on G/H, one can first restrict to this case. In fact, all the « classical » geometries are covered by this :

— Affine geometry : $G = \mathrm{Aff}(n) = \mathbb{R}^n \rtimes GL_n(\mathbb{R})$, $H = GL_n(\mathbb{R})$,

— Euclidean geometry : $G = \mathbb{R}^n \rtimes SO_n(\mathbb{R})$, $H = SO_n(\mathbb{R})$,

— Projective geometry : $G = GL_{n+1}(\mathbb{R})$ and:

$$H = \left\{ \begin{pmatrix} \lambda & v \\ 0 & A \end{pmatrix} \in GL_{n+1}(\mathbb{R}), \lambda \in \mathbb{R}^*, A \in GL_{n+1}(\mathbb{R}) \right\},$$

— Lorentzian geometry, $G = \mathbb{R}^n \rtimes SO_+(1,n)$, $H = SO_+(1,n)$.

In all the above examples, G is a Lie group and H a closed subgroup of G; therefore the quotient space G/H has a natural manifold structure. The reader may be surprised by the groups used in our description of projective geometry, as she probably expected to see the usual projective group $GL_{n+1}(\mathbb{R})/N$ where N is the normal subgroup formed by the homothetic transformations of $\mathbb{R}^{n+1}$. There is, in fact, a diffeomorphism : $GL_{n+1}(\mathbb{R})/H \cong (GL_{n+1}(\mathbb{R})/N)/(H/N)$; the advantage of the groups above is that they are matrix Lie groups.

E. Cartan took Klein's program further, by defining curved versions of Klein geometries, known as Cartan geometries, of which more familiar pseudo-Riemannian geometry is actually an example. Before exploring this direction further in paragraph 4.1.1, let us first note that on the same manifold X, it is completely possible to consider the smooth action of different groups nested in one another. In our particular case it will be interesting to consider a larger group than the initial group G. This will possibly result in reducing the number of geometric invariants, causing a probable loss of information, but, it is sometimes the case that the weakened structure has an extension to a larger space containing X. It is in this spirit, that in General Relativity, we sometimes seek what is known as geometric *compactifications* of a pseudo-Riemannian geometry (M, g). The most prominent example being that of conformal compactifications. In a sense to be later made precise, a spacetime (M, g) is a curved version of Lorentzian geometry as described above. To construct a possible conformal compactification, we weaken the geometric structure by allowing conformal variations of the metric $g \to \Omega^2 g$. The metric g itself is therefore no longer a geometric invariant, but its conformal class $[g]$ is. Via this operation we have implicitly replaced the G of Lorentzian geometry by the larger conformal group $C(1,n)$.

A conformal compactification is possible when M can be made into the interior of a larger manifold with boundary $\overline{M} = M \cup \partial M$ such that the conformal class extends to the boundary ∂M. The boundary is then referred to as the (conformal) infinity of spacetime and one can use $\overline{M}$ to study the asymptotic behaviour of objects living in M. The reader will find an excellent introduction to this topic in [18].

Conformal geometry is very rich, and conformal compactifications can be useful in the study of the wave equation. The underlying reason for this success is, in addition to the existence of a conformally invariant Laplace operator, that the light-cone structure remains a geometric invariant. However, for fields with mass on asymptotically flat space-times, it is likely that this is not the right structure, because the part of the boundary where we would want to encode the asymptotic information is generally reduced to two points !

As a possible remedy to this drawback, in what follows we shall consider a different type of geometric compactification, first introduced in [ČG14], known as a projective compactification. The picture is very similar to the conformal case: we want to weaken the geometric structure of (M, g), but, instead of just the light cone, we would like the set of all oriented unparametrised geodesics to be a geometric invariant of the new geometry. Therefore it is the Levi-Civita connection of g, rather than g itself, that will be at the heart of our considerations. The idea is to consider the class $[\nabla_g]$ of all torsion-free connections $\hat{\nabla}$ on TM that have the same unparametrised geodesics as ∇_g. We will find that, from the perspective of Klein/Cartan, the group $\mathbb{R}^{n+1} \rtimes SO_+(1,n)$ is enlarged to $SL_{n+1}(\mathbb{R})$.

Minkowski spacetime has a projective compactification, that we will describe in Section 4.4, and the projective infinity has a very interesting structure: it splits up into different orbits, each of which one can identify with either timelike, spacelike and lightlike infinity, and, what is more, time-like infinity is not just a singleton. This turns out to be characteristic of possible projective compactifications of scalar flat pseudo-Riemannian manifolds, cf. [FG18]. It is our hope that, thanks to this, the projective compactification will enable us to develop analogous techniques to those of conformal geometry to massive equations on asymptotically flat spacetimes.

Other results corroborate the hope we place in projective compactifications, and in particular, a result due to L. Hörmander [Hör97, Theorem 7.2.7] with an implicit projective flavour. Hörmander derives an asymptotic expansion of solutions to the Klein-Gordon equation on Minkowski space time $M \cong \mathbb{R}^{1+n}$ in which the coefficients are only dependent on the projective parameter $\frac{x}{t}$, which leads us to believe that they can be interpreted in terms of the projective compactification. However, his proof relies on a decomposition of the field into positive and negative frequency parts. This is achieved through the diagonalisation, in Fourier space, of the Klein-Gordon operator with domain $L^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ defined by:

$$\mathrm{KG} = \begin{pmatrix} 0 & \Delta - m^2 \\ 1 & 0 \end{pmatrix}.$$

This « diagonalisation » reduces the study of the equation $\frac{\partial\psi}{\partial t} = \mathrm{KG}\psi$ to that of two uncoupled equations involving the pseudo-differential operators $\pm i\sqrt{-\Delta+m^2}$. The asymptotic expansion results from a precise study of an integral formula for the solutions to these equations. Precisely, Hörmander shows that if $u \in S'(\mathbb{R}^{1+n})$ is a solution of :

$$\begin{cases} \partial_t u = i(-\Delta+1)^{\frac{1}{2}}u, \\ u(0,x) = \varphi(x), \varphi \in S(\mathbb{R}^n), \end{cases}$$

then $u(t,x) = U_0(t,x) + U_+(t,x)e^{\frac{i}{\rho}}$, with $U_0 \in S(\mathbb{R}^{1+n})$, $\rho = (t^2-|x|^2)^{-\frac{1}{2}}$, and [1]:

$$U_+(t,x) \sim (+0+i\rho)^{\frac{n}{2}} \sum_0^\infty \rho^j w_j(t,x).$$

Setting $\tilde{x} = \rho x$ and writing the Fourier transform of φ, $\hat{\varphi}$, we have furthermore:

$$w_0(t,x) = \begin{cases} (2\pi)^{-n/2}\sqrt{1+|\tilde{x}|^2}\hat{\varphi}(-\tilde{x}) & \text{if } t^2 > |x|^2, \\ 0 & \text{otherwise.} \end{cases}$$

From this, we notice immediately that w_0 is a function of $\frac{x}{t}$; it can be shown that this is also the case for $w_j, j > 0$. Moreover, the functions $(\rho, \tilde{x})$ define a coordinate system on the interior of the future lightcone $\{t > |x|\}$. These coordinates are regular at $\rho = 0$, which can be identified with future timelike infinity in the projective compactification. This gives further motivation to the conjecture that the coefficients w_j can be interpreted geometrically in terms of the compactified space.

However, a major difficulty of projective geometry in the study of equations with physical meaning, is that the connections in a projective class $\boldsymbol{p} = [\nabla]$ are not necessarily the Levi-Civita connection for some metric. Hence, unlike in conformal geometry, there is no induced change on the metric when we change connection : g is fixed and may, or may not, have an extension to the boundary. By consequence, once we have parted from the physical Levi-Civita connection, not of all the operators we can write down will have a clear-cut physical interpretation. For instance, even if $\hat{\nabla}$ is projectively equivalent to the Levi-Civita connection of some metric g, then $g^{ab}\hat{\nabla}_a\hat{\nabla}_b$ is not, a priori, the Laplace operator for some other pseudo-Riemannian metric.

For a general projective class $\boldsymbol{p}$, the presence of a Levi-Civita connection is governed

1. $(+0+i\rho)^{n/2} = \lim_{\varepsilon\to 0^+} \exp(\frac{n}{2}\log(\varepsilon+i\rho)) = \rho^{n/2}e^{in\pi/4}$ where log the Principal complex logarithm.

by the so-called Metrisability equation [EM08]:

$$\nabla_c \sigma^{ab} - \frac{2}{n+1} \nabla_d \sigma^{d(a} \delta_c^{b)} = 0. \tag{ME}$$

In the above, the unknown is $\sigma^{ab} \in S^2TM(-2)$ [2]. The Metrisability equation is projectively invariant, i.e. if σ^{ab} satisfies (ME) for some connection $\nabla \in \boldsymbol{p}$ then it satisfies it for any connection $\hat{\nabla}$ in the class. When there is a solution the projective class is said to be *metric*.

Any solution σ^{ab} to the Metrisability equation induces a symmetric bilinear form $g = \sigma \sigma^{ab}$ whose Levi-Civita connection is in the projective class. Here :

$$\sigma = \varepsilon^2_{a_1 a_2 \dots a_n b_1 \dots b_n} \sigma^{a_1 b_1} \dots \sigma^{a_n b_n},$$

and $\varepsilon^2_{a_1 a_2 \dots a_n b_1 \dots b_n}$ is the canonical [3] map $\Lambda^n TM \otimes \Lambda^n TM \to \mathscr{E}(2n+2)$.

The precise geometry of solutions to (ME) and the relationship with projective compactifications are studied in [FG18]. In our particular problem, we start with a Lorentzian manifold (M, g), which leads to a solution of (ME). So we are mainly interested in the consequences that this has, in particular, the fact that each solution σ^{ab} corresponds to a section H^{AB} of a certain « tractor » bundle. These bundles will be introduced in Section 4.3 and we will discuss Equation (ME) in more detail in Section 4.5.1. For now, we note simply that they can be defined as tensor powers of the 1-jet prolongation of $\mathscr{E}(1)$, $J^1\mathscr{E}(1)$ and its dual; this is the approach adopted in [BEG94; ČGM14; ČG14]. Our path will be closer in spirit to Cartan's work.

The fact that we work with a class of affine connections, rather than one in particular, means that many of the expressions involving a covariant derivative that we write on TM will not make sense on the projective manifold $(M, \boldsymbol{p})$. This is because they will depend on the connection used to write them down. It is sometimes possible to recover invariance if we work with weighted tensors, as was the case in the Metrisability equation (ME). Another example of this is the Projective Killing Equation :

$$\nabla_{(a} v_{b)} = 0, \tag{4.1}$$

2. Recall that if $\mathcal{B}$ is a vector bundle on M, $\mathcal{B}(\omega)$ is the tensor bundle $\mathcal{B} \otimes \mathscr{E}(\omega)$ where $\mathscr{E}(\omega)$ is the bundle of projective densities of weight $\omega \in \mathbb{R}$, cf. Definition 1.4.1.

3. $\Lambda^n TM \otimes \Lambda^n TM$ is canonically oriented, so this map exists even if M is not orientable, if M is oriented, then we can consider the « square » of a volume form.

which is projectively invariant if $v_b \in \mathscr{E}_b(2)$. Alternatively, a very important property of the tractor bundles is that a projective class $\boldsymbol{p}$ of affine connections on TM determines *an* affine connexion on them. This connection enables us to define an operator on projective densities and weighted tractors, known as the Thomas D-operator, D_A, which satisfies the Leibniz rule and can be iterated. It is a major tool for producing projectively invariant differential operators. In particular, equipped with a solution to the Metrisability equation, one can consider the operator $H^{AB}D_A D_B$; a natural candidate for a projective Laplacian operator. Unfortunately, for scalar flat metrics, H^{AB} is degenerate and the operator $H^{AB}D_A D_B$, will contain no terms that we can assimilate to a mass.

The work presented in this final chapter are the first steps towards understanding how one apply these results to the asymptotic analysis of fields. The original ambition was to obtain a geometric proof, via methods of projective differential geometry, of Hörmander's result in Minkowski spacetime. However, it turned out that Minkowski spacetime and, more generally, scalar flat metrics are an exceptional case where a part of the structure degenerates, and it is not quite clear as to how one can overcome the obstructions this entails. This realisation lead me to study in deeper detail the non-scalar flat case, where a number of results already exist in conformal geometry and to question to what extent there are projective analogues. The main bulk of this work is presented in Section 4.7 where a projective exterior tractor calculus is developed that enables us to obtain a formal solution operator for the Proca equation, extending to projective geometry results parallel to those developed in [GW14; GLW15] in Conformal geometry.

My understanding of the topic was significantly advanced during a trip to Auckland in New Zealand financed by the University of Western Brittany, the Brittany region and ED MathSTIC to whom I express once more my gratitude.

4.1.1 Cartan geometries

In his article [Car23], E. Cartan gives an alternative definition of affine connections to the one used in standard textbooks on Differential Geometry. This definition is also distinct, although closer in spirit, to the Principal Connection version we discussed in Section 1.3 : it is based on a $\mathfrak{aff}(n)$- valued differential form (as opposed to a $\mathfrak{gl}_n$ valued one). The picture behind Cartan's definition is to first imagine attaching to each point of a manifold a copy of affine space. In Cartan's mind, the information then required to locally identify a small open subset of the manifold to an open subset of affine space is a « rule » that describes how to merge into one the affine spaces attached to neighbouring points. Assume we assign to each point an affine frame, based at that point, of the space attached to it in such a way that they vary smoothly. After merging the affine spaces attached to two infinitely neighbouring points, then, the relationship between the two frames is an affine transformation « infinitely close » to the Identity map : this can be encoded in an element of $\mathfrak{aff}(n)$, and, in fact, all the useful information of the « merging rule » is contained in this form.

Cartan generalises these ideas to the projective group in [Car24] and later, to other Klein geometries G/H. Over the space G/H there is a canonical structure that globally encodes the information of the « infinitesimal transformations » described above: the Maurer-Cartan form of G. Recall that it is the $\mathfrak{g}$-valued differential form defined for any vector field $X \in \Gamma(TG)$ by:

$$\theta_G(X)(g) = (dL_{g^{-1}})_g(X).$$

With the canonical projection of G onto G/H, G can be seen as a smooth H-principal fibre bundle over the base G/H, which we will think of as a frame bundle. With respect to this structure, θ_G satisfies the following :

1. $\forall h \in H, R_h^*\theta_G = \mathfrak{ad}_{h^{-1}}\theta_G$.
2. For any $X \in \mathfrak{h}$, the fundamental field [4] X^* satisfies:
$$\theta_G(X^*) = X.$$
3. For each $p \in G$, $\theta_{G_p} : T_pG \to \mathfrak{g}$ is a vector space isomorphism.
4. $$\mathrm{d}\theta_G + \frac{1}{2}[\theta_G \wedge \theta_G] = 0. \text{ [5]} \tag{4.2}$$

To recover Cartan's local gauge version, we can simply take a section $G/H \to G$, and use it to pull-back θ_G over X. The section can be thought of as the smooth assignment of a frame in the affine space attached to each of the points we mentioned in the introduction.

The above properties have a close ressemblance to Definition 1.3.6, but, θ_G is clearly not a principal connection on G (seen as a H-principal bundle over G/H) since it maps to $\mathfrak{g}$ and not $\mathfrak{h}$. Furthermore, Condition 3 shows that at each point the kernel is $\{0\}$ and

4. cf. Definition 1.3.5
5. If α, β are $\mathfrak{g}$ valued 1-forms then for all vector fields X, Y, we set: $[\alpha \wedge \beta](X, Y) = [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)]$ where $[\ ,\ ]$ is the Lie bracket in $\mathfrak{g}$, see also Appendix C.

so it does not define an interesting horizontal distribution.

Equation (4.2), in fact, characterises locally the manifold G/H; it is an integrability condition. Due to this, the equation should not be imposed in the curved version: the failure to respect this condition is a measure of curvature. We are therefore lead, to the following definition (cf. [Sha97, Definition 5.3.1])

Definition 4.1.1. Let G be a Lie group, H a closed subgroup of G and $\mathfrak{g}, \mathfrak{h}$ be their respective Lie algebras. A Cartan geometry, (P, ω) modeled on (G, H) is a smooth manifold M and:

— a H-principal fibre bundle over M, (P, π, M),

— a $\mathfrak{g}$-valued differential form, ω, on P that satisfies :

1. for any $h \in H, R_h^* \omega = \mathfrak{ad}_{h^{-1}} \omega$,
2. for any $X \in \mathfrak{h}$, $\omega(X^*) = X$,
3. for each $p \in P$, $\omega_p : T_p P \to \mathfrak{g}$ is a vector space isomorphism.

Although the Cartan connection ω is not a principal connection on P, it induces a principal connection α on the associated fibre bundle $Q = P \times_H G$ where H acts on G by left multiplication. [6].

Note that Definition (4.1.1) has an equivalent version analogous to Definition 1.3.4:

Definition 4.1.2. Let G be a Lie group, H a closed subgroup and $\mathfrak{g}, \mathfrak{h}$ their respective Lie algebras. A Cartan connection on M is a family of $\mathfrak{g}$-valued differential forms, (ω_U), associated with an open cover $\mathscr{U}$ of the manifold M, and a family of transition functions $h_{UV} : U \cap V \to H, U, V \in \mathscr{U}$ such that $\omega_{U x} \mathrm{mod}\ \mathfrak{h} : T_x M \to \mathfrak{g}/\mathfrak{h}$ is a vector space isomorphism for each $x \in U$ and:

$$\omega_V = h_{UV}^* \theta_H + \mathfrak{ad}_{h_{UV}^{-1}} \omega_U, \tag{4.3}$$

where θ_H is the Maurer-Cartan form of H.

Remark 4.1.1. The definition given here is apparently more restrictive than the one in [Sha97] because we require that the transition functions h_{UV} be given in advance. Sharpe [Sha97] shows that this is really superfluous when the largest normal subgroup of G contained in H is simply $\{e\}$; if this condition is satisfied, the geometry is called an effective Cartan geometry.

6. For instance, we can construct ω in local bundle charts of Q.

In conclusion to this introduction, we would like to explain the link between Cartan's notion of affine connection and the classical Definition 1.3.1. Suppose that P is a Cartan geometry on a manifold M modeled on affine space. The key is to note that the so-called adjoint representation of $GL_n(\mathbb{R})$ on $\mathfrak{aff}_n(\mathbb{R})$ is reducible and has the following stable direct sum decomposition :

$$\mathfrak{aff}_n(\mathbb{R}) = \mathfrak{gl}_n(\mathbb{R}) \oplus \mathbb{R}^n. \tag{4.4}$$

Consequently, the Cartan connection ω can be split as: $\omega = \alpha + \eta$. Since $\mathfrak{gl}_n(\mathbb{R})$ is stable under the adjoint action of $GL_n(\mathbb{R})$ on $\mathfrak{aff}_n(\mathbb{R})$, α is actually a Principal Connection on P. Similarly, studying the restriction of this action on the stable subspace $\mathbb{R}^n$, we find that η is an equivariant[7] ($R_g^*\eta = g^{-1}\eta$) and horizontal (if $X \in \ker \mathrm{d}\pi_p, \eta_p(X) = 0$) $\mathbb{R}^n$-valued one form[8] known as the solder form. The pair (α, η) is completely equivalent to ω. However, on the frame bundle $L(TM)$ of the tangent bundle, there is a canonical choice for η, specifically $p = (x, u_x) \in L(TM)$, $X \in T_pL(TM)$,

$$\eta_p(X) = u_x^{-1}(\mathrm{d}\pi_p(X)).$$

It follows that one only needs to specify a principal connection on $L(TM)$ in order to define an affine connection in Cartan's sense. Note that this generalises *mutatis mutandis* to the frame bundle of any vector bundle. Our usual affine connections on vector bundles are therefore equivalent to Cartan affine connections, with the canonical choice of solder form.

4.2 Projective differential geometry

4.2.1 The Model

Unlike Cartan we will not quite work with classical projective space per se. In fact our model is an oriented version of projective geometry : $SL_{n+1}(\mathbb{R})/H$. Here :

$$H = \left\{ \begin{pmatrix} (\det(A))^{-1} & v \\ 0 & A \end{pmatrix} \in SL_{n+1}(\mathbb{R}), A \in GL_n^+(\mathbb{R}), v \in M_{1,n}(\mathbb{R}) \cong (\mathbb{R}^n)^* \right\}. \tag{4.5}$$

7. For the fundamental representation of $GL_n(\mathbb{R})$.
8. This means that it is a TM-valued one form, see Appendix C.

The model is easier to imagine as the set of all oriented rays [9],$P_+(\mathbb{R}^{n+1})$, in $\mathbb{R}^{n+1}$, or, equivalently, the quotient space of $\mathbb{R}^{n+1}\backslash\{0\}$ under the natural action of the multiplicative group $\mathbb{R}_+^*$ on $\mathbb{R}^{n+1}$. The reader can find a very complete introduction to the topic in [Sto87].

For our needs, a major advantage is that the action of $SL_{n+1}(\mathbb{R})$ on $P_+(\mathbb{R}^{n+1})$, defined by the commutative diagram in Figure 4.1, is effective. One can also remark that the largest normal subgroup of $SL_{n+1}(\mathbb{R})$ contained in H is $\{\mathrm{Id}\}$, and so it is also an effective Klein geometry [10].

$$\begin{tikzcd}
\mathbb{R}^{n+1} \setminus \{0\} \arrow[r, "B"] \arrow[d, "\pi"] & \mathbb{R}^{n+1} \setminus \{0\} \arrow[d, "\pi"] \\
P_+(\mathbb{R}^{n+1}) \arrow[r, "\rho(B)"] & P_+^n(\mathbb{R}^{n+1}).
\end{tikzcd}$$

Figure 4.1 – Definition of the action of $SL_{n+1}(\mathbb{R})$ on $P_+(\mathbb{R}^{n+1})$, π is, as usual, the canonical projection. $B \in SL_{n+1}(\mathbb{R})$.

Topology-wise, $P_+(\mathbb{R}^{n+1})$ is homeomorphic to the n-sphere S^n and is a two-sheeted covering of usual Projective space; for this reason we will also refer to $P_+(\mathbb{R}^{n+1})$ as the « projective sphere ». The notion of homogenous coordinates naturally carries over : the fibre of $\pi : \mathbb{R}^{n+1}\backslash\{0\} \to P_+(\mathbb{R}^{n+1})$ above a point p, is the oriented ray $\mathbb{R}_+^* u = \{\lambda u, \lambda \in \mathbb{R}_+^*\}$ for some certain non-zero vector u. The homogenous coordinates of p are defined to be the equivalence class of all the coordinates of non-zero vectors in the ray, with respect to the canonical basis of $\mathbb{R}^{n+1}$; we will simply write : $[u_1, \ldots, u_{n+1}]$. This leads to a natural description of a differential atlas made up of the open sets :

$$U_{i_0}^+ = \{p \in P_+(\mathbb{R}^{n+1}), u_{i_0} > 0, u \in \pi^{-1}(\{p\})\},$$
$$U_{i_0}^- = \{p \in P_+(\mathbb{R}^{n+1}), u_{i_0} < 0, u \in \pi^{-1}(\{p\})\},$$

where the coordinate maps are defined by:

$$[u_1, \ldots, u_{i_0}, \ldots, u_{n+1}] \mapsto \left(\frac{u_1}{|u_{i_0}|}, \ldots, \frac{u_{i_0-1}}{|u_{i_0}|}, \frac{u_{i_0+1}}{|u_{i_0}|}, \ldots, \frac{u_{n+1}}{|u_{i_0}|} \right) \in \mathbb{R}^n.$$

For later reference, we would like to point out that on, for instance, U_{n+1}^+, there is a

9. as opposed to the set of all lines
10. cf. [Sha97, Chapter 4, §3].

local section $SL_{n+1}(\mathbb{R}) \to SL_{n+1}/H \cong P_+(\mathbb{R}^{n+1})$ given by:

$$\sigma_{U^+_{n+1}} = [u_1, \dots, u_{n+1}] \longmapsto \begin{pmatrix} 1 & 0 \\ \gamma([u_1, \dots, u_{n+1}]) & I_n \end{pmatrix},$$

$$\gamma([u_1, \dots, u_{n+1}]) = \begin{pmatrix} \frac{u_1}{u_{n+1}} \\ \vdots \\ \frac{u_n}{u_{n+1}} \end{pmatrix}.$$

Moreover, if $p = [u_1, \dots, u_{n+1}] \in U^+_{n+1}$, then setting $X_i(p) = \frac{u_i}{u_n}$, the pull-back by $\sigma_{U^+_{n+1}}$ of the Maurer-Cartan form ω_G can be expressed as :

$$\sigma^*_{U^+_{n+1}} \omega_G = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \mathrm{d}X_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{d}X_n & 0 & \dots & 0 \end{pmatrix}. \tag{4.6}$$

It will be useful to have this in mind when we discuss the projective compactifications of affine and Minkowski space in Paragraph 4.4.

Similarly to classical projective geometry, there is also an affine model of oriented projective space : consider in $\mathbb{R}^{n+1}$ the two affine hyperplanes $H_\pm = \{x_{n+1} = \pm 1\}$. Any oriented ray, not lying in the hyperplane $H_\infty = \{x_{n+1} = 0\}$, meets exactly one of $H_\pm$ at exactly one point. $P_+(\mathbb{R}^{n+1}) \setminus \pi(H_\infty \setminus \{0\})$ is hence in one-to-one correspondence with the union of these two planes. $\pi(H_\infty \setminus \{0\})$ plays once more the role of « hyperplane at infinity ». One can think of $H_\pm$ as the two faces of a same sheet of paper, the front-side being positively oriented and the back-side negatively so.

The primitive objects of this geometry are « oriented subspaces ». In order to preserve the useful notion of projective duality, if $\mathscr{B}$ and $\mathscr{B}'$ are two bases of the same linear subspace $V \subset \mathbb{R}^{n+1}$ we are lead to distinguish the image of the subspace generated by $\mathscr{B}$ from that of $\mathscr{B}'$ when the two bases have different orientation. The important point for us, specifically for our later description of the projective compactification of Minkowski spacetime, is that if we restrict to one of the faces of oriented projective space, we recover a model for an oriented vector space. Additionnally it is compactified by, intuitively, appending two points to each line.

In what follows, we will refer to a Cartan geometry over M with model geometry $(SL_{n+1}(\mathbb{R}), H)$ (H is defined by (4.5)) as a *projective structure* over M.

4.2.2 Projectively equivalent affine connections

Given a smooth manifold M and an affine connection ∇ on its tangent bundle, the unparametrised geodesics of ∇ give rise to a projective structure, in the sense defined above, on M. This result was discovered independently by É. Cartan [Car24] and T. Thomas [Tho34]. In his work, Cartan focusses on the projective connection, whereas Thomas defines an affine connection in the usual sense on a new vector bundle, later to be called the « tractor bundle » when the theory was rediscovered by T.Bailey, M. Eastwood and A. R. Gover. In their founding article [BEG94], they translate Thomas' work into more modern language and give an efficient introduction to the theory.

In this section, we will adopt an intermediate perspective between that in [Car24] and [Tho34]. Unless otherwised specified, M is a smooth *orientable* manifold; the orientability assumption serving only as a means to simplify our discussion. We recall that an affine connection is *torsion-free* when :

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

In terms of the local connection form $(\omega^i_j)_{1\leq i,j\leq n}$ associated to an arbitrary local moving frame (e_i) of TM with dual basis (ω^i) this is expressed as:

$$d\omega^i + \omega^i_k \wedge \omega^k = 0, \quad \text{(First structure equation)}. \tag{4.7}$$

Remark 4.2.1. If η is the canonical solder form, then the torsion free assumption can be expressed as : $d^\nabla \eta = 0$, where d^∇ is the exterior covariant derivative. (cf. appendix C).

Definition 4.2.1. Let $\nabla, \hat{\nabla}$ be two torsion-free affine connections (as defined in Definition 1.3.1) on the tangent bundle TM. We will say that $\nabla, \hat{\nabla}$ are *projectively equivalent* if and only if they have the same unparametrised geodesics. A projective structure on M, written, $\mathbf{p}$ or $[\nabla]$, is an equivalence class of projectively equivalent torsion-free affine connections. We will refer to $(M, \boldsymbol{p})$ as a projective manifold.

We begin our study of this structure, with a useful characterisation, due to Weyl, of projectively equivalent connections.

Proposition 4.2.1. *Two torsion-free affine connections ∇ et $\hat{\nabla}$ are projectively equivalent if and only if one can find a form $\Upsilon \in \Gamma(T^*M)$ such that for any $\eta, \xi \in \Gamma(TM)$:*

$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\eta)\xi + \Upsilon(\xi)\eta. \tag{4.8}$$

We will write:

$$\hat{\nabla} = \nabla + \Upsilon. \tag{4.9}$$

Remark 4.2.2. In the abstract index notation, Equation (4.8) can be written:

$$\hat{\nabla}_a \eta^b = \nabla_a \eta^b + \Upsilon_c \eta^c \delta_a^b + \Upsilon_a \eta^b = \nabla_a \eta^b + 2\Upsilon_{(a}\delta_{c)}^b \eta^c. \tag{4.10}$$

Proof. For the proof, we will continue to use Penrose's abstract index notation, which greatly clarifies the main arguments. First note that the map $\eta^b \mapsto \hat{\nabla}_a \eta^b - \nabla_a \eta^b$ is tensorial [11] , so, one can find $\Gamma^b_{ac} \in \mathcal{E}^b_{ac}$ such that for each η^b:

$$\hat{\nabla}_a \eta^b - \nabla_a \eta^b = \Gamma^b_{ac}\eta^c.$$

The torsion-free assumption leads to the symmetry : $\Gamma^b_{ac} \in \mathcal{E}^b_{(ac)}$, i.e. $\Gamma^b_{ac} = \Gamma^b_{ca}$. Hence it is sufficient to determine $\Gamma^b_{ac}\eta^a\eta^c$ for any $\eta \in \Gamma(TM)$; Γ^b_{ac} is determined by the polarisation identity.

The value of $\Gamma^b_{ac}\eta^a\eta^c = \hat{\nabla}_\eta \eta - \nabla_\eta \eta$ at a given point $p \in M$ only depends on that of $\eta(p)$, so we can evaluate it by following a geodesic ∇ that satisfies the initial conditions $\gamma(0) = p, \dot{\gamma}(0) = \eta(p)$. By the geodesic equation, we have :

$$\Gamma^b_{ac}\eta^a\eta^c = \lambda\eta^b,$$

for some scalar field λ. Since the left-hand side is quadratic, one can find Υ_a such that $\lambda = 2\Upsilon_a\eta^a$ and, hence :

$$\Gamma^b_{ac} = 2\Upsilon_{(a}\delta^b_{c)}.$$

11. In this context, a $C^\infty(M)$ linear map.

4.2.3 Thomas' projective differential invariant and the projective connection associated with a projective class $[\nabla]$

We will now choose a local section σ_U of the linear frame bundle $L(TM)$ and denote by (e_i) the associated local frame on U. Concordantly, we write (ω^i) for the dual frame. Equation (4.10) can be re-written in terms of the respective local connection forms $\omega_U = ({\omega^i}_j)$ and $\hat{\omega}_U = ({\hat{\omega}^i}_j)$ of ∇ et $\hat{\nabla}$ in the following way :

$$ {\hat{\omega}^i}_j = {\omega^i}_j + \Upsilon(e_j)\omega^i + \Upsilon\delta^i_j. \tag{4.11} $$

If we take the trace in this expression, then :

$$ {\hat{\omega}^k}_k - {\omega^k}_k = (n+1)\Upsilon, \tag{4.12} $$

Moreover, if we reinject (4.12) into (4.11), we find that:

$$ {\hat{\omega}^i}_j - \frac{1}{n+1}{\hat{\omega}^k}_k(e_j)\omega^i - \frac{1}{n+1}{\hat{\omega}^k}_k\delta^i_j = {\omega^i}_j - \frac{1}{n+1}{\omega^k}_k(e_j)\omega^i - \frac{1}{n+1}{\omega^k}_k\delta^i_j. \tag{4.13} $$

Thus, the quantity:

$$ {\omega^i}_j - \frac{1}{n+1}{\omega^k}_k(e_j)\omega^i - \frac{1}{n+1}{\omega^k}_k\delta^i_j, $$

is independent of the choice of connection in the class $[\nabla]$. Let us call it : $\Pi_U = (\Pi^i_j)$. Note that, like ω_U, Π_U satisfies Equation (4.7), i.e.

$$ d\omega^i + \Pi^i_k \wedge \omega^k = 0. \tag{4.14} $$

Nonetheless, the family (Π_U) does *not* define an affine connection on $L(TM)$, unless we restrict to transition functions with positive unit determinant. Indeed, if $\sigma_V = \sigma_U g, g : U \cap V \to GL_n(\mathbb{R})$, then on $U \cap V$:

$$ \Pi_V = g^{-1}dg + g^{-1}\Pi_U g - \frac{1}{n+1}\mathrm{tr}(g^{-1}dg)I_n - \frac{1}{n+1}g^{-1}Ag, \tag{4.15} $$

where A is the matrix-valued differential form $A^i_j = \mathrm{tr}(g^{-1}dg(e_j))\omega^i$.

However, it is possible to view Π_U as a submatrix of a projective connection ϑ_U with values in $\mathfrak{sl}_{n+1}(\mathbb{R})$. To this end, we use the orientability assumption to restrict to transition functions $g(x) \in GL_n(\mathbb{R})$ with positive determinant, $\det(g(x)) > 0$. Our question is whether or not one can lift each $g(x)$ to an element $h(x) \in H$ *and* find $\mathfrak{sl}_{n+1}(\mathbb{R})$ valued forms, ϑ_U, subject to the transformation rule given by Equation (4.3). Let us set :

$$h = \begin{pmatrix} \det(g)^{-\frac{1}{n+1}} & \gamma \\ 0 & \det(g)^{-\frac{1}{n+1}} g \end{pmatrix} \in H \text{ and } \vartheta_U = \begin{pmatrix} 0 & \alpha_U \\ \beta_U & \Pi_U \end{pmatrix} \in \mathfrak{sl}_{n+1}(\mathbb{R}),$$

where γ, α_U and β_U are unknowns. Evaluating $h^*\theta_H = h^{-1}dh$, one finds:

$$h^{-1}dh = \begin{pmatrix} -\frac{\operatorname{tr}(g^{-1}dg)}{n+1} & \det(g)^{\frac{1}{n+1}} d\gamma - \det(g)^{\frac{1}{n+1}} \gamma g^{-1}\mathrm{dg} + \det(g)^{\frac{1}{n+1}} \frac{\operatorname{tr}(g^{-1}dg)}{n+1}\gamma \\ 0 & g^{-1}dg - \frac{\operatorname{tr}(g^{-1}dg)}{n+1} I \end{pmatrix}.$$

Additionally, $\mathfrak{ad}_{h^{-1}}\vartheta_U = h^{-1}\vartheta_U h$ is:

$$\begin{pmatrix} -\det(g)^{\frac{1}{n+1}}\gamma g^{-1}\beta_U & \alpha_U g - \det(g)^{\frac{2}{n+1}}\gamma g^{-1}\beta_U\gamma - (\det(g))^{\frac{1}{n+1}}\gamma g^{-1}\Pi_U g \\ g^{-1}\beta_U & \det(g)^{\frac{1}{n+1}} g^{-1}\beta_U\gamma + g^{-1}\Pi_U g. \end{pmatrix}. \tag{4.16}$$

Their sum must be equal to θ_V. We inspect each component separately, beginning with Π_U. Our constraints translate to the fact that β_U et γ must satisfy :

$$\det(g)^{\frac{1}{n+1}} g^{-1}\beta_U\gamma + g^{-1}\Pi_U g + g^{-1}dg - \frac{\operatorname{tr}(g^{-1}dg)}{n+1} I = \Pi_V,$$

taking into account Equation (4.15) this reduces to:

$$\det(g)^{\frac{1}{n+1}} g^{-1}\beta_U\gamma = -\frac{1}{n+1} g^{-1}Ag. \tag{4.17}$$

According to (4.16), under change of basis β_U behaves like a column vector of 1-forms, in which each component transforms like an element of the dual basis. We therefore have a solution if we set :

$$\beta_U = \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix}, \quad \gamma = -\det(g)^{\frac{-1}{n+1}} \frac{\gamma'}{n+1} g, \tag{4.18}$$

where we define : $\gamma' = \begin{pmatrix} \operatorname{tr}(g^{-1}dg(e_1)) & \dots & \operatorname{tr}(g^{-1}dg(e_n)) \end{pmatrix}$. In this case we also have :

$$-\det(g)^{\frac{1}{n+1}}\gamma g^{-1}\beta_U = \frac{\operatorname{tr}(g^{-1}dg)}{n+1};$$

and the condition on the first component of the matrix is immediately satisfied. It only remains to choose α_U ; a straight-forward computation shows that:

$$d\gamma = -\frac{\mathrm{tr}(g^{-1}dg)}{n+1}\gamma - \det(g)^{\frac{-1}{n+1}}\frac{d\gamma'}{n+1}g + \frac{\gamma g^{-1}dg}{n+1},$$

and,

$$\gamma g^{-1}\beta_U\gamma = -\det(g)^{\frac{-1}{n+1}}\frac{\mathrm{tr}(g^{-1}dg)}{n+1}\gamma.$$

Therefore, we must have :

$$\alpha_V = \alpha_U g - \frac{d\gamma'}{n+1}g - \frac{\mathrm{tr}(g^{-1}dg)}{(n+1)^2}\gamma' g + \frac{\gamma'}{n+1}\Pi_U g. \tag{4.19}$$

The solution now is to use Π_U to form quantities of the right nature and study their transformation rules. We also note that when, for any $x \in U \cap V$, $\det g(x) = 1$, i.e. when Π_U behaves like an affine connection, then Equation (4.19) reduces to $\alpha_V = \alpha_U g$.

There is a natural quantity one can construct from Π_U, namely :

$$\Omega_U = d\Pi_U + \Pi_U \wedge \Pi_U;$$

when Π_U behaves like a connection this is its usual curvature form. The interesting point is that the column vector :

$$\frac{1}{n-1}\Omega^i{}_j(\cdot, e_i),$$

where $\Omega^i{}_j$ are the components of Ω_U, transforms exactly according to Equation (4.19). The proof of this is given in Appendix D, as it gives no further insight.

Putting these steps together, we have constructed a solution to our initial problem :

$$\theta_U = \begin{pmatrix} 0 & \frac{1}{n-1}\Omega^i{}_j(\cdot, e_i) \\ \omega^i & \Pi_U \end{pmatrix}, \tag{4.20}$$

Lifting a transition function g to h defined by:

$$\begin{pmatrix} (\det g)^{\frac{-1}{n+1}} & -(\det g)^{\frac{-1}{n+1}}\mathrm{tr}(g^{-1}dg(e_j)) \\ 0 & (\det g)^{\frac{-1}{n+1}} g \end{pmatrix}. \tag{4.21}$$

One can show that the resulting family satisfies Proposition 1.3.2 and can be used to construct the H-principal bundle over M of Definition 4.1.1. For this, one considers the quotient space $\coprod U \times H/\sim$ of the coproduct $\coprod U \times H$ indexed by an orientation atlas of the base M for the equivalence relation $\sim$ defined on $\coprod U \times H$ by :

$$(x, h_1) \in U \times H \sim (x, h_2) \in V \times H \Leftrightarrow \begin{cases} x \in U \cap V, \\ h_2 = h_{UV}(x)^{-1} h_1, \end{cases}$$

where $h_{UV}(x)$ is the lift of g_{UV} defined according to Equation (4.21).

Incidentally and contrary to the impression given by some of the apparently arbitrary choices made in our construction, this projective connection is in fact uniquely determined by conditions we will describe in the next section.

Remark 4.2.3. The matrix notation we used throughout this section corresponds to the following direct sum decomposition of $\mathfrak{sl}_{n+1}(\mathbb{R})$:

$$\mathfrak{sl}_{n+1}(\mathbb{R}) = \mathbb{R}^n \oplus \mathfrak{gl}_n(\mathbb{R}) \oplus (\mathbb{R}^n)^* = \mathfrak{g}_{-1} \oplus \ \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

This is in fact a $|1|$-grading of the Lie algebra. Although it may seem quite inessential in our presentation, such a $|k|$-grading of the Lie algebra is the theoretical origin for some of the nicer properties of parabolic geometries. We refer the interested reader to the textbook reference [AJ09]. It should be noted that this decomposition should be distinguished from the one in Equation (4.4), because it is not stable under the adjoint action of H on $\mathfrak{sl}_{n+1}(\mathbb{R})$.

4.2.4 The geodesics of a projective connection

We will now describe in what sense the above projective connection induced by a class of projectively equivalent affine connections is unique. Following Cartan, we will generalise the notion of geodesics to a projective connection. An alternative description can be found in [Sha97].

Let M be a smooth manifold with projective structure (P, ω). Let us consider the associated bundles :

$$Q = P \times_H SL_{n+1}(\mathbb{R}) \text{ and } E = Q \times_{SL_{n+1}(\mathbb{R})} SL_{n+1}(\mathbb{R})/H.$$

In the second case, $SL_{n+1}(\mathbb{R})$ acts on $SL_{n+1}(\mathbb{R})/H$ in the usual way by left multiplication; the bundle E is probably the closest thing to Cartan's idea of gluing a copy of projective space to each point of M. In this paragraph, we describe how to use the Cartan connection to define a Parallel Transport operator on E. This enables us to carry points from nearby fibres into the one above the point x_0. To simplify notation, we set in this paragraph $G = SL_{n+1}(\mathbb{R})$. The reader might note that this part of the discussion is essentially independent of the Lie groups G and H.

Firstly, E has a canonical section, analagous to a « choice of origin » in each of its fibres. To see this, first note that E and Q are respectively quotient spaces of the manifolds $Q \times G/H$ and $P \times G$. Thus, elements of E are in fact equivalence classes $\{(q, gH)\}, q \in Q, g \in G$ such that $\{qg', g'^{-1}gH\} = \{q, gH\}$ for any $g' \in G$. Similarly, elements of Q are equivalence classes : $[p, g], p \in P, g \in G$ that satisfy $[ph, h^{-1}g] = [p, g]$ for any $h \in H$. With this notation, set for any $x \in M$:

$$s(x) = \{[p, e], eH\}.$$

In the above p an arbitrary element of the fibre over x in P. $s(x)$ is completely independent of the choice of p as if $h \in H$ then:

$$\{[ph, e], eH\} = \{[p, h], eH\} = \{[p, e] \cdot h, eH\} = \{[p, e], hH\} = \{[p, e], eH\}.$$

We now pause to discuss how a Cartan connection defines a parallel transport on E. Let $x_0 \in M$ and γ a curve on M subject to the initial condition $\gamma(0) = x_0$. Recall that, ω induces a principal connection on Q which allows us to horizontally lift vector fields over M to vector fields over Q. Therefore, for each element q in the fibre $Q_{\gamma(0)}$ over $\gamma(0)$, we can formulate the Cauchy problem :

$$\begin{cases} \dot{\tilde{\gamma}}(t) = \mathrm{Hor}(\dot{\gamma}(t)), \\ \tilde{\gamma}(0) = q. \end{cases},$$

where $\mathrm{Hor}(\dot{\gamma(t)})$ is the horizontal lift of the velocity field of γ. By consequence, for sufficiently small values of t, one can define an invertible operator,

$$T^{\gamma}_{\gamma(0),\gamma(t)} : Q_{\gamma(0)} \to Q_{\gamma(t)}.$$

Since a maximal solution to the Cauchy problem is unique, one has :

$$T^{\gamma}_{\gamma(0),\gamma(t)}(qg) = T^{\gamma}_{\gamma(0),\gamma(t)}(q)g. \tag{4.22}$$

Parallel transport on E, can then be defined by :

$$\{q, gH\} \mapsto \{T^{\gamma}_{\gamma(0),\gamma(t)}(q), gH\}.$$ [12]

Equation (4.22) guarantees that this is well-defined.

Using the section s for any sufficiently small values of t, one can pull $s(\gamma(t))$ from the fibre $E_{\gamma(t)}$ over $\gamma(t)$ into the fibre above $\gamma(0)$, $E_{\gamma(0)}$. This results in a curve $\tilde{\gamma}$ in $E_{\gamma(0)}$, known as the development of γ. A « geodesic » of the projective connection ω will be defined as a curve γ whose development is contained in a projective line. In paragraph 4.2.3, the projective geodesics of our projective connection are the unparametrised geodesics of the class of affine connections $[\nabla]$.

In his article [Car24], E. Cartan studies under what circumstances two projective connections ω and ω' have the same geodesics. He shows that there is in fact a certain amount of freedom in the projective curvature form $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$. Exploiting this freedom to simplify the projective curvature form, Cartan then determines a unique projective connection known as the Cartan « normal » connection.

To describe this « simplification », Cartan first notes that it is always possible to choose a connection such that Ω is in fact $\mathfrak{h}$-valued. When this condition is satisfied, the section is said to be « torsion free » [13]. Equation (4.14) is the local coordinate expression of the fact that the projective connection (4.20) is torsion-free.

Cartan then shows that one of the « traces » of Ω can be set to 0. More specifically, if $\tilde{\Omega}^i{}_j$ is the local form of the curvature Ω, then, for any $j \in [\![1, n]\!]$, one can impose the condition that:

$$\sum_{i=1}^{n} \tilde{\Omega}^i{}_j(\cdot, e_i) = 0, \tag{4.23}$$

without changing the projective geodesics. Here, the basis (e_i) is not arbitrary (otherwise the condition would not be invariant), it is determined by the direct sum decomposition:

$$\mathfrak{sl}_{n+1}(\mathbb{R}) = \mathbb{R}^n \oplus \mathfrak{gl}_n(\mathbb{R}) \oplus (\mathbb{R}^n)^*.$$

Breaking up the connection form according to this decomposition, the $\mathbb{R}^n$ term is a vector of 1-forms (ω^i) that constitue a local frame of T^*M [14] ; the basis (e_i) in the above formula

12. The formula can be understood as follows: the parallel transport of a point expressed as gH in the frame q, is the point that is expressed as gH in the frame obtained by parallel transporting q along γ.
13. Definition **5**.3.1 in [Sha97]
14. Recall that the local connection forms restrict to isomorphisms $T_xM \to \mathfrak{g}/\mathfrak{h}$.

is the dual basis to (ω^i). It is easy to check that our projective connection in 4.2.3 satisfies this condition and is therefore Cartan's normal connection. One might also remark that, a Cartan connection satisfying the above described conditions determines an equivalence class of affine connections. Suppose the local form of U is given by :

$$\begin{pmatrix} -\mathrm{tr}(\Pi) & \alpha_j \\ \omega^i & \Pi \end{pmatrix}.$$

If $\mathrm{tr}(\Pi) \neq 0$ then one can remove it by performing a local projective transformation of the form

$$\begin{pmatrix} 1 & v \\ 0 & \mathrm{I}_n \end{pmatrix},$$

where the components of the column vector v correspond exactly to the coordinates of $\mathrm{tr}(\Pi)$ in the basis (ω^i). After this, one can use Equation (4.13) to determine an affine connection ${\omega^i}_j$ for any arbitrary choice of $\Upsilon \equiv \omega^k_k$.

Remark 4.2.4. Our « normalisation » condition (4.23) is a literal translation of the one Cartan gave in [Car24]; in this form, to the author, it remains relatively inextricable. According to [Sha97], there is nevertheless a more geometric interpretation if one studies a little more the curvature form.

4.3 Projective tractors and their calculus

4.3.1 Definition

We finally have all the material required to introduce the tractor bundle alluded to in the introduction. Let us define, once more, $Q = P \times_H SL_{n+1}(\mathbb{R})$, where H acts by left multiplication.

Definition 4.3.1. The tractor bundle $\mathcal{T}$ is the associated vector bundle $Q \times_{SL_{n+1}(\mathbb{R})} \mathbb{R}^{n+1}$ where $SL_{n+1}(\mathbb{R})$ acts on $\mathbb{R}^{n+1}$ in the usual canonical way. In abstract index notation the module of smooth sections of $\mathcal{T}$ will be written $\mathcal{E}^A$ [15].

The specific form of our transition functions defined by Equation (4.21) furnishes important information about the structure of the bundle $\mathcal{T}$. Using our notation from Paragraph 1.4.2 and Definition 1.4.1, one has :

15. i.e. capital latin letters will be used to denote tractor indices.

Proposition 4.3.1. *There is a short exact vector bundle sequence:*

$$0 \longrightarrow \mathcal{E}(-1) \xrightarrow{X} \mathcal{T} \xrightarrow{Z} TM(-1) \longrightarrow 0. \tag{4.24}$$

Furthermore, choosing a connection in the class $\nabla \in \mathbf{p}$*, the sequence right-splits and we have the non-canonical isomorphism:*

$$\mathcal{T} \overset{\nabla}{\cong} \mathcal{E}(-1) \oplus TM(-1).$$

Proof. The maps in the sequence can be read off from Equation (4.21) and are all constructed in the same manner as the map giving the isomorphism, so we will only prove this final point. Let $\nabla \in \mathbf{p}$ and consider the following collection :

$$\mathscr{A} = \{(U, \sigma_U),\ U \text{ open }, \sigma_U : U \to L(TM) \text{ is a local section}\}.$$

We assume that the open sets U cover M. Let g_{UV} denote the transition functions and $(\omega_U)^i_j$ the local connection forms. One can think of $TM(-1)$ as the quotient space

$$\left(\coprod_{U \in \mathscr{A}} U \times \mathbb{R}^n\right) / \sim,$$

where $(x, v_1) \in U_1 \times \mathbb{R}^n$ and $(x, v_2) \in U_2 \times \mathbb{R}^n$ are equivalent if:

$$v_2 = \det(g_{U_1U_2}(x))^{\frac{1}{n+1}} g_{U_1U_2}(x)^{-1} v_1, x \in U_1 \cap U_2.$$

By construction, $\mathcal{T}$ itself can be described as $\coprod\limits_{U \in \mathscr{A}} U \times \mathbb{R}^{n+1} / \sim$ for the equivalence relation:

$$(x, V_1) \in U_1 \times \mathbb{R}^{n+1} \sim (x, V_2) \in U_2 \times \mathbb{R}^{n+1} \Leftrightarrow V_2 = h_{U_1U_2}(x)^{-1} V_1, x \in U_1 \cap U_2,$$

where $h_{U_1U_2}$ is obtained from $g_{U_1U_2}$ using Equation (4.21). Now, the right-inverse we need to construct can be described by first defining for each $x \in U$ a linear map :

$$\begin{array}{cccc} \phi^U_x : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^{n+1} \\ & v & \longmapsto & \begin{pmatrix} ((\omega_U)^i_i)_x(v) \\ v \end{pmatrix}. \end{array}$$

For each $U \in \mathscr{A}$ we set : $\phi_U : U \times \mathbb{R}^n \to U \times \mathbb{R}^{n+1}$ where $\phi_U(x, V_1) = (x, \phi_x^U V_1)$. The change of chart rule for the connection forms implies that :

$$\phi_x^V((\det(g_{UV}(x)))^{\frac{1}{n+1}} g_{UV}(x)^{-1}v) = h_{UV}(x)^{-1}\phi_x^U(v),$$

which is sufficient to show that (ϕ_U) can be smoothly glued together and factor to a vector bundle morphism $TM(-1) \to \mathcal{T}$.

According to this result a choice of connection ∇_a in the projective class, enables us to write sections of $\mathcal{T}$ – tractor fields – as columns:

$$\begin{pmatrix} \nu^a \\ \rho \end{pmatrix},$$

where $\nu^a \in \Gamma(TM(-1)), \rho \in \Gamma(\mathcal{E}(-1))$. When we change connection according to $\hat{\nabla}_a = \nabla_a + \Upsilon_a$ then we get a new description ,

$$\begin{pmatrix} \hat{\nu}^a \\ \hat{\rho} \end{pmatrix},$$

related to the previous one by:

$$\begin{cases} \hat{\nu}^a = \nu^a, \\ \hat{\rho} = \rho - \Upsilon_a \nu^a. \end{cases} \tag{4.25}$$

Finally it will be convenient to identify the maps X and Z in Proposition 4.3.1 with sections $X^A \in \mathcal{E}^A(1)$, $Z_A^a \in \mathcal{E}_A^a(-1)$; these maps are canonical as they do not depend on a choice of connection. The non-canonical maps that split the sequence will be identified with sections $Y_A \in \mathcal{E}_A(-1)$ and $W_a^A \in \mathcal{E}_a^A(1)$. Note that :

$$X^A Y_A = 1,\ Z_A^a W_b^A = \delta_b^a,\ Z_A^a X^A = 0,\ W_a^A Y_A = 0.$$

In this notation, one has:

$$\begin{pmatrix} \nu^a \\ \rho \end{pmatrix} = \rho X^A + \nu^a W_a^A.$$

4.3.2 Change of connection

Although tractors have an invariant meaning, in practice, to work with them, we will often choose a connection and split the short exact sequence. In this context, it is important to relate various non-invariant quantities, such as curvature or covariant derivatives, between two projectively equivalent affine connections, ∇ and $\hat{\nabla}$. Recall that we write $\hat{\nabla} = \nabla + \Upsilon$, $\Upsilon \in \Gamma(T^*M)$, when for any vector fields η, ξ:

$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\eta)\xi + \Upsilon(\xi)\eta.$$

As we have previously remarked in Equation (4.11), this implies that the connection forms are related by :

$$\hat{\omega}^i{}_j = \omega^i{}_j + \Upsilon(e_j)\omega^i + \Upsilon\delta^i_j. \qquad \text{(4.11 revisited)}$$

We can deduce from this the change in covariant derivative of any section of any associated vector bundle to the frame bundle $L(TM)$, simply by applying the induced Lie algebra morphism to (4.11) in order to determine the local connection forms. For instance, projective densities of weight ω correspond to the representation $\rho : A \mapsto |\det A|^{\frac{\omega}{n+1}}$, the induced Lie algebra morphism is $\rho_* : M \mapsto \frac{\omega}{n+1}\mathrm{tr}(M)$. Hence, for any density σ of projective weight ω,

$$\hat{\nabla}_a \sigma = \nabla_a \sigma + \frac{\omega}{n+1}\left(\Upsilon(e_j)\omega^j_a + n\Upsilon_a\right)\sigma = \nabla_a\sigma + \omega\Upsilon_a\sigma.$$

Similarly, for one forms μ_a, which correspond to the contragredient representation $A \mapsto {}^tA^{-1}$, the corresponding Lie algebra morphism is : $M \mapsto -{}^tM$ and we find that :

$$\hat{\nabla}_b \mu_a = \nabla_b \mu_a - \Upsilon_a \mu_b - \Upsilon_b \mu_a.$$

We move now to curvature. Generally, the Riemann tensor of any torsion-free affine connection admits a unique decomposition as :

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + 2\delta^c_{[a}P_{b]d} + \beta_{ab}\delta^c{}_d, \tag{4.26}$$

where : $W_{ab}{}^c{}_d$ is trace-free and β_{ab} is antisymmetric. Taking traces of the above expression

β_{ab} and P_{ab} are easily shown to be related to the Ricci tensor $R_{bd} = R_{ab}{}^{a}{}_{d}$:

$$\begin{cases} (n-1)P_{ab} = R_{ab} + \beta_{ab}, \\ \beta_{ab} = -\frac{2}{n+1} R_{[ab]}. \end{cases} \tag{4.27}$$

We refer to $W_{ab}{}^{c}{}_{d}$ as the *projective Weyl tensor* and to P_{ab} as the *projective Schouten tensor*. The following lemma describes how these quantities are related between two torsion-free projectively equivalent affine connections :

Lemma 4.3.1. *Let* $\hat{\nabla} = \nabla + \Upsilon$ *then :*

- $\hat{W}_{ab}{}^{c}{}_{d} = W_{ab}{}^{c}{}_{d}$,
- $\hat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b$,
- $\hat{\beta}_{ab} = \beta_{ab} + 2\nabla_{[a}\Upsilon_{b]}$.

4.3.3 Special connections

In projective geometry, densities (cf. Definition 1.4.1) play an important role. As we have previously remarked, for some equations, like the geodesic equation [GST20] or the Killing equation (Equation (4.1)), considering weighted tensors, i.e. sections of $\mathcal{B} \otimes \mathcal{E}(\omega)$ for some tensor bundle $\mathcal{B}$ and some weight $\omega \in \mathbb{R}^*$, as opposed to usual tensors, can make the equation projectively invariant. Densities also appear naturally in the splitting of the tractor bundle in Proposition 4.3.1. We discuss here a further application : generalising the notion of *scale*, naturally present in conformal geometry.

A connection ∇ in a projective class $\boldsymbol{p}$ is said to be *special* if it preserves a nowhere vanishing density σ. Such a density is unique up to a constant factor and will be said to be the *scale* determined by ∇. Although all connections are not special, there is always a special connection in any projective class, as given any nowhere vanishing density $\sigma \in \Gamma(\mathcal{E}(\omega))$ and $\nabla \in \boldsymbol{p}$, the connection $\hat{\nabla} = \nabla - \frac{1}{\omega}\sigma^{-1}\nabla_a \sigma$ preserves σ, i.e. $\hat{\nabla}_a \sigma = 0$. Correspondingly, $\hat{\nabla}$ is said to be the scale determined by σ.

Special connections have useful properties, particularly with regards to curvature. We note first that the projective density bundles are flat, indeed, if ∇ preserves a nowhere vanishing density $\sigma \in \Gamma(\mathcal{E}(\omega))$, then any other such section ρ can be expressed as $\rho = f\sigma$ for some smooth function f, thus :

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\rho = ((\nabla_a \nabla_b - \nabla_b \nabla_a) f)\sigma = 0,$$

since we work with torsion-free connections. This has consequences on the Riemann tensor, as if we recall the decomposition :

$$R_{ab}{}^{c}{}_{d} = W_{ab}{}^{c}{}_{d} + 2\delta^{c}_{[a}P_{b]d} + \beta_{ab}\delta^{c}{}_{d},$$

then, one can show that for any $\omega \in \mathbb{R}^*$ and any section $\rho \in \Gamma(\mathcal{E}(\omega))$:

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\rho = \omega\beta_{ab}\rho. \tag{4.28}$$

We thus conclude that for special connections: $\beta_{ab} = -\frac{2}{n+1}R_{[ab]} = 0$. This implies that the Ricci and Schouten tensors are both symmetric, and that all the density bundles are flat. For future reference, we summarise this in:

Lemma 4.3.2. *If ∇ is a special connection, then the corresponding Ricci and Schouten tensors are symmetric and all the density bundles are flat.*

4.3.4 The Tractor Connection

The local connection forms (Equation (4.20)) of the Cartan connection on P induce a principal connection α on Q. Adapting our construction in Paragraph 1.3.4, α, in turn, according to the procedure described in Paragraph 1.4.3, induces an affine connection $\nabla^{\mathcal{T}}$ – the tractor connection – on the vector bundle $\mathcal{T}$ and, a fortiori, on its tensor algebra. Choosing a connection in the projective class and identifying $\mathcal{T} \overset{\nabla}{\cong} \mathcal{E}(-1) \oplus TM(-1)$, the tractor connection can be shown to act as follows :

Proposition 4.3.2. *Let $\nabla \in \mathbf{p}$. In terms of the isomorphism $\mathcal{T} \overset{\nabla}{\cong} \mathcal{E}(-1) \oplus TM(-1)$, the connection $\nabla^{\mathcal{T}}$ acts on the tractor $T \overset{\nabla}{=} \begin{pmatrix} \nu^a \\ \rho \end{pmatrix}$ according to the equation :*

$$\nabla^{\mathcal{T}}_b \begin{pmatrix} \nu^a \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_b \nu^a + \delta^a_b \rho \\ \nabla_b \rho - P_{ba}\nu^a \end{pmatrix}. \tag{4.29}$$

In the above, P_{ab} is the projective Schouten tensor defined in Section 4.3.2. The notation: $T \overset{\nabla}{=} \begin{pmatrix} \nu^a \\ \rho \end{pmatrix}$ should be understood to mean that the column vector on the right-hand side corresponds to the components of the tractor T, after splitting $\mathcal{T}$ with ∇.

After a choice of connection $\nabla \in \boldsymbol{p}$, one can equip any tensor bundle :

$$\otimes^k \mathcal{T} \otimes^l \mathcal{T}^* \otimes^p TM \otimes^q T^*M \otimes \mathcal{E}(\omega),$$

with a natural connection by mixing $\nabla^{\mathcal{T}}$ and ∇ [16], by abuse of notation, we will call this connection ∇. Doing this enables us to summarise the action of $\nabla^{\mathcal{T}}$ quite succinctly in terms of the splitting tractors X^A, Z^a_A, W^A_a, Y_A :

$$\begin{cases} \nabla_a Y_A = P_{ab} Z^b_A, \\ \nabla_a Z^b_A = -\delta^b_a Y_A, \\ \nabla_a X^A = W^A_a, \\ \nabla_a W^A_c = -P_{ac} X^A. \end{cases} \tag{4.30}$$

Tractor curvature

It will be convenient to have at our disposal the expression of the tractor curvature tensor ${\Omega_{ab}}^C{}_D$ in a splitting determined by a connection $\nabla \in \boldsymbol{p}$. It is a short computation that we carry out here to illustrate working with the splitting tractors introduced in Section 4.3.4. Let us work with a fixed connection $\nabla \in \boldsymbol{p}$, and observe that any $(1,1)$-tractor $L^C{}_D$ can be decomposed as :

$$L^C{}_D = f X^C Y_D + \mu_d X^C Z^d_D + \upsilon^c W^A_c Y_D + \lambda^c{}_d W^C_c Z^d_D, \tag{4.31}$$

for $f \in C^\infty(M)$, $\upsilon \in \Gamma(T^*M)$, $\mu \in \Gamma(TM)$, $\lambda \in \mathrm{End}(TM)$; the components are not weighted. In order to determine the components, we only need to calculate the action of ${\Omega_{ab}}^C{}_D$ on an arbitrary tractor $T^D = \rho X^D + \nu^b W^B_b$. By definition :

$$ {\Omega_{ab}}^C{}_D T^D = 2\nabla_{[a}\nabla_{b]} T^C,$$

16. we impose the Leibniz rule on simple tensors.

let us now evaluate the right-hand side, using Equation (4.30) :

$$\begin{aligned}\nabla_a\nabla_b T^C &= \nabla_a\left(\nabla_b\rho X^C + \rho\underbrace{\nabla_b X^C}_{W_b^C} + \nabla_b\nu^c W_c^C + \nu^c\underbrace{\nabla_b W_c^C}_{-P_{bc}X^C}\right),\\ &= \nabla_a\left((\nabla_b\rho - P_{bc}\nu^c)X^C) + (\nabla_b\nu^c + \rho\delta_b^c)W_c^C\right),\\ &= (\nabla_a\nabla_b\rho - (\nabla_a P_{bc})\nu^c - 2P_{(a|c|}\nabla_{b)}\nu^c - \rho P_{ab})X^C\\ &\qquad + (2\delta^c_{(a}\nabla_{b)}\rho + \nabla_a\nabla_b\nu^c - P_{bd}\nu^d\delta^c_a)W_c^C.\end{aligned}$$

Thus, using Equation (4.28) and the fact that ρ has weight -1, we find :

$$\begin{aligned}2\nabla_{[a}\nabla_{b]}T^C &= (2\nabla_{[a}\nabla_{b]}\rho - Y_{abd}\nu^d - 2\rho P_{[ab]})X^C + (2\nabla_{[a}\nabla_{b]}\nu^c - 2\delta_{(a}{}^cP_{b)d}\nu^d)W_c^C,\\ &= -Y_{abd}\nu^d X^C + (R_{ab}{}^c{}_d\nu^d - \beta_{ab}\nu^c - 2\delta_{(a}{}^cP_{b)d}\nu^d)W_c^C,\\ &= -Y_{abd}\nu^d X^C + (W_{ab}{}^c{}_d\nu^d)W_c^C.\end{aligned}$$

In the above, we have introduced the projective Cotton tensor $Y_{abc} = 2\nabla_{[a}P_{b]c}$, and, in the second equation we have used the fact that : $2P_{[ab]} = -\beta_{ab}$.

Applying T^D to Equation (4.31), one can then identify the components of the tractor curvature that we find to be, very simply :

$$\Omega_{ab}{}^C{}_D \overset{\nabla}{=} -Y_{abd}X^CZ_D^d + W_{ab}{}^c{}_dW_c^CZ_D^d. \tag{4.32}$$

Of course, the simplicity of the curvature tensor is a direct consequence of the choices made in Section 4.2.4.

4.4 Projective Compactifications

We come now to the notion of projective compactifications. In Paragraph 4.2.3 we have shown that to any pseudo-Riemannian (M, g) with Levi-Civita connection ∇, or, more generally, any smooth manifold equipped with a torsion-free affine connection, one can ascribe a Cartan projective structure whose projective geodesics are precisely the unparametrised geodesics of ∇. The structure is unique, under the condition that we require that the restrictions described by Paragraph 4.2.4 are satisfied.

The question underlying the ideas of projective compactification can be expressed as follows: let $\overline{M} = M \cup \partial M$ be a manifold with boundary, whose interior, M, is equipped with a pseudo-Riemannian structure that does not extend [17] to its boundary ∂M, is it possible that the associated projective structure extends nevertheless to ∂M and, if so, what can be said of the geometry of ∂M? Our starting point will be the following definition of [ČGM14; ČG14]:

Definition 4.4.1. Let $\bar{M} = M \cup \partial M$ be a smooth manifold with boundary, whose interior is M, and let ∇ be an affine connection on M. A boundary defining function is a map ρ that satisfies :

1. $\mathcal{Z}(\rho) = \{x \in \bar{M}, \rho(x) = 0\} = \partial M$,
2. $\mathrm{d}\rho \neq 0$ on ∂M.

We will say that ∇ is *projectively compact of order* $\alpha \in \mathbb{R}^*_+$ if for every point $x_0 \in \partial M$, one can find a neighbourhood U of x_0 in $\bar{M}$ and a boundary defining function ρ such that the connection [18] on $U \cap M$:

$$\hat{\nabla} = \nabla + \frac{\mathrm{d}\rho}{\alpha\rho}, \tag{4.33}$$

has a smooth extension to the boundary, i.e. for instance, the local connection forms of $\hat{\nabla}$, defined on $U \cap M$, in any frame (e_i) on U that is smooth up to the boundary, extend to ∂M.

The definition is independent of the choice of defining function ρ, as any other defining function on U can be written $\tilde{\rho} = e^k\rho$ and in this case $\mathrm{d}\tilde{\rho} = \tilde{\rho}\mathrm{d}k + e^k\mathrm{d}\rho \propto \mathrm{d}\rho$ on $U \cap \partial M$.

On the other hand, the parameter α, cannot be removed. This is clearer if we introduce the notion of boundary defining densities :

Definition 4.4.2. A boundary defining density is a global section of $\sigma \in \mathcal{E}(\omega)$ for a fixed weight $\omega \in \mathbb{R}^*_+$ vanishing exactly on ∂M and such that its expression in any local trivialisation on a neighbourhood of a boundary point $x_0 \in \partial M$ is a boundary defining function.

The parameter ω is fixed : suppose that $\sigma \in \mathcal{E}(\omega), \hat{\sigma} \in \mathcal{E}(\omega')$ are two defining densities such that $\omega \leq \omega'$. Let τ be a density of weight ω that is non-vanishing on a neighbourhood U of a boundary point $x_0 \in \partial M$, and write $\sigma = \rho\tau$ on $M \cap U$, ρ is therefore a boundary defining function. Since $\sigma^{\omega'/\omega}$ is also non-vanishing on $M \cap U$, it follows that $\hat{\sigma} = e^f\sigma^{\omega'/\omega} =$

17. For instance, because it is geodesically complete
18. cf. Proposition 4.2.1.

$e^f\rho^{\omega'/\omega}\tau^{\omega'/\omega}$. $\tau^{\omega'/\omega}$ is also non-vanishing on U, so we conclude that $e^f\rho^{\omega'/\omega}$ is a boundary defining function, therefore :

$$d(e^f\rho^{\omega'/\omega}) = e^f\rho^{\omega'/\omega}df + \rho^{\omega'/\omega-1}d\rho e^f \neq 0 \text{ on } \partial M,$$

so $\omega' = \omega$.

The following lemma relates boundary defining densities and projectively compact connections of order α:

Lemma 4.4.1 ([ČG14, Proposition 2.3 *(ii)*]). *Let $\bar{M}$ be a manifold with boundary equipped with a projective structure $[\nabla]$ on the interior M that extends to the boundary ∂M. Suppose that $\sigma \in \mathcal{E}(\alpha)$ is a boundary defining density and let $\hat{\nabla}$ be the scale determined by σ on M, then: $\hat{\nabla}$ is projectively compact of order α.*

The cases covered the most in examples and in the literature are $\alpha \in \{1, 2\}$, this is because there are well understood model cases. We also quote the following completeness result:

Proposition 4.4.1 ([ČG14, Proposition 2.4]). *Let ∇ be an affine connection on M which is projectively compact of order $\alpha \leq 2$. Assume that γ is a projective geodesic that reaches $x_0 \in \partial M$ with tangent vector transverse to ∂M. In this case, one can find an affine parametrisation $c : [0, \infty[\to \bar{M}$ (with respect to ∇) of part of γ such that $c([0, \infty[) \subset M$ and $\lim\limits_{t\to+\infty} c(t) = x_0$.*

In the next paragraph, we will study in detail the case of Minkowski spacetime, which we will find to be projectively compact of order 1. We will also explain how its metric structure is encoded projectively in a parallel 2-tractor and that its projective infinity inherits a metric projective structure.

4.4.1 Affine space

It is no surprise that the projective compactification of affine space A^n and, by extension, that of Minkowski spacetime, involves the central projection of $\mathbb{R}^{n+1}$. We will first look at this using the language of Paragraph 4.2.3 : identify A^n to $\mathbb{R}^n$ with its canonical affine structure, let $(e_1, \dots, e_n)$ be the canonical basis and $(\omega^i) = (\mathrm{d}x^i)$ the dual basis; they are all parallel with respect to the canonical affine connection (which corresponds exactly to the Maurer-Cartan form of the affine group). Thomas' projective invariant Π

(cf. Equation (4.13)), that is globally defined, vanishes and the (local) normal Cartan connection is nothing more than :

$$\begin{pmatrix} 0 & 0 \\ \omega^i & 0 \end{pmatrix}.$$

This is identical to the expression of the Maurer-Cartan form of the (oriented) projective group in an affine chart that we gave in Equation (4.6)! So we recover the expected result by reasoning locally.

One can also interpret this globally in terms of tractors. The first point we make towards this is that the tractor bundle $\mathcal{T}$ on the projective sphere is trivial. Indeed, unlike the H-principal bundle $(P = SL_{n+1}(\mathbb{R}), \pi : SL_{n+1}(\mathbb{R}) \to SL_{n+1}(\mathbb{R})/H)$, the associated bundle $Q = P \times_H SL_{n+1}(\mathbb{R})$ has a canonical global section : $pH \mapsto [p, p^{-1}]$ furnishing an inverse to the bundle map :

$$\begin{array}{ccc} P \times_H G & \longrightarrow & SL_{n+1}(\mathbb{R})/H \ \times SL_{n+1}(\mathbb{R}) \\ {[p,g]} & \longmapsto & (\pi(p), pg). \end{array}$$

In the above, we view elements of $P \times_H G$ as equivalence classes of pairs $(p, g) \in SL_{n+1}(\mathbb{R}) \times SL_{n+1}(\mathbb{R})$. The map is well-defined as for any $h \in H$:

$$\pi(ph) = \pi(p)\pi(h) = \pi(p) \text{ and } (ph)(gh)^{-1} = phh^{-1}g = pg.$$

Additionally, the Maurer-Cartan form of $SL_{n+1}(\mathbb{R})$ induces the trivial connection on $SL_{n+1}(\mathbb{R})/H \times SL_{n+1}(\mathbb{R})$, as can be seen by examining the local connection forms. Since Q is the frame bundle of the tractor bundle, it follows that : $\mathcal{T} = S^n \times \mathbb{R}^{n+1}$. We can therefore view parallel tractors on the projective sphere as constant vectors of $\mathbb{R}^{n+1}$.

Exploiting its definition as a quotient space, other geometric quantities on the projective sphere have similar interpretations in terms of $\mathbb{R}^{n+1}$. For instance, functions on S^n are in one-to-one correspondence with functions on $\mathbb{R}^{n+1} \setminus \{0\}$ that are invariant under the natural action of $\mathbb{R}^*_+$ on $\mathbb{R}^{n+1}$, i.e. $f(tx) = f(x)$ for all $x \in \mathbb{R}^{n+1}$ and any $t \in \mathbb{R}^*_+$; densities of weight ω can be identified with ω-homogenous functions on $\mathbb{R}^{n+1} \setminus \{0\}$, i.e. $f(tx) = t^\omega f(x)$, $x \in \mathbb{R}^{n+1}$, $t \in \mathbb{R}^*_+$. This last identification follows from the fact that the frame bundle of densities of weight 1 can be identified with $\mathbb{R}^{n+1} \setminus \{0\}$, for instance as follows :

$$\begin{array}{ccl} \mathbb{R}^{n+1} \setminus \{0\} & \longrightarrow & S^n \times \mathbb{R}^*_+ \\ (x_1, \dots, x_{n+1}) & \longmapsto & ([x_1, \dots, x_{n+1}], |x_1| + \dots + |x_{n+1}|). \end{array}$$

In a similar vein, the map X in the short exact sequence (4.24) of Proposition 4.3.1, can be thought of as the homogenous coordinates of a point on S^n. Finally, Z can be identified using the usual interpretation of vector fields as differential operators : it is the map that restricts a differential operator v on $\mathbb{R}^{n+1} \setminus \{0\}$ to the space of smooth $\mathbb{R}^*_+$-invariant functions. However, the result of v is not a vector field on S^n. Instead, $v(f)$ is a homogenous function on $\mathbb{R}^{n+1} \setminus \{0\}$ of weight -1, hence we have a weighted vector field with weight -1.

Consider now the hyperplane $x_{n+1} = -1$ in $\mathbb{R}^{n+1}$. In order for the canonical action of $SL_{n+1}(\mathbb{R})$ on the projective sphere to induce, via central projection, a group action on the plane, we must reduce the group by demanding that the co-vector $I = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}$ be preserved. This is tantamount to introducing a parallel co-tractor I_A on the projective sphere. The elements of the subgroup of $SL_{n+1}(\mathbb{R})$ that preserves I have the form :

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}, A \in SL_n(\mathbb{R}), b \in \mathbb{R}^n;$$

so it is easily recognised to be the affine group, and it acts as such on the hyperplane. The projective compactification of affine space can therefore be understood to be obtained directly from the projective sphere by choosing a parallel cotractor I_A and demanding that it be preserved by the structure. This splits the projective sphere into three orbits, two of which can be identified with the hyperplane, (points with homogenous coordinates $[x_1, \dots, x_n, \pm 1]$) and the third, formed by points with homogeneous coordinates $[x_1, \dots, x_n, 0]$, is what we will identify as the boundary at infinity. Note that the weight one density, $\sigma = X^A I_A$, corresponding to the homogeneous function of weight 1 given by the $(n+1)$-th homogeneous coordinate, x_{n+1}, is a defining density for the boundary.

4.4.2 Minkowski spacetime

To introduce additional structure on the hyperplane, for instance a pseudo-Euclidean structure, one should reduce the projective group $SL_{n+1}(\mathbb{R})$ further by requiring that a constant metric on $(\mathbb{R}^{n+1})^*$, H, be preserved. Let :

$$H = \begin{pmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and note that $H^{AB}I_A = 0$. The subgroup that preserves H and I is given by : $\mathbb{R}^n \rtimes SO(p,q)$ that we identify with the matrices :

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}, A \in SO(p,q), b \in \mathbb{R}^n.$$

Studying the orbits of the action of this group on the projective sphere, one observes that, there are again two orbits that can be identified with pseudo-Euclidean space, however, in addition, the boundary at infinity - i.e. points with homogenous coordinates $[x_1,\dots,x_n,0]$ - splits into orbits classified by the sign of $H(x,x)$, where x is the vector in $\mathbb{R}^{n+1}$ formed by the homogenous coordinates. In the specific case of a Lorentzian signature $(-,+,\dots,+)$, the 3-orbits can be interpreted as timelike $(H(x,x)<0)$, spacelike, $(H(x,x)>0)$ and null infinity $(H(x,x)=0)$.

Once more, $\sigma = I_A X^A$ is a weight one boundary defining density, therefore, $(d+1)$-dimensional Minkowski spacetime is projectively compact of order $\alpha = 1$. The bilinear form H is to be identified with a parallel 2-tractor on the projective sphere. The structure described here is, in fact, the generic model for projectively compact metrics of order 1; this is explained in [FG18].

Up to now, we have viewed things from the point of view of an ambient projective sphere, producing a projective compactification of pseudo-Euclidean space by embedding it directly into the n-sphere. Nevertheless, one can argue that it is more natural to adopt the opposite point of view, and attempt to construct a compactification from the inside out.

For definiteness, let us restrict now our discussion to $n = d+1$ dimensional Minkowski spacetime, that is to say : $\mathbb{R}^n = \mathbb{R}^{1+d}$ with its usual Cartesian coordinates $X_0,\dots,X_d$ and the usual $(+,-,\dots,-)$ Minkowski metric. Generic coordinates of points in $\mathbb{R}^{n+1} = \mathbb{R}^{d+2}$ shall be written $(x_0,\dots,x_{d+1})$ and homogenous coordinates in $P_+(\mathbb{R}^{n+1}) = P_+(\mathbb{R}^{d+2})$, $[x_0,\dots,x_{d+1}]$.

Trivialising the density bundle with various densities enables us to deduce boundary defining *functions* that can be used to define local coordinates charts well-adapted to the projective compactification. For instance, consider the usual Euclidean norm $||\cdot||_2$; it defines a homogenous function of weight 1 on $\mathbb{R}^{d+2}$, hence a projective density of weight 1 on S^{d+1}. As before, the boundary defining density $\sigma = X^A I_A$ corresponds to the homogenous function given by the last coordinate in x_{d+1} in $\mathbb{R}^{d+2}$. In the local affine chart

$U^-_{d+1} \cong \mathbb{R}^{d+1}$ of S^{d+1} :

$$x_{d+1} = \rho_1 || \cdot ||_2,$$

with :

$$\rho_1 : \begin{array}{ccc} U^+_{d+1} \cong \mathbb{R}^{d+1} & \longrightarrow & \mathbb{R} \\ (X_0, \dots, X_d) & \longmapsto & \frac{-1}{\sqrt{1+X_0^2+\dots+X_d^2}}. \end{array}$$

Another interesting example is the local trivialisation given by the homogenous function :

$$f(x_0, \dots, x_{d+1}) = -\sqrt{|x_0^2 - x_2^2 - \dots - x_d^2|},$$

on points such that $x_{d+1} < 0$ and $x_0^2 - x_2^2 - \dots - x_d^2 \neq 0$. Writing $x_{d+1} = \rho f$, we find:

$$\rho(X_0, \dots, X_d) = \frac{1}{\sqrt{|X_0^2 - X_2^2 - \dots - X_d^2|}}. \tag{4.34}$$

The function ρ can be used to construct future timelike infinity directly. Indeed, the surfaces $\rho = c$ furnish a foliation of the interior of the future light-cone $\mathscr{I}^+$ represented in Figure 4.2. In the coordinate chart $(\rho, \tilde{x}_1, \dots, \tilde{x}_d)$ on $\mathscr{I}^+$ defined by : $\tilde{x}_i = \rho X_i$, which

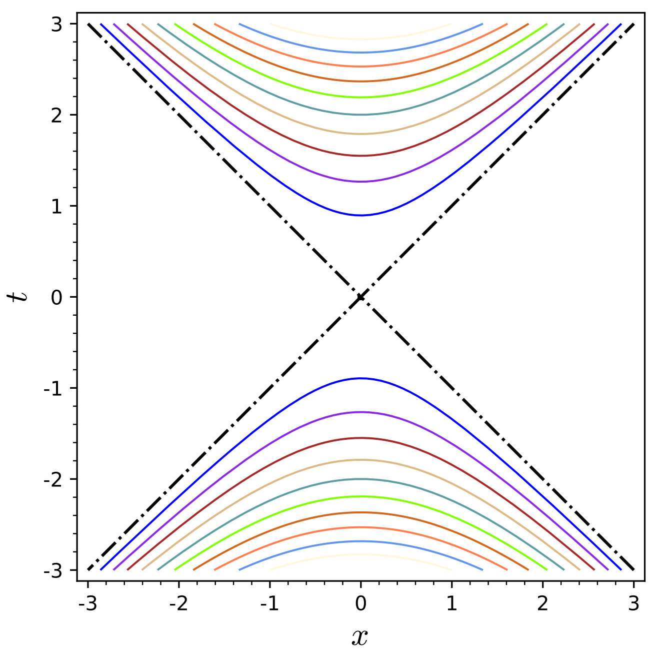

Figure 4.2 – Foliation of the interior light-cone by hyperbolic sheets

is adapted to this foliation, the usual Minkowski metric takes the form :

$$g = \frac{\mathrm{d}\rho^2}{\rho^4} - \frac{1}{\rho^2}\sum_{i,j}\underbrace{\left(\delta_{ij} - \frac{\tilde{x}_i\tilde{x}_j}{1+|\tilde{x}|^2}\right)}_{\rho^2 g_{ij}}\mathrm{d}\tilde{x}_i\mathrm{d}\tilde{x}_j, \quad |\tilde{x}| = \sum_{i=1}^{d}\tilde{x}_i^2. \tag{4.35}$$

In fact, [ČG14, Theorem 2.6] shows that projective compactness of order 1 follows directly from this expression; the compactification being obtained by appending the hypersurface $\{\rho = 0\}$. In this simple case, it is nonetheless straightforward to verify this directly: the coordinates $(\rho, \tilde{x}_1; \dots, \tilde{x}_d)$ identify $\mathscr{S}^+$ with $]0, +\infty[\times\mathbb{R}^n$, that is naturally viewed as an open-subset of $[0, +\infty[\times\mathbb{R}^n$. To prove projective compactness of order 1, it is necessary to show that the connection $\hat{\nabla} = \nabla + \frac{d\rho}{\rho}$, has a smooth extension to points where $\rho = 0$. The associated coordinate basis $(\frac{\partial}{\partial\rho}, (\frac{\partial}{\partial\tilde{x}_i})_{i\in[\![1,d]\!]})$ on $\mathscr{S}+$, has a natural extension to $[0, \infty[\times\mathbb{R}^d$, hence it is sufficient to calculate the local connection form in this local frame. First, those of the Levi-Civita connection are :

$$\begin{aligned}
\omega^0{}_0 = -2\frac{\mathrm{d}\rho}{\rho}; \quad \omega^i{}_0 = -\frac{\mathrm{d}\tilde{x}_i}{\rho},\\
\omega^0{}_j = -\rho\mathrm{d}\tilde{x}_j + \frac{\rho\tilde{x}_j}{1+|\tilde{x}|^2}\sum_k\tilde{x}_k\mathrm{d}\tilde{x}_k = \rho^3\sum_k g_{jk}\mathrm{d}\tilde{x}_k,\\
\omega^i{}_j = -\tilde{x}_i\mathrm{d}\tilde{x}_j + \frac{\tilde{x}_i\tilde{x}_j}{1+|\tilde{x}|^2}\sum_k\tilde{x}_k\mathrm{d}\tilde{x}_k - \delta_{ij}\frac{\mathrm{d}\rho}{\rho} = \tilde{x}_i\rho^2\sum_k g_{jk}\mathrm{d}\tilde{x}_k - \delta_{ij}\frac{\mathrm{d}\rho}{\rho}.
\end{aligned} \tag{4.36}$$

Those of the connection $\hat{\nabla} = \nabla + \frac{\mathrm{d}\rho}{\rho}$ are obtained by applying Equation (4.11):

$$\begin{aligned}
\hat{\omega}^0{}_0 = \hat{\omega}^i{}_0 = 0,\\
\hat{\omega}^0{}_j = \omega^0{}_j,\\
\hat{\omega}^i{}_j = \tilde{x}_i\rho^2\sum_k g_{jk}\mathrm{d}\tilde{x}_k,
\end{aligned} \tag{4.37}$$

and clearly have smooth extensions to points where ρ vanishes, which proves projective compactness of order 1.

Using Equation (4.35), one easily identifies a metric on the boundary :

$$h = \rho^2\left(g - \frac{d\rho^2}{\rho^4}\right) = \sum_{i,j}\left(\delta_{ij} - \frac{\tilde{x}_i\tilde{x}_j}{1+|\tilde{x}|^2}\right)\mathrm{d}\tilde{x}_i\mathrm{d}\tilde{x}_j.$$

Although this is not obvious from the outset, one can show that $\hat{\nabla}$ restricted to the boundary is the Levi-Civita connection for h.

Minor modifications of ρ can be used to construct past-timelike infinity and spacelike infinity by a similar coordinate based method. On the other hand, projective null infinity requires a slightly different treatment. It can in fact be obtained by projectively compactifying the incomplete spacelike and/or timelike projective infinity, which is projectively compact of order 2.

4.4.3 de-Sitter spacetime

Another interesting example is that of de-Sitter space, a Minkowski signature analogue of the Euclidean sphere. In 4 dimensions, it is the hypersurface $\{\eta(x,x) = k^2, x \in \mathbb{R}^5\}$ in $\mathbb{R}^5$ equipped with the standard $(-,+,+,+,+)$ signature metric η. As with the sphere, the parameter $k \in \mathbb{R}$ is a scaling of curvature and has no importance for us, we will henceforth set $k = 1$. The geometry can be described as $\mathbb{R}_\psi \times S^3$ with the metric :

$$g = -\mathrm{d}\psi^2 + \cosh^2\psi d\sigma^3, \tag{4.38}$$

where $d\sigma^3$ is the usual Euclidean metric of the unit 3-sphere in $\mathbb{R}^4$. A coordinate based approach to the compactification is to consider, for instance, the scalar field $\rho = \frac{1}{2\cosh^2\psi}$. In this case:

$$g = -\frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{2\rho}\left(-\frac{\mathrm{d}\rho^2}{1-2\rho} + \mathrm{d}\sigma^3\right).$$

Since $h = -\frac{\mathrm{d}\rho^2}{1-2\rho} + \mathrm{d}\sigma^3$ extends smoothly to $s = 0$, [ČG14, Theorem 2.6] allows us to conclude immediately that de-Sitter space is projectively compact of order 2. This fact can of course be verified directly upon inspection of the connection forms given in Appendix F.

A more geometric view of the compactification is to first recall that de-Sitter space is the homogeneous space $SO(4,1)/SO(3,1)$. Thanks to its embedding in $\mathbb{R}^5$, it is easily identified with a subset of the projective sphere S^4 via central projection. Consider now the projective sphere with its canonical projective structure and introduce the standard signature $(4,1)$ metric H^{AB} on $\mathbb{R}^{5*}$, i.e. a parallel 2-tractor on S^4. Demanding that the structure preserve the metric results in the reduction of $SL_5(\mathbb{R})$ to $SO(4,1)$ and enables us to retrieve the usual geometric structure on de-Sitter space. Within the projective sphere, de Sitter corresponds to the set of points with homogenous coordinates X such that the weight 2 density $\sigma = H_{AB}X^AX^B > 0$. This density is also a natural defining density for the boundary and, as with Minkowski space, trivialising it with respect to other non-vanishing 2-densities on the projective sphere yields boundary defining functions.

The defining function used above in the coordinate based approach comes from trivialising σ with the 2-density corresponding to the weight 2 homogenous function on $\mathbb{R}^5$: $X_1^2 + \cdots + X_4^2$, one has :

$$\sigma = -X_0^2 + X_1^2 + X_3^2 + X_4^2 = \left(\underbrace{1 - \frac{X_0^2}{X_1^2 + X_3^2 + X_4^2}}_{\text{function defined on the projective sphere}} \right) (X_1^2 + \cdots + X_4^2).$$

In the curved coordinate chart (ψ, ϑ), $\psi \in \mathbb{R}$, $\vartheta \in S^3$ this function is exactly $1 - \tanh^2\psi = \frac{1}{\cosh^2\psi}$; had we chosen to trivialise with the 2-density $||\cdot||_2^2$ we would have found the boundary defining function $\tilde{\rho} = \frac{1}{\cosh(2\psi)}$.

A striking difference with the compactification of Minkowski spacetime is that the action of $SO(4,1)$ on the projective sphere does not restrict to an action on the boundary $\sigma = 0$ of de-Sitter space and, consequently, we do not get a projective structure on the boundary, instead it inherits a conformal structure.

The above examples are model cases for the local geometry of solutions to the so-called Metrisability equation. In both cases, a symmetric bilinear form on the cotractor manifold H^{AB} plays an important role, and will also be an important tool in the lifting of equations on the base to the tractor bundle. In the next section, the reader will find a brief review of the theory of the Metrisability equation establishing the correspondence between solutions to the Metrisability equation and symmetric bilinear forms on the cotractor bundle.

4.5 A brief primer on the Metrisability equation

4.5.1 General theory

In [EM08], the presence of a connection within a given projective class $\boldsymbol{p}$ on a projective manifold $(M, \boldsymbol{p})$ was shown to be governed by the existence of solutions to the following (overdetermined) projectively invariant equation with unknown $\sigma^{ab} \in \Gamma(\mathcal{E}(-2))$:

$$\nabla_c \sigma^{ab} - \frac{2}{n+1} \nabla_d \sigma^{d(a} \delta_c^{b)} = 0. \tag{ME}$$

Equation (ME) states simply that the trace-free part of $\nabla_c \sigma^{ab}$ must vanish, and its projective invariance follows directly from this observation since if $\hat{\nabla} = \nabla + \Upsilon$:

$$\hat{\nabla}_c \sigma^{ab} = \nabla_c \sigma^{ab} + \Upsilon_d \sigma^{db} \delta^a_c + \Upsilon_d \sigma^{ad} \delta^b_c = \nabla_c \sigma^{ab} + \underbrace{2\Upsilon_d \sigma^{d(a} \delta^{b)}_c}_{\text{trace term}}.$$

Hence, the trace-free parts are identical. If σ^{ab} is a non-degenerate solution to this equation, then we define:

$$\sigma = \varepsilon^2_{a_1 \dots a_n b_1 \dots b_n} \sigma^{a_1 b_1} \dots \sigma^{a_n b_n} := \mathbf{det}\, \sigma^{ab}.$$

Even on non-orientable manifolds M the bundle $\Lambda^n TM \otimes \Lambda^n TM$ is canonically oriented and we denote by $\varepsilon^2_{a_1 \dots a_n b_1 \dots b_n}$ the canonical section that identifies $\Lambda^n TM \otimes \Lambda^n TM$ to the density bundle $\mathcal{E}(2n+2)$. In case there is an orientation, we can instead use the dual of the volume form $\omega^{a_1 \dots a_n}$ and identify the bundles using $\omega^{a_1 \dots a_n} \omega^{b_1 \dots b_n}$. In any case, σ is a weight two density, and, if σ^{ab} is non-degenerate, σ is nowhere vanishing so one can define a metric, g, by: $g_{ab} = \sigma^{-1} \sigma_{ab}$. The scale determined by σ, ∇^σ, can then be shown to be the Levi-Civita connection for g.

It turns out that the solutions to (ME) are in one-to-one correspondence with 2-tractors H^{AB} that satisfy the equation :

$$\nabla_c H^{AB} + \frac{2}{n} X^{(A} W_{cE}{}^{B)}{}_F H^{EF} = 0. \tag{ME2}$$

This result can be obtained by *prolongation*[19] of (ME), which consists, briefly, in adding variables in order to obtain a closed system. Here, we recall the explanation given in [ČGM14]

19. [Bra+06]

using the projective tractor calculus. The first remark is that there is a canonical projectively invariant operator : $T^*M \longrightarrow \mathrm{End}(\mathcal{T})$ that acts on 1-forms as :

$$u_b \mapsto X^A Z_B^b u_b.$$

This induces an operator $\partial^* : \Lambda^k T^*M \otimes S^2\mathcal{T} \longrightarrow \Lambda^{k-1}T^*M \otimes S^2\mathcal{T}$ that acts as :

$$H^{AB}_{b_1\dots b_k} \mapsto X^{(A}H^{B)C}_{cb_2\dots b_k}Z^c_C.$$

To understand how this operators acts in the column vector notation, we note first that after splitting the sequence (4.24) with a choice of connection ∇, an arbitrary section of $S^2\mathcal{T}$ can be written :

$$H^{AB} = \zeta^{ab}W_a^A W_b^B + 2\lambda^b X^{(A}W_b^{B)} + \tau X^A X^B,$$

where $\zeta \in \Gamma(S^2TM(-2))$, $\lambda \in \Gamma(TM(-2))$,$\tau \in \Gamma(\mathcal{E}(-2))$. Therefore, if we write :

$$H^{AB}_{b_1\dots b_k} = \zeta^{ab}_{b_1\dots b_k}W_a^A W_B^b + 2\lambda^b_{b_1\dots b_k}X^{(A}W_b^{B)} + \tau_{b_1\dots b_k}X^A X^B,$$

it follows that:

$$\begin{aligned}(\partial^* H)^{AB}_{b_2\dots b_k} &= X^{(A}H^{B)C}_{cb_2\dots b_k}Z^c_C \\ &= \zeta^{cb}_{cb_2\dots b_k}X^{(A}W_b^{B)} + \lambda^c_{cb_2\dots b_k}X^A X^B.\end{aligned} \tag{4.39}$$

This enables us to verify that, in fact : $\partial^* \circ \partial^* = 0$, and hence that we have a chain complex. It turns out that in addition to this we have so-called splitting operators:

Theorem 4.5.1. *Let $\zeta^{ab} \in \Gamma(S^2TM(-2))$, then there is a unique section $L(\zeta)$ of $S^2\mathcal{T}$ that satisfies:*

- $Z_A^a Z_B^b L(\zeta)^{AB} = \zeta^{ab}$,
- $\partial^*(\nabla^{\mathcal{T}} L(\zeta)) = 0$.

If $\nabla \in \boldsymbol{p}$ then we have :

$$L(\zeta) \overset{\nabla}{=} \zeta^{ab}W_a^A W_b^B - 2\frac{\nabla_a\zeta^{ab}}{n+1}X^{(A}W_b^{B)} + \frac{P_{ab}\zeta^{ab}(n+1) + \nabla_a\nabla_b\zeta^{ab}}{n(n+1)}X^A X^B. \tag{4.40}$$

Furthermore :

$$Z_A^a Z_B^b \nabla_c L(\zeta)^{AB} = \nabla_c \zeta^{ab} - \frac{2}{n+1}\nabla_d \zeta^{d(a}\delta_c^{b)}.$$

We seek now to prove the equivalence between solutions to (ME2) and those of (ME). This proof is already available in [ČGM14], but it is an interesting exercise to carry out these steps here as it helps to apprehend the articulations between the different notions of curvature, as well as the roles they each play in the projective structure. First let us observe that (ME2) can in fact be rewritten:

$$\nabla_c H^{AB} = -\frac{2}{n}\partial^*(\Omega_{ce}{}^{(A}{}_F H^{B)F}).$$

Hence, any H^{AB} satisfying (ME2), satisfies $\partial^*\nabla_c H^{AB} = 0$, which implies that $H^{AB} = L(\zeta)$ where $\zeta^{ab} = Z^a_A Z^b_B H^{AB}$. Furthermore, since it is in the image of ∂^*, $Z^a_A Z^b_B \nabla_c H^{AB} = 0$. Therefore, any H^{AB} that satisfies (ME2) is $L(\zeta)$ for a solution ζ of the metrisability equation. The proof of the converse is more involved. Let us assume that ζ is a solution to (ME) and denote by H^{AB} the tractor $L(\zeta)$. Let $\nabla \in \boldsymbol{p}$ and rewrite (4.40):

$$H^{AB} \overset{\nabla}{=} \begin{pmatrix} \zeta^{ab} \\ \lambda^b \\ \tau \end{pmatrix} = \begin{pmatrix} \zeta^{ab} \\ -\frac{\nabla_a\zeta^{ab}}{n+1} \\ \frac{P_{ab}\zeta^{ab}}{n} + \frac{\nabla_a\nabla_b\zeta^{ab}}{n(n+1)} \end{pmatrix}.$$

A simple calculation leads to:

$$\nabla_c H^{AB} \overset{\nabla}{=} \begin{pmatrix} \nabla_c\zeta^{ab} + 2\lambda^{(b}\delta^{a)}_c \\ \nabla_c\lambda^b - \zeta^{ab}P_{ca} + \tau\delta^b_c \\ \nabla_c\tau - 2P_{cb}\lambda^b \end{pmatrix}.$$

If ζ^{ab} is solution to the metrisability equation then the top slot cancels. Let us now calculate $\nabla_c\nabla_d\zeta^{ad}$:

$$\begin{aligned}
\nabla_c\nabla_d\zeta^{ad} &= \nabla_d\nabla_c\zeta^{ad} + R_{cd}{}^a{}_f\zeta^{df} + R_{cd}{}^d{}_f\zeta^{af} - 2\beta_{cd}\zeta^{ad}, \\
&= \frac{2}{n+1}\nabla_d\nabla_f\zeta^{f(a}\delta^{d)}_c + R_{cd}{}^a{}_f\zeta^{df} - \underbrace{(R_{cf}+\beta_{cf})}_{(n-1)P_{cf}}\zeta^{af} - \beta_{cf}\zeta^{af}, \\
&= \frac{\nabla_c\nabla_f\zeta^{fa}}{n+1} + \frac{\nabla_d\nabla_f\zeta^{fd}\delta^a_c}{n+1} + W_{cd}{}^a{}_f\zeta^{df} + 2\delta^a_{[c}P_{d]f}\zeta^{df} - (n-1)P_{cf}\zeta^{af}, \\
&= \frac{\nabla_c\nabla_f\zeta^{fa}}{n+1} + \frac{\nabla_d\nabla_f\zeta^{fd}\delta^a_c}{n+1} + W_{cd}{}^a{}_f\zeta^{df} + \delta^a_c P_{df}\zeta^{df} - nP_{cf}\zeta^{af}.
\end{aligned}$$

Therefore:

$$\begin{aligned}\frac{n}{n+1}\nabla_c\nabla_d\zeta^{ad} &= \frac{\nabla_d\nabla_f\zeta^{fd}\delta^a_c}{n+1} + W_{cd}{}^a{}_f\zeta^{df} + \delta^a_c P_{df}\zeta^{df} - nP_{cf}\zeta^{af},\\ &= n(\tau\delta^a_c - P_{cf}\zeta^{fa}) + W_{cd}{}^a{}_f\zeta^{df}.\end{aligned}$$

Thus, after rearrangement :

$$n(\nabla_c\lambda^a - P_{cf}\zeta^{fa} + \tau\delta^a_c) = -W_{cd}{}^a{}_f\zeta^{df}.$$

We now aim to perform a similar calculation for $\nabla_c\tau = \nabla_c\left(\frac{P_{ab}\zeta^{ab}}{n} + \frac{\nabla_a\nabla_b\zeta^{ab}}{n(n+1)}\right)$. To begin with :

$$\begin{aligned}n(\nabla_c\tau) &= (\nabla_c P_{ab})\zeta^{ab} + P_{ab}\nabla_c\zeta^{ab} + \frac{\nabla_c\nabla_a\nabla_b\zeta^{ab}}{n+1},\\ &= (\nabla_c P_{ab})\zeta^{ab} - 2P_{ab}\lambda^{(a}\delta^{b)}_c + \frac{\nabla_c\nabla_a\nabla_b\zeta^{ab}}{n+1},\\ &= (\nabla_c P_{ab})\zeta^{ab} + \frac{\nabla_c\nabla_a\nabla_b\zeta^{ab}}{n+1} - \underbrace{P_{ac}\lambda^a}_{P_{ca}\lambda^a - \beta_{ac}\lambda^a} - P_{cb}\lambda^b,\\ &= (\nabla_c P_{ab})\zeta^{ab} + \frac{\nabla_c\nabla_a\nabla_b\zeta^{ab}}{n+1} - 2P_{cb}\lambda^b + \beta_{ac}\lambda^a.\end{aligned}$$

Let us focus now on: $\nabla_c\nabla_a\nabla_b\zeta^{ab}$:

$$\begin{aligned}\nabla_c\nabla_a\nabla_b\zeta^{ab} &= \nabla_a\nabla_c\underbrace{\nabla_b\zeta^{ab}}_{-(n+1)\lambda^a} + R_{ca}{}^a{}_f\nabla_b\zeta^{fb} + R_{ca}{}^b{}_f\nabla_b\zeta^{af} - R_{ca}{}^f{}_b\nabla_f\zeta^{ab} - 2\beta_{ca}\nabla_b\zeta^{ab},\\ &= -(n+1)\nabla_a\nabla_c\lambda^a + (n+1)R_{cf}\lambda^f + 2(n+1)\beta_{ca}\lambda^a.\end{aligned}$$

Therefore:

$$\frac{\nabla_c\nabla_a\nabla_b\zeta^{ab}}{n+1} = -\nabla_a\nabla_c\lambda^a + (n-1)P_{cf}\lambda^f + \beta_{cf}\lambda^f.$$

From our previous computation :

$$\begin{aligned}\nabla_a\nabla_c\lambda^a &= \nabla_a\left(P_{cf}\zeta^{fa} - \tau\delta^a_c - \frac{1}{n}W_{cd}{}^a{}_f\zeta^{fd}\right)\\ &= -\nabla_c\tau + (\nabla_a P_{cf})\zeta^{fa} - (n+1)P_{cf}\lambda^f - \frac{1}{n}W_{cd}{}^a{}_f\nabla_a\zeta^{fd} - \frac{1}{n}\nabla_a W_{cd}{}^a{}_f\zeta^{fd}.\end{aligned}$$

We appeal now to (E.1) which shows that :

$$\nabla_a W_{cd}{}^a{}_f\zeta^{fd} = (n-2)Y_{cdf}\zeta^{fd},$$

thus :

$$\begin{aligned}\nabla_a\nabla_c\lambda^a &= -\nabla_c\tau + (\nabla_a P_{cf})\zeta^{fa} - (n+1)P_{cf}\lambda^f - \frac{1}{n}W_{cd}{}^a{}_f\nabla_a\zeta^{fd} - \frac{n-2}{n}Y_{cdf}\zeta^{fd},\\ &= -\nabla_c\tau + (\nabla_a P_{cf})\zeta^{fa} - (n+1)P_{cf}\lambda^f + \underbrace{\frac{2}{n}W_{cd}{}^a{}_f\lambda^{(f}\delta_a^{d)}}_{=0\text{ since }W\text{ is tracefree}} - \frac{n-2}{n}Y_{cdf}\zeta^{fd},\\ &= -\nabla_c\tau + (\nabla_a P_{cf})\zeta^{fa} - (n+1)P_{cf}\lambda^f - \frac{n-2}{n}Y_{cdf}\zeta^{fd}.\end{aligned}$$

Overall :

$$\frac{\nabla_c\nabla_a\nabla_b\zeta^{ab}}{n+1} = \nabla_c\tau - (\nabla_a P_{cf})\zeta^{fa} + 2nP_{cf}\lambda^f + \frac{n-2}{n}Y_{cdf}\zeta^{df} + \beta_{cf}\lambda^f.$$

Therefore:

$$\begin{aligned}(n-1)(\nabla_c\tau - 2P_{cb}\lambda^b) &= (\nabla_c P_{ab})\zeta^{ab} + \frac{n-2}{n}Y_{cdf}\zeta^{df} - (\nabla_a P_{cf})\zeta^{fa},\\ &= 2\frac{n-1}{n}Y_{cdf}\zeta^{df}.\end{aligned}$$

Hence:

$$(\nabla_c\tau - 2P_{cb}\lambda^b) = \frac{2}{n}Y_{cdf}\zeta^{df}.$$

Finally, we conclude that :

$$\nabla_c H^{AB} \stackrel{\nabla}{=} \frac{1}{n}\begin{pmatrix}0\\ -W_{cd}{}^a{}_f\zeta^{df}\\ 2Y_{cdf}\zeta^{df}\end{pmatrix}.$$

Now, using Equation (4.32) we see that :

$$2\Omega_{ce}{}^{(A}{}_F H^{B)F} = -2Y_{cef}\zeta^{bf}X^{(A}W_b^{B)} + 2W_{ce}{}^a{}_f\zeta^{bf}W_a^A W_b^B - 2\lambda^f Y_{cef}X^A X^B.$$

Hence, according to (4.39):

$$\partial^*(2\Omega_{ce}{}^{(A}{}_F H^{B)F}) = -2\zeta^{ef}Y_{cef}X^A X^B + 2\zeta^{ef}W_{ce}{}^b{}_f X^{(A}W_b^{B)} \stackrel{\nabla}{=} \begin{pmatrix}0\\ \zeta^{ef}W_{ce}{}^b{}_f\\ -2Y_{cef}\zeta^{ef}\end{pmatrix}.$$

Which shows that Equation (ME2) is satisfied and proves the equivalence. Due to this result, we will sometimes also refer to (ME2) as the metrisability equation.

4.5.2 Normal solutions

If we inspect the hypothesis of Theorem 4.5.1, we observe that there is a special class of solutions to the metrisability equation: those ζ such that $H^{AB} = L(\zeta)$ is parallel for the tractor connection. They will be referred to as *normal* solutions, and it is shown in [ČGM14] that they are intimately related to Einstein manifolds. Indeed, recall that:

$$H^{AB} \overset{\nabla}{=} \begin{pmatrix} \zeta^{ab} \\ \lambda^b \\ \tau \end{pmatrix} = \begin{pmatrix} \zeta^{ab} \\ -\frac{\nabla_a \zeta^{ab}}{n+1} \\ \frac{P_{ab}\zeta^{ab}}{n} + \frac{\nabla_a \nabla_b \zeta^{ab}}{n(n+1)} \end{pmatrix}, \ \nabla_c H^{AB} \overset{\nabla}{=} \begin{pmatrix} \nabla_c \zeta^{ab} + 2\lambda^{(b}\delta_c^{a)} \\ \nabla_c \lambda^b - \zeta^{ab} P_{ca} + \tau \delta_c^b \\ \nabla_c \tau - 2P_{cb}\lambda^b \end{pmatrix}.$$

In the scale ∇^σ determined by $\sigma = \mathbf{det}\, \zeta^{ab}$, which is the Levi-Civita connection of the metric $g_{ab} = \sigma^{-1}\zeta_{ab}$, away from where σ vanishes, $\lambda^b = 0$ and the condition $\nabla_c H^{AB} = 0$ implies that :

$$\zeta^{ab} P_{ca} = \frac{P_{ef}\zeta^{ef}}{n}\delta_c^b,$$

hence :

$$P_{cd} = \frac{P_{ef}\zeta^{ef}}{n}\zeta_{cd}. \tag{4.41}$$

Furthermore, since $n\tau = \zeta^{ab}P_{ab}$ and $\nabla_c \tau = 0$, it follows that ζ_{cd} is Einstein.

4.6 Projective Laplace operator and Boundary calculus

4.6.1 The Thomas D-operator

We have now laid out the basic tools that we have at our disposal on projectively compact pseudo-Riemannian manifolds and can at last proceed to discuss how one may seek to use these new tools in the asymptotic analysis of second order partial differentiation equations from Physics. The main idea is to construct a tractor version of the usual differential operators, that can be used to write down a similar equation on the tractor bundle. Equations obtained in this way are projectively invariant by construction and one can hope that it is possible to extract asymptotic information from the version expressed in the scale $\hat{\nabla}$ that extends smoothly to the boundary. A basic tool in the construction of these projectively invariant operators is the Thomas D-operator:

Definition 4.6.1. Let F° be an arbitrary section of a weighted tractor bundle [20] of weight ω. In a given scale ∇, the projective Thomas D-operator is defined as :

$$D_A F^\circ \stackrel{\nabla}{=} \omega Y_A F^\circ + \nabla_a F^\circ Z^a_A.$$

The definition is independent of the choice of scale since if $\hat{\nabla} = \nabla + \Upsilon$ then : $\hat{\nabla}_a F = \nabla_a F + \omega \Upsilon_a F$ and $\hat{Y}_A = Y_A - \Upsilon_a Z^a_A$, so :

$$\omega Y_A F + Z^a_A \nabla_a F = \omega \hat{Y}_A F + (\nabla_a F + \omega \Upsilon_a F) Z^a_A = \omega \hat{Y}_A F + \hat{\nabla}_a F Z^a_A.$$

We note here that the operator $X^A D_A$ [21] is the weight operator $\boldsymbol{\omega}$: $\mathscr{F}(\omega) \to \mathscr{F}(\omega)$ defined on an arbitrary weighted tractor bundle $\mathscr{F}(\omega)$ by : $F \mapsto \omega F$.

The Thomas D-operator is closely analogous to a covariant derivative with tractor indices and satisfies, in particular, the Leibniz rule:

$$D_A(F^\circ G^\circ) = (D_A F^\circ) G^\circ + F^\circ (D_A G^\circ).$$

It is interesting to note that this was not the case for the conformal equivalent of the Thomas D-operator. It holds here because the projective structure is, in some sense, first

20. $\circ$ denotes an arbitrary set of tractor indices.
21. X^A is the canonical tractor.

order whereas the conformal structure is second order : projective tractors are 1-jets and conformal tractors, 2-jets.

4.6.2 A projective Laplace operator and its boundary calculus

First attempt

Our original interest in Laplace-type operators that can act on tractors arose from the hope that they may provide a framework for the geometric interpretation of Hörmander's scattering result at timelike infinity for Klein-Gordon fields in Minkowski space-time discussed in the introduction. The Laplace operator $g^{ab}\nabla_a\nabla_b$ is not, of course, projectively invariant. However, much like in the conformal case, it can be made projectively invariant by working with projective densities of arbitrary weight.

Consider a Lorentzian manifold (M, g) of dimension n and denote by $[\nabla]$ the projective class of its Levi-Civita connection. If $\sigma \in \Gamma(T^*M(\omega))$, then observe that $g^{ab}\nabla_a\sigma_b$ transforms under a change of connection $\hat{\nabla} = \nabla + \Upsilon$ according to:

$$\begin{aligned} g^{ab}\hat{\nabla}_a\sigma_b &= g^{ab}\nabla_a\sigma_b + \omega\Upsilon^b\sigma_b - \Upsilon^b\sigma_b - \sigma_b\ \Upsilon^b \\ &= g^{ab}\nabla_a\sigma_b + (\omega-2)\Upsilon^b\sigma_b. \end{aligned}$$

It is therefore immediately invariant if $\omega = 2$, however, we can avoid fixing the weight (that we hope to identify with a mass term) if we consider instead an operator of the form $\nabla_a + \zeta_a$, where the form ζ depends on the connection in the class and transforms according to:

$$\hat{\zeta}_a = \zeta_a - (\omega-2)\Upsilon_a.$$

It is possible to construct such a co-vector from any non-degenerate symmetric tensor h_{ab}, indeed, $\zeta_a = \frac{\omega-2}{n+3}h^{ac}\nabla_c h_{ab}$, is a suitable choice since:

$$\begin{aligned} h^{ac}\hat{\nabla}_c h_{ab} &= h^{ac}\nabla_c h_{ab} - 2h^{ac}h_{ab}\Upsilon_c - h^{ac}h_{ac}\Upsilon_b - \Upsilon_a h^{ac}h_{cb} \\ &= h^{ac}\nabla_c h_{ab} - 2\Upsilon_b - n\Upsilon_b - \Upsilon_b \\ &= h^{ac}\nabla_c h_{ab} - (n+3)\Upsilon_b. \end{aligned}$$

With any such choice of ζ, the quantity:

$$g^{ab}(\hat{\nabla}_a + \zeta_a)\sigma_b,$$

is projectively invariant. Studying similarly the transformation rule for $\nabla_a \tau$ we find that for any $\tau \in \mathcal{E}(\omega)$, $\xi_a = h^{ac}\nabla_c h_{ab}$, $h \in S^2(T^*M)$, non-degenerate:

$$g^{ab}\left(\nabla_a + \frac{\omega-2}{n+3}\xi_a\right)\left(\nabla_b + \frac{\omega}{n+3}\xi_b\right)\tau,$$

is also projectively invariant. The operator can be extended to weighted tractors simply by coupling with the tractor connection.

Since we already have a metric g at our disposal, it seems natural to set $h = g$ in the above. The resulting operator is a candidate for a D'Alembertian type operator, however, in the Levi-Civita scale the term ξ_a vanishes and does not, therefore, provide a mass term...

Restricting the problem to Minkowski spacetime, we can try to solve the mass issue by exploiting some of the freedom left in the construction outlined above. Let ρ be the boundary defining function defined by Equation (4.34) on the future region of the future light cone of the origin of Minkowski spacetime. We note immediately that, with the Levi-Civita connection:

$$\Box\rho = -(n-3)\rho^3, \quad \nabla^a\rho\nabla_a\rho = \rho^4.$$

Set $h_{ab} = f(\rho)g_{ab}$, so that the new $\tilde{\xi}_a$ is given by:

$$\tilde{\xi}_a = \xi_a + (f(\rho))^{-1}f'(\rho)\nabla_a\rho.$$

Studying the form of $\nabla^a\tilde{\xi}_a$ and $\tilde{\xi}^a\xi_a$, it transpires that an interesting choice for f is $f(\rho) = e^{-\frac{\alpha}{\rho}}$, for some $\alpha \in \mathbb{C}$. For such a choice, expressed in the Levi-Civita scale:

$$\tilde{\xi}_a = \alpha\,\frac{\nabla_a\rho}{\rho^2}, \quad \tilde{\xi}^a\tilde{\xi}_a = \alpha^2, \quad \nabla^a\tilde{\xi}_a = -\alpha(n-1)\rho,$$

so that, if we write: $\beth\tau = g^{ab}\left(\nabla_a + \frac{\omega-2}{n+3}\tilde{\xi}_a\right)\left(\nabla_b + \frac{\omega}{n+3}\tilde{\xi}_b\right)\tau$, then in the Levi-Civita scale:

$$\beth\tau = g^{ab}\nabla_a\nabla_b\tau - \alpha(n-1)\rho\omega\tau + \frac{2(\omega-1)}{n+3}g^{ab}\xi_a\nabla_b\tau + \frac{(\omega-2)\omega\alpha^2}{(n+3)^2}\tau.$$

Setting $\omega = 1$ rids us of the first order term and setting $\alpha = im(n+3)$ we arrive, again in the Levi-Civita scale, at:

$$\sqsupset\tau + im(n-1)\rho\tau = (\Box + m^2)\tau.$$

This has apparently given us what we sought; but perhaps in a trivial way. Indeed, there is no guarantee here that $\sqsupset$ has an interesting extension to the boundary (or is defined there), and, our development depends in a non-trivial way on the boundary defining function, which is inherently dissatisfying. Finally, since τ is of weight 1, and Minkowski spacetime is projectively compact of order 1, then $\tau = \phi\sigma$, where $\sigma \in \mathcal{E}(1)$ is a boundary defining density. Hence, if ϕ extends to the boundary, so does τ but with vanishing boundary value. Working out the action of the operator on the component of τ expressed in a trivialisation that is valid up to the boundary, we find unfortunately that our concerns are founded as it has smooth coefficients that tend to 0 at the boundary and reduces there to multiplication by m^2.

A more natural operator

On a n-dimensional projective manifold $(M, \boldsymbol{p})$ equipped with a solution $H^{AB} = L(\zeta)$ to the metrisability equation (ME2), D provides a natural candidate for a projectively invariant Laplacian operator, namely :

$$\Delta^{\mathcal{T}} = H^{AB} D_A D_B.$$

During my trip to Auckland University, A.R. Gover suggested to me that this operator would likely play a key role and may be a more successful candidate. We also note that it has already appeared in the literature [GS18]. Of course, the results in Section 4.5.1 corroborate this, given the geometric significance of H^{AB}. This is also a closer analogue to the operator « $I \cdot D$ » in conformal tractor calculus [18, Chapter 3, §3.9] than the previous attempt.

The analogy with the conformal case is in fact complete in the case where H^{AB} is non-degenerate. In the early stages of his on-going thesis work, Samuel Porath, a student of R. Gover, developed a boundary calculus in this case, that is in all points analogous to the results in [GW14]. In fact, the important point is that $\Delta^{\mathcal{T}}$ defined above is part of an $\mathfrak{sl}_2$ algebra.

Proposition 4.6.1 (S. Porath). *Suppose (M, g) is projectively compact of order $\alpha = 2$,*

H^{AB} non-degenerate, and let:

— *x be the operator of multiplication by a boundary defining density σ,*

— *$y = -\frac{1}{\sigma^{-1}I^2}\Delta^{\mathcal{T}}$, with $I^2 = H^{AB}D_A\sigma D_B\sigma$,*

— *$h = \boldsymbol{\omega} + \frac{d+2}{2}$.*

Then x, y, h form an $\mathfrak{sl}_2$-triple i.e.

$$[x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y.$$

It is interesting to note that Proposition 4.6.1 is not specific to normal solutions to the Metrisability equation. The consequence of this that interests to us, is that following the same procedure as in [GW14], S. Porath developed a Boundary Calculus from which follows a formal solution operator that we relate to the asymptotics of solutions, we will discuss this in Section 4.6.4.

Let us begin our discussion here by studying how $\Delta^{\mathcal{T}}$ acts on weighted densities. Let $f \in \Gamma(\mathcal{E}(\omega))$ and $\nabla \in \boldsymbol{p}$, then:

$$D_AD_Bf \overset{\nabla}{=} (\omega-1)\omega f Y_AY_B + 2(\omega-1)\nabla_b f Y_{(A}Z_{B)}^b + (\nabla_a\nabla_b f + \omega P_{ab}f)Z_A^aZ_B^b.$$

Hence, writing $H^{AB} \overset{\nabla}{=} \zeta^{ab}W_a^AW_b^B - 2\frac{\nabla_a\zeta^{ab}}{n+1}X^{(A}W_b^{B)} + \left(\frac{P_{ab}\zeta^{ab}}{n} + \frac{\nabla_a\nabla_b\zeta^{ab}}{n(n+1)}\right)X^AX^B$, we find that:

$$\Delta^{\mathcal{T}}f \overset{\nabla}{=} \omega(\omega-1)\left(\frac{P_{ab}\zeta^{ab}}{n} + \frac{\nabla_a\nabla_b\zeta^{ab}}{n(n+1)}\right)f + \zeta^{ab}(\nabla_a\nabla_b f + \omega P_{ab}f) - 2\frac{\omega-1}{n+1}\nabla_a\zeta^{ab}\nabla_b f.$$

In the scale ∇_ζ the expression reduces to :

$$\Delta^{\mathcal{T}}f \overset{\nabla_\zeta}{=} \frac{\omega(\omega+n-1)P_{ab}\zeta^{ab}}{n}f + \zeta^{ab}\nabla_a\nabla_b f. \tag{4.42}$$

This indicates that in the case where the density $P_{ab}\zeta^{ab}$ is parallel for ∇_ζ, $\Delta^{\mathcal{T}}$ is a projectively invariant generalisation of the Klein-Gordon operator, with the proviso that the order-0 term be identified with the mass. Unfortunately, in the case of scalar-flat metrics like Minkowski spacetime, the term vanishes altogether and we have but a projective wave operator. We will see that scalar-flatness is also an obstruction to our next developments, as well as Proposition 4.6.1. The above formulae generalise to the case where f is a weighted tractor by coupling a connection on M with the tractor connection.

Let us now restrict to the case in which our projective manifold is the projective compactification $\bar{M}$ of a connected oriented $n = (1+d)$-dimensional Lorentzian manifold (M, g). That is to say, we assume that:

Hypothesis A.

— *$\bar{M}$ is a manifold with boundary ∂M such that int $\bar{M} = M$,*

— *M is a connected, oriented manifold equipped with a smooth Lorentzian metric g,*

— *The Levi-Civita connection ∇^g of g does not extend smoothly to any point on the boundary,*

— *The projective class $[\nabla^g]$ extends to the boundary.*

Let ω_g denote the volume density for g and set for $\alpha \in \{1, 2\}$, $\sigma = (\omega_g)^{-\frac{\alpha}{n+2}}$. Then $\zeta = \sigma^{-\frac{2}{\alpha}} g^{-1}$ is a solution to the metrisability equation (ME) on M, and gives rise to a tractor H^{AB} that is a solution of Equation (ME2), equally on M. Since Equation (ME2) can be rewritten as $\tilde{\nabla}_c H^{AB} = 0$ for an obvious modification $\tilde{\nabla}$ of the tractor connection, we can observe that our assumptions imply that $\tilde{\nabla}$ extends smoothly to the boundary and, consequently, H^{AB} can be extended by parallel transport to ∂M. Projecting onto the invariant component, $Z^a_A Z^b_B H^{AB} = \zeta^{ab}$, shows that ζ itself extends smoothly to the boundary, furthermore, its extension is degenerate on ∂M since if this was not the case ∇_g would extend to the boundary. According to whether H^{AB} is non-degenerate ($\alpha = 2$) or g is Ricci-flat ($\alpha = 1$) we are now in one of the situations described in [FG18, Theorems 3.6 or 3.14] and σ is a boundary defining density in each case.

Consider now as in Proposition 4.6.1, x, the operator acting on weighted tractors that multiplies by σ and define the weight $\alpha - 1$ co-tractor $I_A = D_A\sigma$, then :

Lemma 4.6.1.

$$[x, \Delta^{\mathcal{T}}] = -\frac{\sigma^{-1} I^2}{\alpha}(2\boldsymbol{\omega} + d + \alpha)$$

where: $I^2 = H^{AB} I_A I_B$ and $\boldsymbol{\omega} = X^A D_A$ is the weight operator.

Proof. In the scale ∇_g, σ is parallel, so it commutes with ∇_g. However, it does not commute with the weight operator as it increases weight by α. Hence, if F is an arbitrary tractor of weight ω then :

$$[x, \Delta^{\mathcal{T}}]F = (\omega(\omega + d) - (\omega + \alpha)(\omega + \alpha + d)\frac{P_{ab}\zeta^{ab}}{d+1}\sigma F.$$

Again, in the scale ∇_g, $I_A = \alpha\sigma Y_A$ and $I^2 = \alpha^2\sigma^2\frac{P_{ab}\zeta^{ab}}{d+1}$ and the result ensues.

If g is Ricci-flat ($\alpha = 1$) then on M, $P_{ab} = 0$ and $I^2 = 0$. Furthermore, I_A is parallel for the tractor connection and extends naturally to $\bar{M}$, hence I^2 also extends smoothly to 0 on $\bar{M}$. So $[x, \Delta^{\mathcal{T}}] = 0$ in this case. On the other hand, if H^{AB} is non-degenerate on M ($\alpha = 2$), the function $\sigma^{-1}I^2$ is non-vanishing on $\bar{M}$ and, defining $y = -\frac{1}{\sigma^{-1}I^2}\Delta^{\mathcal{T}}$, we have reproduced Proposition 4.6.1.

We see here directly the unfortunate consequences of Ricci-flatness, Proposition 4.6.1 cannot hold in the case $\alpha = 1$ because x and $\Delta^{\mathcal{T}}$ commute and generate a trivial sub-Lie algebra.

4.6.3 Asymptotics of solutions to the Klein-Gordon equation in de-Sitter spacetime

As mentioned above, in the non-degenerate case, the $\mathfrak{sl}_2$-triple in Proposition 4.6.1 is the basis of a so-called Boundary Calculus. This enables us to formally generate approximate solutions to $yf = 0$ off the boundary. To study this, we first consider the specific case of $(1+d)$-dimensional de-Sitter spacetime and return to the notations introduced in Section 4.4.3. Let $\sigma \in \mathcal{E}(2)$ be the defining density for the boundary constructed from the volume form ω_g by $\sigma = |\omega_g|^{-\frac{2}{d+2}}$ and $\zeta^{ab} = \sigma^{-1}g^{ab}$. In the scale defined by σ :

$$\zeta^{ab}P_{ab} = \frac{1}{d}\zeta^{ab}R_{ab} = \sigma^{-1}(d+1)$$

so Equation (4.42) becomes :

$$\Delta^{\mathcal{T}} f \overset{\nabla_\zeta}{=} \sigma^{-1}\left(\omega(\omega+d)f + g^{ab}\nabla_a\nabla_b f\right). \tag{4.43}$$

This is the Klein-Gordon operator with mass defined by the relation $\omega(\omega+d) = -m^2$. Vice versa, for a given value of m there are therefore two weights on which $\Delta^{\mathcal{T}}$ acts exactly as the Klein-Gordon operator with mass m :

$$\omega_m \in \left\{\frac{1}{2}\left(-d+\xi\right), \xi^2 = d^2 - 4m^2\right\},$$

generically, ξ is complex.

On de-Sitter spacetime, the operator y is simply $y = -\Delta^{\mathcal{T}}$, therefore the equation $yf = 0$ for $f \in \mathcal{E}(\omega_m)$, in the scale determined by σ, is the Klein-Gordon equation for a classical scalar field with mass m. More precisely, solutions to the Klein-Gordon equation with mass m on de-Sitter space are in one-to-one correspondence with densities of weight ω_m in $\ker y$; the correspondance being accomplished naturally via the map $\phi \mapsto \phi\sigma^{\frac{\omega_m}{2}} \equiv f_\phi$. The operator $y = \frac{-1}{I^2}H^{AB}D_AD_B$ has the advantage that it is well-defined on the boundary and therefore, expressed in a scale that is regular at the boundary, it can be used to study the asymptotic behaviour of solutions to the Klein-Gordon equation. Let us work this out explicitly on our example. Introduce the coordinate functions $(\psi, \vartheta), \psi \in \mathbb{R}, \vartheta \in S^n$ and recall that $\rho = \frac{1}{2\cosh^2\psi}$ is a boundary defining function. Each local frame $(\omega^0, \dots, \omega^d)$ on T^*M defines a positive density of any weight ω that we will call $|\omega^0 \wedge \dots \wedge \omega^d|^{-\frac{\omega}{d+2}}$. For simplicity : let $\omega^1, \dots, \omega^d$ be dual to an orthonormal frame on TS^d and write $\omega^1 \wedge \dots \wedge \omega^d = \mathrm{d}\Omega^d$. Then $|\mathrm{d}\rho \wedge d\Omega^d|^{\frac{-\omega}{d+2}}$ is smooth up to the boundary and:

$$\sigma = 2\rho(1-2\rho)^{\frac{1}{d+2}}|\mathrm{d}\rho \wedge \mathrm{d}\Omega^n|^{\frac{-2}{d+2}}.$$

By construction, the connection $\nabla^s = \nabla_g + \frac{\mathrm{d}\rho}{2\rho}$ extends to the boundary and preserves the 2-density $s = \frac{1}{\rho}\sigma$; the scale s can therefore be used to study y near the boundary $\sigma = 0$. Using the change of connection formulae, we find:

$$H^{AB} \overset{\nabla_s}{=} \begin{pmatrix} s^{-1}\rho^{-1}g^{ab} \\ 2s^{-1}(1-2\rho)\partial^a_\rho \\ 2s^{-1} \end{pmatrix}.$$

Hence, if expressed in terms of the connection ∇^s, for $f \in \mathcal{E}(\omega)$:

$$yf = 2s^{-1}(\omega-1)\omega f + 4s^{-1}\rho(1-2\rho)(\omega-1)\partial^a_\rho{\nabla^s}_a f + \zeta^{ab}\left(\nabla^s_a\nabla^s_b f + \omega P^s_{ab} f\right). \tag{4.44}$$

Writing $f = \phi s^{\frac{\omega}{2}}$, and using the fact that $\nabla^s s = 0$, Lemma F.1.3, shows that:

$$\begin{aligned} yf &= s^{\frac{\omega}{2}-1}\rho^{-1}\Box_s\phi + 4s^{-1}(\omega-1)(1-2\rho)\partial_\rho\phi + 2s^{-1}(\omega+d-1)\omega\phi) \\ &= 2s^{\frac{\omega}{2}-1}\bigg(-2\rho(1-2\rho)\partial^2_\rho\phi + [2(\omega-1)(1-2\rho) + (2\rho(1-d)+d)]\partial_\rho\phi \\ &\qquad\qquad + \Delta_{S^d}\phi + \omega(\omega+d-1)\phi\bigg). \end{aligned} \tag{4.45}$$

Therefore, near the boundary, $yf = 0, f = \phi s^{\frac{\omega}{2}}$ is equivalent to:

$$-2\rho(1-2\rho)\partial^2_\rho\phi + (1 + (1-2\rho)(d-3+2\omega))\partial_\rho\phi + \Delta_{S^d}\phi + \omega(\omega+d-1)\phi = 0.$$

Exploiting the spherical symmetry of the above equation by decomposing onto a spherical harmonic $\lambda = l(l+d-1)$, the problem is reduced to the ODE:

$$-2\rho(1-2\rho)\partial_\rho^2\phi + (1+(1-2\rho)(d-3+2\omega))\partial_\rho\phi + (\omega(\omega+d-1)-\lambda)\phi = 0. \quad (4.46)$$

Since all coefficients are analytic functions of ρ, it is well adapted to the Frobenius method [Inc56] and so we seek solutions of the form :

$$\phi = \rho^\nu \sum_{k\geq 0} \alpha_k \rho^k.$$

Plugging this ansatz into Equation (4.46) yields the indicial equation :

$$\nu(2\omega + n - 2\nu) = 2\nu(h_0 - \nu - 1) = 0, \quad (4.47)$$

where we have introduced $h_0 = hf = \omega + \frac{d+2}{2}$ and h is the operator defined in Lemma 4.6.1. Hence:

$$\nu = 0 \text{ or } \nu = h_0 - 1.$$

For $k \geq 1$, the coefficients α_k satisfy the following recurrence relation :

$$2(\nu+k)(h_0-k-1-\nu))\alpha_k = (2(\nu+k-1)(2\nu+k+1-2h_0) + \omega(\omega+n-1)-\lambda)\,\alpha_{k-1},$$

which is readily solved for any given α_0 provided that for all $k \in \mathbb{N}$, $k \neq h_0 - 1$ (when $\nu = 0$) or $k \neq -(h_0-1)$ when $(\nu = h_0 - 1)$. In a generic case $h_0 \in \mathbb{C}\setminus\mathbb{R}$, and there is no obstruction to the existence of the series. Under the assumption that we avoid these special cases, the Frobenius method yields two independent solutions to the equation, and generic smooth solutions can be written :

$$\phi = \phi_0 + \rho^{h_0-1}\phi_1,$$

where ϕ_0, ϕ_1 are regular up to the boundary. Now, returning to the scale ∇_g,

$$f = \tilde{\phi}\sigma^{\frac{\omega}{2}} = \tilde{\phi}\rho^{\frac{\omega}{2}}s^{\frac{\omega}{2}}.$$

Hence:

$$\tilde{\phi} = \phi_0\rho^{-\frac{\omega}{2}} + \rho^{h_0-\frac{\omega}{2}-1}\phi_1 = \phi_0\rho^{-\frac{\omega}{2}} + \rho^{\frac{\omega}{2}+\frac{d}{2}}\phi_1. \quad (4.48)$$

Choosing $\omega \in \{\frac{1}{2}(-d+\xi), \xi^2 = d^2 - 4m^2\}$, Equation (4.48) describes the asymptotic behaviour of solutions to the Klein-Gordon equation near the projective boundary. Observe from (4.48) that the precise choice of weight in the identification $\phi \mapsto \phi\sigma^{\frac{\omega}{2}}$ is inconsequential and switching between the two possible values at fixed mass m amounts to exchanging ϕ_0 and ϕ_1. Overall, solutions behave asymptotically as :

$$\tilde{\phi} = \phi_0 \rho^{\frac{1}{4}(d-\sqrt{d^2-4m^2})} + \phi_1 \rho^{\frac{1}{4}(d+\sqrt{d^2-4m^2})}.$$

Where $\sqrt{d^2-4m^2}$ is a (perhaps complex) square root of d^2-4m^2. This result should be compared with [Vas10, Theorem 1.1].

4.6.4 Formal solution operator

As stated previously, using Proposition 4.6.1, one can *formally* generalise the previous result, in the same manner as [GW14] in the conformal case, to the more general framework of the rather general hypotheses A with the additional assumption that the solution H^{AB} to the Metrisability equation (ME2) is non-degenerate; recall that this implies that the order of the compactification is 2. The idea is to look for *formal* operators A, generated by $x^\alpha, \alpha \in \mathbb{C}$ $(x = \sigma)$ and y, that annihilate y from the right, i.e. that satisfy $yA = 0$. Inspired by the Frobenius method, one can seek solutions of the form:

$$A = x^\nu \sum_{k=0}^{\infty} \alpha_k x^k y^k.$$

Now, note the same reasoning outlined in the proof of Lemma 4.6.1, can be used to prove that for any complex $\nu \in \mathbb{C}$,

$$[x^\nu, y] = x^{\nu-1}\nu(h+\nu-1). \tag{4.49}$$

Hence, formally :

$$\begin{aligned} yA = yx^\nu \sum_{k=0}^{\infty} \alpha_k x^k y^k &= x^\nu \sum_{k=1}^{\infty} \alpha_k y x^k y^k - x^{\nu-1} \sum_{k=0}^{\infty} \nu(h+\nu-1)\alpha_k x^k y^k, \\ &= x^\nu \sum_{k=0}^{\infty} \alpha_k x^k y^{k+1} - \sum_{k=0}^{\infty} x^{k-1} k(h+k-1)\alpha_k y^k - x^{\nu-1} \sum_{k=0}^{\infty} \nu(h+\nu-1)\alpha_k x^k y^k \end{aligned}$$

Considering the action of A on an eigenspace of the operator $h = \boldsymbol{\omega} + \frac{d+2}{2}$ with a fixed eigenvalue h_0, we have :

$$
\begin{aligned}
yA &= x^{\nu}\sum_{k=0}^{\infty}\alpha_k x^k y^{k+1} - \sum_{k=0}^{\infty} x^{k-1}k(h_0-k-1)\alpha_k y^k - x^{\nu-1}\sum_{k=0}^{\infty}\nu(h_0+\nu-1)\alpha_k x^k y^k, \\
&= x^{\nu-1}\left(\sum_{k=1}^{\infty}\alpha_{k-1}x^k y^k - (h_0-1)\sum_{k=0}^{\infty}k\alpha_k x^k y^k\right. \\
&\qquad\qquad\left. + \sum_{k=0}^{\infty}k^2\alpha_k x^k y^k - \nu(h_0+\nu-1)\sum_{k=0}^{\infty}\alpha_k x^k y^k\right).
\end{aligned}
$$

In order to ensure $Ay = 0$, we demand that the operator between brackets vanish identically. To find a solution, it is necessary to be a little more precise about how we would like A to act. In fact, the idea would be to take some smooth data f_0 on the boundary, extend it arbitrarily to $\bar{f}_0$ over M and $A\bar{f}_0$ should satisfy $yA\bar{f}_0 = 0$ and $x^{-\nu}A\bar{f}_0$ should restrict to f_0 on the boundary. In other words, $\alpha_0 = 1$. Thus, after rewriting the above equation in terms of a formal series $F(z) := \sum_{k=0}^{\infty}\alpha_k z^k \in \mathbb{C}[[z]]$, where $z^k =: (xy)^k := x^k y^k$, we see that necessarily:

$$\nu(h_0+\nu-1) = 0. \tag{4.50}$$

Taking this into account, we find that the formal series F satisfies the ODE:

$$(zF')' - (h_0-1)F' + F = 0.$$

Equation (4.50) should be compared with the indicial equation (4.47) we obtained when applying the Frobenius method in de-Sitter space.

4.7 Exterior tractor calculus

In this section, we will enrich the boundary calculus in Lemma 4.6.1 for tractor forms in order to develop similar projective methods for Proca style equations on k-forms. We return to the general setting of a n-dimensional projective manifold $(M, \boldsymbol{p})$. The first stage is to understand the exterior algebra of projective co-tractor k-forms, we begin by describing how the splitting of the exact sequence in Proposition (4.3.1) induces a splitting of $\Lambda^k \mathcal{T}^*$:

Lemma 4.7.1. *Let $(M, \boldsymbol{p})$ be a projective manifold of dimension n, $k \in [\![1, n+1]\!]$ and $\nabla \in \boldsymbol{p}$ then:*

$$\Lambda^k \mathcal{T}^* \overset{\nabla}{\cong} (\Lambda^{k-1} T^* M)(k) \oplus (\Lambda^k T^* M)(k).$$

Any section $F_{A_1 \dots A_k}$ can be expressed as:

$$F_{A_1 \dots A_k} = \begin{pmatrix} \mu_{a_2 \dots a_k} \\ \xi_{a_1 \dots a_k} \end{pmatrix} = k \mu_{a_2 \dots a_k} Y_{[A_1} Z_{A_2}^{a_2} \cdots Z_{A_k]}^{a_k} + \xi_{a_1 \dots a_k} Z_{A_1}^{a_1} Z_{A_2}^{a_2} \cdots Z_{A_k}^{a_k}. \tag{4.51}$$

(The second component vanishes if $k = n+1$). Under the change of connection $\hat{\nabla} = \nabla + \Upsilon$ the components transform according to:

$$\begin{cases} \hat{\mu} = \mu, \\ \hat{\xi} = \xi + \Upsilon \wedge \mu. \end{cases} \tag{4.52}$$

The reader will find a proof of Lemma 4.7.1 in Appendix E.2.

4.7.1 Wedge product and exterior derivative

The next stage is to describe how the usual operations of exterior calculus work with respect to the representation in Lemma 4.7.1. The wedge product is relatively simple:

Lemma 4.7.2. *Let $F \in \Lambda^k \mathcal{T}^*$, $G \in \Lambda^l \mathcal{T}^*$, and $\nabla \in \boldsymbol{p}$ on a projective manifold $(M, \boldsymbol{p})$. Suppose that:*

$$F \overset{\nabla}{=} \begin{pmatrix} \mu \\ \xi \end{pmatrix}, \quad G \overset{\nabla}{=} \begin{pmatrix} \nu \\ \eta \end{pmatrix},$$

then:

$$F \wedge G \overset{\nabla}{=} \begin{pmatrix} \mu \wedge \eta + (-1)^k \xi \wedge \nu \\ \xi \wedge \eta \end{pmatrix}. \tag{4.53}$$

A tractor analogue of the exterior derivative is, as for the Laplacian, provided, by the Thomas D-operator (Definition 4.6.1). The result can be stated as follows:

Proposition 4.7.1. *Let D_A denote the projective Thomas D-operator then one can define a co-chain complex:*

$$\cdots \longrightarrow \mathcal{E}_{[A_1,\dots,A_k]}(\omega) \xrightarrow{\mathscr{D}} \mathcal{E}_{[A_1,\dots,A_{k+1}]}(\omega-1) \xrightarrow{\mathscr{D}} \mathcal{E}_{[A_1,\dots,A_{k+2}]}(\omega-2) \longrightarrow \cdots$$

The operator $\mathscr{D}$ is defined on a section $F \in \mathcal{E}_{[A_1,\dots,A_k]}(\omega)$ by

$$\mathscr{D}F = (k+1)D_{[A_1}F_{A_2\cdots A_{k+1}]}.$$

Furthermore, in terms of Lemma 4.7.1, if $F \stackrel{\nabla}{=} \begin{pmatrix} \mu_{a_2\cdots a_k} \\ \xi_{a_1\cdots a_k} \end{pmatrix}$ then:

$$\mathscr{D}F \stackrel{\nabla}{=} \begin{pmatrix} (\omega+k)\xi_{a_2\cdots a_{k+1}} - k\nabla_{[a_2}\mu_{a_3\cdots a_{k+1}]} \\ (k+1)\nabla_{[a_1}\xi_{a_2\cdots a_{k+1}]} + \frac{(k+1)!}{(k-1)!}P_{[a_1a_2}\mu_{a_3\cdots a_{k+1}]} \end{pmatrix}. \tag{4.54}$$

Proof. First, we prove the expression for $\mathscr{D}F$ in the splitting associated with some connexion $\nabla \in \boldsymbol{p}$. Let $F \in \Lambda^k\mathcal{T}^*(\omega)$ be such that:

$$F_{A_1A_2\dots A_k} \stackrel{\nabla}{=} k\mu_{a_2\dots a_k}Y_{[A_1}Z^{a_2}_{A_2}\dots Z^{a_k}_{A_k]} + \xi_{a_1\dots a_k}Z^{a_1}_{A_1}Z^{a_1}_{A_2}\dots Z^{a_k}_{A_k}.$$

By definition:

$$D_AF_{A_1A_2\dots A_k} = \omega F_{A_1A_2\dots A_k}Y_A + Z^a_A\nabla_aF_{A_1A_2\dots A_k}.$$

Let us first concentrate on $\nabla_aF_{A_1A_2\dots A_k}$. Using Equation (4.30), we find that:

$$\begin{aligned}\nabla_aF_{A_1A_2\dots A_k} &= k\nabla_a\mu_{a_2\dots a_k}Y_{[A_1}Z^{a_2}_{A_2}\dots Z^{a_k}_{A_k]} + kP_{a[a_1}\mu_{a_2\dots a_k]}Z^{a_1}_{A_1}\dots Z^{a_k}_{A_k} \\ &\quad + \nabla_a\xi_{a_1\dots a_k}Z^{a_1}_{A_1}\dots Z^{a_k}_{A_k} - \xi_{a_1\dots a_k}\sum_{i=1}^{k} Z^{a_1}_{A_1}\dots Z^{a_{i-1}}_{A_{i-1}}\delta^{a_i}_a Y_{A_i}Z^{a_{i+1}}_{A_{i+1}}\dots Z^{a_k}_{A_k},\end{aligned}$$

in which the last term simplifies to:

$$-k\xi_{aa_2\dots a_k}Y_{[A_1}Z^{a_2}_{A_2}\dots Z^{a_k}_{A_k]}.$$

Hence, in column vector form this can be written as:

$$\nabla_a F = \begin{pmatrix} \nabla_a \mu_{a_2 \ldots a_k} - \xi_{a a_2 \ldots a_k} \\ \nabla_a \xi_{a_1 \ldots a_k} + k P_{a[a_1} \mu_{a_2 \ldots a_k]} \end{pmatrix}.$$

Since $\mathscr{D} F = (k+1) D_{[A} F_{A_1 A_2 \ldots A_k]}$, any terms containing two Y_{A_i} will not contribute in the final expression, furthermore for an arbitrary (weighted) tensor $T_{a_1 \ldots a_k}$:

$$T_{a_1 \ldots a_k} Z^{a_1}_{[A_1} \ldots Z^{a_k}_{A_k]} = T_{[a_1 \ldots a_k]} Z^{a_1}_{A_1} \ldots Z^{a_k}_{A_k}.$$

Hence:

$$\begin{aligned} \mathscr{D} F \overset{\nabla}{=} &(k+1) \omega \xi_{a_1 \ldots a_k} Y_{[A} Z^{a_1}_{A_1} \ldots Z^{a_k}_{A_k]} + \underset{\sim\sim\sim}{k(k+1)(\nabla_a \mu_{a_2 \ldots a_k} - \xi_{a a_2 \ldots a_k}) Y_{[A_1} Z^a_A Z^{a_2}_{A_2} \ldots Z^{a_k}_{A_k]}} \\ &\left((k+1) \nabla_a \xi_{a_1 \ldots a_k} + k(k+1) P_{[a a_1} \mu_{a_2 \ldots a_k]} \right) Z^a_A Z^{a_1}_{A_1} \ldots Z^{a_k}_{A_k}. \end{aligned}$$

Swapping A and A_1 in the underlined term in order to respect the sign conventions laid out implicitly in Lemma 4.7.1 we arrive at the desired result. We now proceed to calculate $\mathscr{D}^2 F$ using 4.54. For readability, we treat each slot in the column vector notation separately. First of all, in the top slot we have:

$$\begin{aligned} &(\omega + k)(k+1) \nabla_{[a_2} \xi_{a_3 \cdots a_{k+2}]} + (\omega + k) \frac{(k+1)!}{(k-1)!} P_{[a_2 a_3} \mu_{a_4 \ldots a_{k+2}]} \\ &\quad - (k+1)(\omega + k) \nabla_{[a_2} \xi_{a_3 \ldots a_{k+2}]} - k(k+1) \nabla_{[a_2} \nabla_{a_3} \mu_{a_4 \ldots a_{k+2}]} \\ &= (\omega + k) \frac{(k+1)!}{(k-1)!} P_{[a_2 a_3} \mu_{a_4 \ldots a_{k+2}]} - k(k+1) \nabla_{[a_2} \nabla_{a_3} \mu_{a_4 \ldots a_{k+2}]}. \end{aligned} \tag{4.55}$$

As for the bottom slot, we have:

$$\begin{aligned} &(k+2)(k+1) \nabla_{[a_1} \nabla_{a_2} \xi_{a_3 \ldots a_{k+2}]} + \frac{(k+2)!}{(k-1)!} \nabla_{[a_1} P_{a_2 a_3} \mu_{a_4 \cdots a_{k+2}]} \\ &+ \frac{(k+2)!}{k!} (\omega + k) P_{[a_1 a_2} \xi_{a_3 \cdots a_{k+2}]} - \frac{(k+2)!}{(k-1)!} P_{[a_1 a_2} \nabla_{a_3} \mu_{a_4 \ldots a_{k+2}]} \\ &= (k+2)(k+1) \nabla_{[a_1} \nabla_{a_2} \xi_{a_3 \ldots a_{k+2}]} + \frac{(k+2)!}{k!} (\omega + k) P_{[a_1 a_2} \xi_{a_3 \cdots a_{k+2}]} \\ &\qquad + \frac{(k+2)!}{(k-1)!} Y_{[a_1 a_2 a_3} \mu_{a_4 \cdots a_{k+2}]}. \end{aligned} \tag{4.56}$$

Where we recall that $Y_{abc} := 2 \nabla_{[a} P_{b]c}$. Expressions (4.55) and (4.56) simplify enormously in the case that ∇ is a special connection (cf. Section 4.3.3). Since there is always a special connection in any projective class [22], there is no loss in generality if we restrict to this case. We appeal now to Lemma 4.3.2, which states that P_{ab} is symmetric so that:

$$\mathscr{D}^2 F \overset{\nabla}{=} \begin{pmatrix} -d^2\mu \\ d^2\xi \end{pmatrix}. \tag{4.57}$$

d denotes here the covariant exterior derivative on the weighted tensor bundles. In general, $d^2 \neq 0$, however, here, using again Lemma 4.3.2, the density bundles are flat so we can conclude that:

$$\mathscr{D}^2 F = 0.$$

Remark 4.7.1. In the preceding proof, one can avoid choosing a particular scale and directly use the fact that $2P_{[ab]} = -\beta_{ab}$ to show that the final expressions in equations (4.55) and (4.56) vanish.

4.7.2 Tractor Hodge duality derived from a solution of the metrisability equation

Towards our aim to formulate a tractor version of the Proca equation, we describe here how one can use a solution to the metrisability equation to define a tractor Hodge star operator. The setting is as follows, we suppose we have a projective manifold with boundary $(\overline{M}, \boldsymbol{p})$, with oriented interior M and boundary ∂M, in addition to a solution ζ of the metrisability equation with degeneracy locus $D(\zeta) = \partial M$ and such that $H^{AB} = L(\zeta^{ab})$ is non-degenerate on $\overline{M}$.

On M, ζ^{ab} is non-degenerate and defines a smooth metric on the weighted cotangent bundle $T^*M(1)$. The orientation on M induces a natural orientation on $(\Lambda^n T^*M)(n)$, so it makes sense to talk of the positive volume form (induced by ζ) $\omega \in \Gamma\left((\Lambda^n T^*M)(n)\right)$. To define an orientation on the tractor bundle, we introduce: $\sigma = |\omega|^{-2} \in \Gamma(\mathcal{E}(2))$; it is a positive defining density for the boundary. Set $I_A = D_A\sigma \in \mathcal{E}_A(1)$, since H^{AB} is non-degenerate, the smooth function $\sigma^{-1}I^2$ is non-vanishing on $\overline{M}$, and so one can define on M:

$$J^0_B = \frac{\sigma^{-\frac{1}{2}}}{\sqrt{|\sigma^{-1}I^2|}} I_B.$$

22. We assume that a manifold's topology is second-countable.

Since any positively oriented orthonormal frame (ω^i_a) induces an orthonormal family of tractors $J^i_B = Z^b_B\omega^i_b$, we define an orientation of the tractor bundle by declaring that $J^0 \wedge \cdots \wedge J^n$ is positive. Since $(J^0, \ldots, J^n)$ is an orthonormal family with respect to H^{AB}, this procedure also yields a local expression of the positive tractor volume form in the splitting of the Levi-Civita connexion ∇_g of $g^{ab} = \sigma\zeta^{ab}$ on M:

$$\Omega_{\mathcal{T}} \stackrel{\nabla_g}{=} \begin{pmatrix} \frac{2\sigma^{\frac{1}{2}}}{\sqrt{|\sigma^{-1}I^2|}}\omega \\ 0 \end{pmatrix}.$$

We can now state the following result:

Proposition 4.7.2. *In the notation of the preceding paragraph, let $\nabla \in \boldsymbol{p}$ and $F \in \Lambda^k\mathcal{T}^*$ be such that: $F_{A_1\ldots A_k} \stackrel{\nabla}{=} k\mu_{a_2\ldots a_k}Y_{[A_1}Z^{a_2}_{A_2}\cdots Z^{a_k}_{A_k]} + \xi_{a_1\ldots a_k}Z^{a_1}_{A_1}\cdots Z^{a_k}_{A_k}$ then on the interior M:*

$$\star F \stackrel{\nabla}{=} \begin{pmatrix} \frac{2\sigma^{\frac{1}{2}}}{\sqrt{|\sigma^{-1}I^2|}}\left((-1)^k \star\xi + T\lrcorner(\star\mu)\right) \\ \frac{\sigma^{-\frac{3}{2}}I^2}{2\sqrt{|\sigma^{-1}I^2|}}\star\mu - \frac{2\sigma^{\frac{1}{2}}}{\sqrt{|\sigma^{-1}I^2|}}\left[(-1)^k T^\flat \wedge (\star\xi) + T^\flat \wedge T\lrcorner(\star\mu)\right] \end{pmatrix}, \tag{4.58}$$

where: $T^b = -\frac{1}{n+1}\nabla_a\zeta^{ab}$, $\flat$ denotes the lowering of indices using ζ_{ab} and $\lrcorner$ denotes contraction.

Proof. Let us first verify that the formula is reasonable on M in the splitting determined by ∇_g. Since T is zero in this scale, the formula reduces to:

$$\star F \stackrel{\nabla_g}{=} \begin{pmatrix} (-1)^k\frac{2\sigma^{\frac{1}{2}}}{\sqrt{|\sigma^{-1}I^2|}}\star\xi \\ \frac{\sigma^{-\frac{3}{2}}I^2}{2\sqrt{|\sigma^{-1}I^2|}}\star\mu \end{pmatrix}.$$

Using Equation (4.53) let us calculate $F \wedge \star F$ in the scale ∇_g. The result is:

$$F \wedge \star F = \begin{pmatrix} \frac{\sigma^{-\frac{3}{2}}I^2}{2\sqrt{|\sigma^{-1}I^2|}}\mu \wedge \star\mu + \frac{2\sigma^{\frac{1}{2}}}{\sqrt{|\sigma^{-1}I^2|}}\xi \wedge \star\xi \\ 0 \end{pmatrix}. \tag{4.59}$$

Recall that the inner product h on $\Lambda^k\mathcal{T}^*$ is defined by:

$$h(F,G) = \frac{1}{k!}H^{A_1B_1}\ldots H^{A_kB_k}F_{A_1\ldots A_k}G_{B_1\ldots B_k}.$$

Since, in the splitting given by ∇_g, we have that: $H^{AB} \stackrel{\nabla_g}{=} \zeta^{ab}W^A_aW^B_b + \frac{1}{n}\zeta^{ab}P_{ab}X^AX^B$, hence:

$$h(F,F) = \frac{1}{n}\zeta^{ab}P_{ab}\zeta(\mu,\mu) + \zeta(\xi,\xi),$$

where, P_{ab} is the projective Schouten tensor *in the scale* ∇_g and, for any section of ν of $(\Lambda^k T^*M)(k+\omega)$, $\zeta(\nu,\nu)$ is shorthand for:

$$\frac{1}{k!}\zeta^{a_1b_1}\cdots\zeta^{a_kb_k}\nu_{a_1\ldots a_k}\nu_{b_1\ldots b_k}.$$

In order to have a totally invariant formula, we observe that $I_A = D_A\sigma \overset{\nabla_g}{=} 2\sigma Y_A$, thus:

$$I^2 = \frac{4\sigma^2}{n}\zeta^{ab}P_{ab},$$

where P_{ab} is calculated in the scale ∇_g. Hence:

$$\frac{\zeta^{ab}P_{ab}}{n} = \frac{I^2\sigma^{-2}}{4}.$$

Consequently:

$$h(F,F) = \frac{I^2\sigma^{-2}}{4}\zeta(\mu,\mu) + \zeta(\xi,\xi)$$

Evaluating the top slot in Equation (4.59), we find:

$$\begin{aligned}\frac{\sigma^{-\frac{3}{2}}I^2}{2\sqrt{|\sigma^{-1}I^2|}}\mu\wedge\star\mu + \frac{2\sigma^{\frac{1}{2}}}{\sqrt{|\sigma^{-1}I^2|}}\xi\wedge\star\xi &= \frac{\sigma^{-\frac{3}{2}}I^2}{2\sqrt{|\sigma^{-1}I^2|}}\zeta(\mu,\mu)\omega + \frac{2\sigma^{\frac{1}{2}}}{\sqrt{|\sigma^{-1}I^2|}}\zeta(\xi,\xi)\omega \\ &= h(F,F)\frac{2\sigma^{\frac{1}{2}}}{\sqrt{|\sigma^{-1}I^2|}}\omega.\end{aligned}$$

Therefore:

$$F\wedge\star F = h(F,F)\Omega_{\mathcal{T}},$$

as desired. To verify that the result is correct for any connection in $\boldsymbol{p}$, we only need to verify that the components in Equation (4.58) transform correctly under a change of connection $\nabla \to \nabla + \Upsilon = \hat{\nabla}$. According to Equation (4.52), the top slot, (TS), must be invariant. To check this, note that $\hat{\xi} = \xi + \Upsilon\wedge\mu$ and $\hat{\mu} = \mu$. Furthermore, $\hat{T}^b = -\frac{1}{n+1}\hat{\nabla}_a\zeta^{ab}$ and:

$$\hat{\nabla}_c\zeta^{ab} = \nabla_c\zeta^{ab} + \Upsilon_d\zeta^{db}\delta^a_c + \Upsilon_d\zeta^{ad}\delta^b_c,$$

which leads to:

$$\hat{\nabla}_a \zeta^{ab} = \nabla_a \zeta^{ab} + (n+1)\Upsilon_a \zeta^{ab}.$$

Therefore:

$$\hat{T} = T - \Upsilon^\sharp \text{ and } \hat{T}^\flat = T^\flat - \Upsilon. \tag{4.60}$$

Since $\hat{\xi} = \xi + \Upsilon \wedge \mu$, Proposition E.3.1 shows that:

$$\star\hat{\xi} = \star\xi + \star(\Upsilon \wedge \mu) = \star\xi + (-1)^k \Upsilon^\sharp \lrcorner \star \mu.$$

Thus, overall:

$$\begin{aligned}(-1)^k \star \hat{\xi} + \hat{T} \lrcorner \star \hat{\mu} &= (-1)^k \star \xi + \Upsilon^\sharp \lrcorner \star \mu - \Upsilon^\sharp \lrcorner \star \mu + T \lrcorner \star \mu, \\ &= (-1)^k \star \xi + T \lrcorner \star \mu,\end{aligned}$$

proving the projective invariance of the top slot as desired.

Referring again to Equation (4.52), we must now show that the bottom slot (BS) of Equation (4.58) satisfies:

$$(\hat{BS}) = (BS) + \Upsilon \wedge (TS).$$

Observe that, since, $\hat{\mu} = \mu$ the first term in (BS) is invariant, moreover, the second term can be written: $-T^\flat \wedge (TS)$, so the desired result follows immediately from Equation (4.60).

4.7.3 Tractor co-differential and Hodge Laplacian

In the previous sections we have sufficiently enhanced the structure on the tractor tensor algebra to introduce a tractor co-differential operator. Towards this, let s denote the sign of the determinant of ζ^{ab} on M, and set:

$$\varepsilon = \operatorname{sgn}(\sigma^{-1} I^2);$$

observe that the sign of the determinant of H^{AB} is then $s\varepsilon$. For future convenience, we introduce the notations:

$$f = \sigma^{-1} I^2, \quad f' = \mathrm{d}f. \tag{4.61}$$

Now:

Definition 4.7.1. By analogy with the usual exterior calculus, we define a tractor codifferential on k-cotractors by:

$$\mathscr{D}^* = (-1)^{(n+1)(k-1)+1} s\varepsilon \star \mathscr{D} \ \star .$$

In the scale ∇_g on M, for any $F \stackrel{\nabla_g}{=} \begin{pmatrix} \mu \\ \xi \end{pmatrix} \in \Gamma[(\Lambda^k \mathcal{T}^*)(\omega)]$:

$$\mathscr{D}^* F \stackrel{\nabla_g}{=} \begin{pmatrix} \frac{1}{2f}(f')^{\#} \lrcorner \mu - \delta\mu \\ \frac{1}{2f}(f')^{\#} \lrcorner \xi + \delta\xi - \frac{(\omega+n+1-k)\sigma^{-1} f}{4}\mu \end{pmatrix}. \tag{4.62}$$

In the above equation, we have introduced the notation $(f')^{\#} = \zeta^{ab}\nabla_b f \in \mathcal{E}(-2)$.

Proof. We prove Equation (4.62): applying successively Equations (4.58) and (4.54) to $F \stackrel{\nabla_g}{=} \begin{pmatrix} \mu \\ \xi \end{pmatrix} \in \Gamma[(\Lambda^k \mathcal{T}^*)(\omega)]$, we find:

$$\begin{aligned}
\mathscr{D} \star F &= \begin{pmatrix} \frac{(\omega+n+1-k)\sigma^{-\frac{1}{2}} f}{2|f|^{\frac{1}{2}}} \star \mu + 2(-1)^{k+1} d\left(\left(\frac{\sigma}{|f|}\right)^{\frac{1}{2}} \star \xi \right) \\ d\left(\frac{\sigma^{-\frac{1}{2}} f}{2|f|^{\frac{1}{2}}} \star \mu \right) \end{pmatrix}, \\
&= \begin{pmatrix} \frac{(\omega+n+1-k)\sigma^{-\frac{1}{2}} f}{2|f|^{\frac{1}{2}}} \star \mu + \frac{2\sigma^{\frac{1}{2}}(-1)^{k+1}}{|f|^{\frac{1}{2}}} \left(-\frac{1}{2f} f' \wedge \star\xi + d(\star\xi) \right) \\ \frac{\sigma^{-\frac{1}{2}}}{4|f|^{\frac{1}{2}}} f' \wedge \star\mu + \frac{\sigma^{-\frac{1}{2}} f}{2|f|^{\frac{1}{2}}} d \star \mu \end{pmatrix}.
\end{aligned}$$

Applying once more the tractor Hodge star to this $n+1-(k-1)$ form of weight $\omega - 1$, we find:

$$\begin{aligned}
\star\mathscr{D} \star F &= \begin{pmatrix} \frac{(-1)^{n-k}}{2|f|} \star (f' \wedge \star\mu) + \frac{(-1)^{n-k} f}{|f|} \star d \star \mu \\ \frac{(\omega+n+1-k)\sigma^{-1} f^2}{4|f|} \star\star\mu + \frac{(-1)^{k+1} f}{|f|} \left(-\frac{1}{2f} \star (f' \wedge \star\xi) + \star d \star \xi \right) \end{pmatrix} \\
&= (-1)^{(n+1)(k+1)} s \begin{pmatrix} \frac{-1}{2|f|}(f')^{\#} \lrcorner \mu + \frac{f}{|f|}\delta\mu \\ \frac{(\omega+n+1-k)\sigma^{-1} f^2}{4|f|}\mu - \frac{f}{|f|}\left(\frac{1}{2f}(f')^{\#} \lrcorner \xi + \delta\xi \right) \end{pmatrix}.
\end{aligned}$$

Equation (4.62) now follows from the fact that $\varepsilon = \mathrm{sgn}(\sigma^{-1} I^2) = \mathrm{sgn}(f)$.

The exterior differential calculus we have developed above leads us to define a new Laplacian operator, analogous to the Hodge or de-Rham Laplacian. In general, it is to be distinguished from $H^{AB} D_A D_B$ that we studied in Section (4.6.2). The remainder of this section is devoted to obtaining an expression for $\mathscr{D}^*\mathscr{D} + \mathscr{D}\mathscr{D}^* = \{\mathscr{D}, \mathscr{D}^*\}$. In order to simplify the computation, we will work exclusively with the Levi-Civita connection ∇_g of $g = \sigma^{-1}\zeta^{-1}$ on M; any identities will extend by density to $\overline{M}$. We first remark that the weight 2 density I^2 is, in general, not parallel in the scale ∇_g.

Lemma 4.7.3. *In an arbitrary scale $\nabla \in \boldsymbol{p}$:*

$$\nabla_c I^2 = \frac{8\sigma^2}{n}\zeta^{ef}Y_{cef} - \frac{4\sigma}{n}\nabla_a\sigma\zeta^{ef}W_{ce}{}^b{}_f$$

Proof.

$$\nabla_c I^2 = (\nabla_c H^{AB})I_A I_B + 2H^{AB}(\nabla_a I_A)I_B.$$

The second term is easily seen to cancel, and we evaluate the first one using Equation (ME2) and the calculations we did in Section 4.5.1. Finally, we have $I_A = 2\sigma Y^A + \nabla_a\sigma Z^a_A$, thus:

$$2X^{(A}W_{cE}{}^{B)}{}_F H^{EF}I_A I_B = -8\sigma^2\zeta^{ef}Y_{cef} + 4\sigma\nabla_a\sigma\zeta^{ef}W_{ce}{}^b{}_f.$$

Let F denote an arbitrary section of $(\Lambda^k\mathcal{T}^*)(\omega)$ given in the scale ∇_g by:

$$F \overset{\nabla_g}{=} \begin{pmatrix} \mu \\ \xi \end{pmatrix}.$$

Equation (4.54) yields directly the expression for $\mathscr{D}F$, which, due to the symmetry of P_{ab}, simplifies to:

$$\mathscr{D}F \overset{\nabla_g}{=} \begin{pmatrix} (\omega+k)\xi - d\mu \\ d\xi \end{pmatrix}.$$

In the above expression d denotes, abusively, the *covariant* exterior derivative[23] on the weighted bundles. Now:

$$\star\,\mathscr{D}F = \begin{pmatrix} (-1)^{k+1}\frac{2\sigma^{\frac{1}{2}}}{|f|^{\frac{1}{2}}} \star d\xi \\ \sigma^{-\frac{1}{2}}\frac{f}{2|f|^{\frac{1}{2}}}\;((\omega+k)\star\xi - \star d\mu) \end{pmatrix}.$$

We apply again $\mathscr{D}$, observing first that $\star\mathscr{D}F$ is a $(n+1-(k+1)) = n-k$ form of weight

23. cf. Annexe C

$\omega - 1$, thus:

$$\mathscr{D} \star \mathscr{D} F = \begin{pmatrix} \frac{(\omega-1+n-k)f}{2(|f|\sigma)^{\frac{1}{2}}}\left((\omega+k)\star\xi - \star d\mu\right) + 2(-1)^k d\left(\frac{\sigma^{\frac{1}{2}}}{|f|^{\frac{1}{2}}}\star d\xi\right) \\ d\left(\sigma^{-\frac{1}{2}}\frac{f}{2|f|^{\frac{1}{2}}}\ \left((\omega+k)\star\xi - \star d\mu\right)\right). \end{pmatrix}$$

For readability, we will now proceed to treat the top and bottom slots separately. Let us begin with the unevaluated differential in the top slot:

$$d\left(\frac{\sigma^{\frac{1}{2}}}{|f|^{\frac{1}{2}}}\star d\xi\right) = -\sigma^{\frac{1}{2}}\frac{f'\wedge\star d\xi}{2f|f|^{\frac{1}{2}}} + \frac{\sigma^{\frac{1}{2}}}{|f|^{\frac{1}{2}}}d\star d\xi.$$

Downstairs we have:

$$\frac{\sigma^{-\frac{1}{2}}}{4|f|^{\frac{1}{2}}}f'\wedge\left((\omega+k)\star\xi - \star d\mu\right) + \frac{\sigma^{-\frac{1}{2}}f}{2|f|^{\frac{1}{2}}}\left((\omega+k)d\star\xi - d\star d\mu\right).$$

Applying again the Hodge star to this $(n+1-k)$-tractor form of weight $(\omega-2)$ leads to a new k-tractor form of weight $\omega - 2$ with, in the top slot:

$$\frac{(-1)^{1+k(n-k)}s}{2|f|}\left((\omega+k)(f')^{\#}\lrcorner\xi - (f')^{\#}\lrcorner d\mu\right) + \frac{(-1)^{k(n-k)}sf}{|f|}((\omega+k)\delta\xi - \delta d\mu).$$

As for the bottom slot, it evaluates to:

$$\frac{(\omega-1+n-k)f^2(-1)^{k(n-k)}s}{4|f|\sigma}((\omega+k)\xi - d\mu) + \frac{(-1)^{k(n-k)+1}s}{|f|}\left(\frac{(f')^{\#}\lrcorner d\xi}{2} + f\delta d\xi\right).$$

Overall:

$$\star\mathscr{D}\star\mathscr{D}F = \begin{pmatrix} \frac{(-1)^{1+k(n-k)}s}{2|f|}\left((\omega+k)(f')^{\#}\lrcorner\xi - (f')^{\#}\lrcorner d\mu\right) + \frac{(-1)^{k(n-k)}sf}{|f|}((\omega+k)\delta\xi - \delta d\mu) \\ \frac{(\omega-1+n-k)f^2(-1)^{k(n-k)}s}{4|f|\sigma}((\omega+k)\xi - d\mu) + \frac{(-1)^{k(n-k)+1}s}{|f|}\left(\frac{(f')^{\#}\lrcorner d\xi}{2} + f\delta d\xi\right) \end{pmatrix}.$$

It remains only to correct the sign ! The result is:

$$\mathscr{D}^*\mathscr{D}F \overset{\nabla_g}{=} \begin{pmatrix} \frac{1}{2f}\left((\omega+k)(f')^{\#}\lrcorner\xi - (f')^{\#}\lrcorner d\mu\right) - ((\omega+k)\delta\xi - \delta d\mu) \\ -\frac{(\omega-1+n-k)f}{4\sigma}((\omega+k)\xi - d\mu) + \left(\frac{(f')^{\#}\lrcorner d\xi}{2f} + \delta d\xi\right) \end{pmatrix}.$$

Calculating $\mathscr{D}\mathscr{D}^*$ is slightly less involved as we have only to apply Equation (4.54) to

Equation (4.62), taking care to note that $\mathscr{D}^* F$ is a $k-1$ form of weight $\omega - 1$.

$$\mathscr{D}\mathscr{D}^* F \overset{\nabla_g}{=} \begin{pmatrix} -(\omega+k-2)(\omega+n+1-k)\dfrac{\sigma^{-1}f}{4}\mu + (\omega+k-2)\left(\dfrac{1}{2f}(f')^{\#}\lrcorner\xi + \delta\xi\right) \\ -d(\dfrac{1}{2f}(f')^{\#}\lrcorner\mu) + d\delta\mu \\ d\delta\xi + d(\frac{1}{2f}(f')^{\#}\lrcorner\xi) - (\omega+n+1-k)\frac{\sigma^{-1}}{4}d(f\mu) \end{pmatrix}.$$

We summarise these computations in:

Proposition 4.7.3. *In the scale ∇_g, $\{\mathscr{D}^*, \mathscr{D}\}$ acts on $F \overset{\nabla_g}{=} \begin{pmatrix}\mu \\ \xi\end{pmatrix} \in \Gamma((\Lambda^k(\mathcal{T}^*)(\omega))$ as:*

$$\begin{pmatrix} \{d,\delta\}\mu - 2\delta\xi + (\omega+k-1)\frac{1}{f}\vec{\nabla} f\lrcorner\mu - \frac{1}{2}\mathcal{L}_{f^{-1}\vec{\nabla}f}\mu - (\omega+k-2)(\omega+n+1-k)\frac{\sigma^{-1}f}{4}\mu \\ \{d,\delta\}\xi - \frac{f\sigma^{-1}}{2}d\mu - (\omega+n+1-k)\frac{\sigma^{-1}}{4}(f'\wedge\mu) + \frac{1}{2}\mathcal{L}_{f^{-1}\vec{\nabla}f}\xi - (\omega+k)(\omega-1+n-k)\frac{f\sigma^{-1}}{4}\xi \end{pmatrix}. \tag{4.63}$$

In the above we have introduced the Lie derivative $\mathcal{L}_X$ extended to weighted vector fields X by the formula:

$$\mathcal{L}_X\xi_{a_1\dots a_k} = X^a\nabla_a\xi_{a_1\dots a_k} + k(\nabla_{[a_1}X^a)\xi_{|a|a_2\dots a_k]}.$$

4.7.4 Weitzenbock identity

Having already introduced the Laplacian type operator $\Delta^{\mathcal{T}} = H^{AB}D_AD_B$ on generic tractor k-coforms in Section 4.6.2, it is interesting to explore how it compares to $\{\mathscr{D}, \mathscr{D}^*\}$. It turns out the relationship between them is completely analogous to that between the Bochner Laplacian and the de-Rham Laplacian on the base manifold. As before, we perform all calculations in the Levi-Civita scale ∇_g. Recall from Equation (4.42) that then:

$$H^{AB}D_AD_BF \overset{\nabla_g}{=} \frac{(\omega+n-1)\omega f\sigma^{-1}}{4}F + \zeta^{ab}\nabla_a\nabla_bF,$$

where for a weighted k-cotractor form in an arbitrary scale ∇:

$$\nabla_a\nabla_bF \overset{\nabla}{=} \begin{pmatrix} \nabla_a\nabla_b\mu_{a_2\dots a_k} - 2\nabla_{(a}\xi_{b)a_2\dots a_k} - kP_{b[a}\mu_{a_2\dots a_k]} \\ \nabla_a\nabla_b\xi_{a_1\dots a_k} + 2kP_{(a|[a_1}\nabla_{|b)|}\mu_{a_2\dots a_k]} + k(\nabla_aP_{b[a_1})\mu_{a_2\dots a_k]} - kP_{a[a_1}\xi_{|b|a_2\dots a_k]} \end{pmatrix}.$$

To simplify computations a little, we restrict now to *normal* solutions of the Metrisability equation (see Section 4.5.2), for which a number of terms in the above expressions, and in particular Equation (4.63), vanish. Furthermore, in the scale ∇_g, Equation (4.41) holds which, recast in terms of our current notations, becomes:

$$P_{cd} = \frac{f\sigma^{-1}}{4}\zeta_{cd}. \qquad ((4.41)\text{-}2)$$

Finally, ∇P_{cd} and all derivatives of f vanish. Overall, performing all the preceding simplifications, we have, for an arbitrary k form:

$$H^{AB}D_AD_BF \overset{\nabla_g}{=} \begin{pmatrix} \Box\mu + 2\delta\xi + (\omega(\omega+n-1)-(n+1-k))\frac{f\sigma^{-1}}{4}\mu \\ \Box\xi + \frac{f\sigma^{-1}}{2}\mathrm{d}\mu + (\omega(\omega+n-1)-k)\frac{f\sigma^{-1}}{4}\xi \end{pmatrix},$$

where we define: $\Box\mu = \zeta^{ab}\nabla_a\nabla_b\mu$.

Similarly, Equation (4.63) simplifies to:

$$\{\mathscr{D},\mathscr{D}^*\}F \overset{\nabla_g}{=} \begin{pmatrix} \{d,\delta\}\mu - 2\delta\xi - (\omega+k-2)(\omega+n+1-k)\frac{\sigma^{-1}f}{4}\mu \\ \\ \{d,\delta\}\xi - \frac{f\sigma^{-1}}{2}d\mu - (\omega+k)(\omega-1+n-k)\frac{f\sigma^{-1}}{4}\xi \end{pmatrix}.$$

From these expressions we will show:

Proposition 4.7.4. *Let $F \in \Gamma((\Lambda^k\mathcal{T}^*)(\omega))$ and suppose that H^{AB} is a normal solution to the Metrisability equation, then:*

$$(\{\mathcal{D},\mathcal{D}^*\}F)_{A_1\dots A_k} = -(H^{AB}D_AD_BF)_{A_1\dots A_k} + k(k+1)H^{AB}\Omega_{[A|B|}{}^{C}{}_{A_1}F_{|C|A_2\dots A_k]}.$$

In the above, $\Omega_{AB}{}^C{}_D = \Omega_{ab}{}^C{}_D Z^a_A Z^a_B$ and $\Omega_{ab}{}^C{}_D$ is the tractor curvature tensor. (cf. Equation (4.32) and the end of Section (4.3.4)).

Proof. We first inspect the difference between the order zero terms in each slot of the two tractors:

$$\begin{aligned}(\omega+k-2)(\omega+n+1-k) &= \omega(\omega+n-1)-(n+1-k)-(k-1)(n+1-k),\\ (\omega+k)(\omega+n-1-k) &= \omega(\omega+n-1)-k+k(n-k).\end{aligned} \qquad (4.64)$$

Moreover, in index notation the usual Weitzenbock identity extended to weighted tensors reads:

$$\begin{aligned}\{d,\delta\}\xi_{a_1\ldots a_k}+\zeta^{ab}\nabla_a\nabla_b\xi_{a_1\ldots a_k}=&\sum_{i=1}^{k}\zeta^{ab}R_{a_ia}{}^{c}{}_{b}\xi_{a_1\ldots a_{i-1}ca_{i+1}\ldots a_k}\\&+\sum_{i=1}^{k}\sum_{\substack{j=1\\j\neq i}}^{k}\zeta^{ab}R_{a_ia}{}^{c}{}_{a_j}\xi_{a_1\ldots a_{j-1}ca_{j+1}\ldots a_{i-1}ba_{i+1}\ldots a_k}.\end{aligned}\tag{4.65}$$

Now, appealing to Equations (4.26) and ((4.41)-2), we have:

$$R_{ab}{}^{c}{}_{d}=W_{ab}{}^{c}{}_{d}+\left(\delta_a^c\zeta_{bd}-\delta_b^c\zeta_{ad}\right)\frac{f\sigma^{-1}}{4}.$$

Therefore:

$$\begin{aligned}\zeta^{ab}R_{a_ia}{}^{c}{}_{b}\xi_{a_1\ldots a_{i-1}ca_{i+1}\ldots a_k}&=\zeta^{ab}W_{a_ia}{}^{c}{}_{b}\xi_{a_1\ldots a_{i-1}ca_{i+1}\ldots a_k}\\&\quad+\frac{f\sigma^{-1}}{4}\underbrace{\zeta^{ab}\left(\delta_{a_i}^c\zeta_{ab}-\delta_a^c\zeta_{a_ib}\right)}_{(n-1)\delta_{a_i}^c}\xi_{a_1\ldots a_{i-1}ca_{i+1}\ldots a_k}\\&=\zeta^{ab}W_{a_ia}{}^{c}{}_{b}\xi_{a_1\ldots a_{i-1}ca_{i+1}\ldots a_k}+\frac{f\sigma^{-1}}{4}(n-1)\xi_{a_1\ldots a_k},\end{aligned}$$

and:

$$\begin{aligned}\zeta^{ab}R_{a_ia}{}^{c}{}_{a_j}\xi_{a_1\ldots a_{j-1}ca_{j+1}\ldots a_{i-1}ba_{i+1}\ldots a_k}&=\zeta^{ab}W_{a_ia}{}^{c}{}_{a_j}\xi_{a_1\ldots a_{j-1}ca_{j+1}\ldots a_{i-1}ba_{i+1}\ldots a_k}\\&\quad+\frac{f\sigma^{-1}}{4}\underbrace{\zeta^{ab}\left(\delta_{a_i}^c\zeta_{aa_j}-\delta_a^c\zeta_{a_ia_j}\right)\xi_{a_1\ldots a_{j-1}ca_{j+1}\ldots a_{i-1}ba_{i+1}\ldots a_k}}_{=-\xi_{a_1\ldots a_k}}.\end{aligned}$$

So Equation (4.65) can be written:

$$\{d,\delta\}\xi_{a_1\ldots a_k}+\zeta^{ab}\nabla_a\nabla_b\xi_{a_1\ldots a_k}=\begin{bmatrix}\text{Terms involving}\\\text{Weyl tensor}\end{bmatrix}+\underbrace{[(n-1)k-k(k-1)]}_{=k(n-k)}\frac{f\sigma^{-1}}{4}\xi_{a_1\ldots a_k}.$$

The second term in the above equation accounts exactly for the differences observed in

Equation (4.64) and it follows that:

$$\{\mathcal{D},\mathcal{D}^*\}F + H^{AB}D_AD_BF \overset{\nabla_g}{=} \begin{pmatrix} \sum_{i=1}^{k-1}\zeta^{ab}W_{a_ia}{}^c{}_b\mu_{a_1\dots a_{i-1}ca_{i+1}\dots a_{k-1}} \\ \qquad + \sum_{i=1}^{k-1}\sum_{\substack{j=1\\ j\neq i}}^{k-1}\zeta^{ab}W_{a_ia}{}^c{}_{a_j}\mu_{a_1\dots a_{j-1}ca_{j+1}\dots a_{i-1}ba_{i+1}\dots a_{k-1}}. \\ \sum_{i=1}^{k}\zeta^{ab}W_{a_ia}{}^c{}_b\xi_{a_1\dots a_{i-1}ca_{i+1}\dots a_k} \\ \qquad + \sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k}\zeta^{ab}W_{a_ia}{}^c{}_{a_j}\xi_{a_1\dots a_{j-1}ca_{j+1}\dots a_{i-1}ba_{i+1}\dots a_k}. \end{pmatrix}$$

We must now attempt to identify the tractor on the right-hand side of the above equation. We claim that it is exactly $k(k+1)H^{AB}\Omega_{[A|B|}{}^C{}_{A_1}F_{|C|A_2\dots A_k]}$. Where:

$$\Omega_{AB}{}^C{}_D = \Omega_{ab}{}^C{}_DZ^a_AZ^b_B \overset{\nabla_g}{=} W_{ab}{}^c{}_dW^C_cZ^a_AZ^b_BZ^d_D.$$

The calculation is « merely » technical and presents no conceptual subtleties, therefore we will only carry it out here fully on the bottom component and leave the upper component to the reader. We first write:

$$k(k+1)H^{AB}\Omega_{[A|B|}{}^C{}_{A_1}F_{|C|A_2\dots A_k]} = H^{AB}\frac{1}{(k-1)!}\sum_{\sigma\in\mathfrak{S}_{k+1}}\varepsilon(\sigma)\Omega_{D_{\sigma(1)}B}{}^C{}_{\sigma(2)}F_{CD_{\sigma(3)}\dots D_{\sigma(k+1)}}.$$

For convenience, we have introduced a new set of indices $\{D_i\}$ defined by:

$$D_1 = A, D_i = A_{i-1}, i\geq 2.$$

Specialising to the Levi-Civita scale ∇_g, the right-hand side is:

$$H^{AB}\frac{1}{(k-1)!}\sum_{\sigma\in\mathfrak{S}_{k+1}}\varepsilon(\sigma)W_{d_1b}{}^c{}_{d_2}W^C_cZ^{d_1}_{D_{\sigma(1)}}Z^b_BZ^{d_2}_{D_{\sigma(2)}}F_{CD_{\sigma(3)}\dots D_{\sigma(k+1)}}.$$

Now: $H^{AB}Z^b_B \overset{\nabla_g}{=} \zeta^{ab}W^a_A$ so we must calculate:

$$T = \sum_{\sigma\in\mathfrak{S}_{k+1}}\varepsilon(\sigma)\zeta^{ab}W^A_aW_{d_1b}{}^c{}_{d_2}W^C_cZ^{d_1}_{D_{\sigma(1)}}Z^{d_2}_{D_{\sigma(2)}}F_{CD_{\sigma(3)}\dots D_{\sigma(k+1)}}.$$

In order to isolate the bottom slot, consider $F_{A_1\dots A_k} \overset{\nabla_g}{=} \xi_{a_1\dots a_k} Z^{a_1}_{A_1}\dots Z^{a_k}_{A_k}$. In this case,

$$
\begin{aligned}
T &= \sum_{\sigma\in\mathfrak{S}_{k+1}} \varepsilon(\sigma)\zeta^{ab} W^A_a W_{d_1 b}{}^{c}{}_{d_2} \xi_{\tilde{c} d_3\dots d_{k+1}} \underbrace{W^C_c Z^{\tilde{c}}_C}_{=\delta^{\tilde{c}}_c} Z^{d_1}_{D_{\sigma(1)}} Z^{d_2}_{D_{\sigma(2)}} Z^{d_3}_{D_{\sigma(3)}} \dots Z^{d_{k+1}}_{D_{\sigma(k+1)}},\\
&= \sum_{\sigma\in\mathfrak{S}_{k+1}} \varepsilon(\sigma)\zeta^{ab} W^A_a W_{d_1 b}{}^{c}{}_{d_2} \xi_{c d_3\dots d_{k+1}} Z^{d_1}_{D_{\sigma(1)}} Z^{d_2}_{D_{\sigma(2)}} Z^{d_3}_{D_{\sigma(3)}} \dots Z^{d_{k+1}}_{D_{\sigma(k+1)}}.
\end{aligned}
$$

Observe now that T can be rewritten:

$$
\begin{aligned}
T &= \sum_{\sigma\in\mathfrak{S}_{k+1}} \varepsilon(\sigma)\zeta^{ab} W^A_a W_{d_{\sigma(1)} b}{}^{c}{}_{d_{\sigma(2)}} \xi_{c d_{\sigma(3)}\dots d_{\sigma(k+1)}} Z^{d_1}_{D_1} Z^{d_2}_{D_2} Z^{d_3}_{D_3} \dots Z^{d_{k+1}}_{D_{k+1}},\\
&= \sum_{\sigma\in\mathfrak{S}_{k+1}} \varepsilon(\sigma)\zeta^{ab} W^A_a W_{d_{\sigma(1)} b}{}^{c}{}_{d_{\sigma(2)}} \xi_{c d_{\sigma(3)}\dots d_{\sigma(k+1)}} Z^{d_1}_{A} Z^{d_2}_{A_1} Z^{d_3}_{A_2} \dots Z^{d_{k+1}}_{A_k},\\
&= \sum_{\sigma\in\mathfrak{S}_{k+1}} \varepsilon(\sigma)\zeta^{d_1 b} W_{d_{\sigma(1)} b}{}^{c}{}_{d_{\sigma(2)}} \xi_{c d_{\sigma(3)}\dots d_{\sigma(k+1)}} Z^{d_2}_{A_1} Z^{d_3}_{A_2} \dots Z^{d_{k+1}}_{A_k}.
\end{aligned}
$$

If $\sigma(1) = 1$, then the summand vanishes leading to:

$$
T = \sum_{\substack{\sigma\in\mathfrak{S}_{k+1}\\ \sigma(1)\neq 1}} \varepsilon(\sigma)\zeta^{d_1 b} W_{d_{\sigma(1)} b}{}^{c}{}_{d_{\sigma(2)}} \xi_{c d_{\sigma(3)}\dots d_{\sigma(k+1)}} Z^{d_2}_{A_1} Z^{d_3}_{A_2} \dots Z^{d_{k+1}}_{A_k}.
$$

We now seek to exploit the antisymmetry of ξ, first we note that:

$$
T = \sum_{i=2}^{k+1} \sum_{\substack{\sigma\in\mathfrak{S}_{k+1}\\ \sigma(1)= i}} \varepsilon(\sigma)\zeta^{d_1 b} W_{d_i b}{}^{c}{}_{d_{\sigma(2)}} \xi_{c d_{\sigma(3)}\dots d_{\sigma(k+1)}} Z^{d_2}_{A_1} Z^{d_3}_{A_2} \dots Z^{d_{k+1}}_{A_k}.
$$

It is interesting to split the inner sum into two further sums as follows:

$$
\begin{aligned}
&\underbrace{\sum_{\substack{\sigma\in\mathfrak{S}_{k+1}\\ \sigma(1)= i\\ \sigma(2)=1}} \varepsilon(\sigma)\zeta^{d_1 b} W_{d_i b}{}^{c}{}_{d_1} \xi_{c d_{\sigma(3)}\dots d_{\sigma(k+1)}} Z^{d_2}_{A_1} Z^{d_3}_{A_2} \dots Z^{d_{k+1}}_{A_k}}_{=T^i_1}\\
&\qquad + \underbrace{\sum_{\substack{j=2\\ j\neq i}}^{k+1} \sum_{\substack{\sigma\in\mathfrak{S}_{k+1}\\ \sigma(2)=j\\ \sigma(1)=i}} \varepsilon(\sigma)\zeta^{d_1 b} W_{d_i b}{}^{c}{}_{d_j} \xi_{c d_{\sigma(3)}\dots d_{\sigma(k+1)}} Z^{d_2}_{A_1} Z^{d_3}_{A_2} \dots Z^{d_{k+1}}_{A_k}}_{=T^i_2}.
\end{aligned}
$$

Consider now a generic term in the sum T_1^i. If we define:

$$\tilde{d}_1 = c, \tilde{d}_l = d_{\sigma(l+1)}, 2 \leq l \leq k;$$

then:

$$\xi_{\tilde{d}_1 \dots \tilde{d}_k} = \varepsilon(s) \xi_{\tilde{d}_{s(1)} \dots \tilde{d}_{s(k)}}, s \in \mathfrak{S}_k.$$

Considering the permutation s given by:

$$s(l) = \begin{cases} \sigma^{-1}(l) - 1 & \text{if } 1 \leq l < i, \\ \sigma^{-1}(l+1) - 1 & \text{if } i \leq l \leq k, \end{cases}$$

which satisfies $s(1) = 1$, and, by Appendix E.2, $\varepsilon(s) = (-1)^{i-1} \varepsilon(\sigma)$, then we have:

$$\xi_{c d_{\sigma(3)} \dots d_{\sigma(k+1)}} = \xi_{\tilde{d}_1 \dots \tilde{d}_k} = (-1)^{i-1} \varepsilon(\sigma) \xi_{\tilde{d}_{s(1)} \dots \tilde{d}_{s(k)}} = (-1)^{i-1} \varepsilon(\sigma) \xi_{c d_2 \dots d_{i-1} d_{i+1} d_{k+1}}.$$

Overall:

$$T_1^i = (k-1)!(-1)^{i-1} \zeta^{d_1 b} W_{d_i b}{}^{c}{}_{d_1} \xi_{c d_2 \dots d_{i-1} d_{i+1} d_{k+1}} Z^{d_2}_{A_1} Z^{d_3}_{A_2} \dots Z^{d_{k+1}}_{A_k}.$$

Hence:

$$\begin{aligned} \frac{1}{(k-1)!} \sum_{i=2}^{k+1} T_k^i &= \sum_{i=2}^{k+1} (-1)^{i-1} \zeta^{d_1 b} W_{d_i b}{}^{c}{}_{d_1} \xi_{c d_2 \dots d_{i-1} d_{i+1} d_{k+1}} Z^{d_2}_{A_1} Z^{d_3}_{A_2} \dots Z^{d_{k+1}}_{A_k}, \\ &= \sum_{i=2}^{k+1} (-1)^{i-1} \zeta^{ab} W_{d_{i-1} b}{}^{c}{}_{a} \xi_{c d_1 \dots d_{i-2} d_i d_k} Z^{d_1}_{A_1} Z^{d_2}_{A_2} \dots Z^{d_k}_{A_k}, \\ &= \sum_{i=1}^{k} \zeta^{ab} W_{d_i b}{}^{c}{}_{a} \xi_{d_1 \dots d_{i-1} c d_{i+1} d_k} Z^{d_1}_{A_1} Z^{d_2}_{A_2} \dots Z^{d_k}_{A_k}. \end{aligned}$$

We move on now to study a generic term $a(\sigma, i, j)$ in T_2^i:

$$a(\sigma, i, j) = \varepsilon(\sigma) \zeta^{d_1 b} W_{d_i b}{}^{c}{}_{d_j} \xi_{c d_{\sigma(3)} \dots d_{\sigma(k+1)}} Z^{d_2}_{A_1} Z^{d_3}_{A_2} \dots Z^{d_{k+1}}_{A_k},$$

It can be handled by the same reasoning as before, but now we should distinguish between the cases $i < j$ and $j > i$. In the first case, $s(j) = 1$ and:

$$a(\sigma, i, j) = (-1)^{i-1} \zeta^{d_1 b} W_{d_i b}{}^{c}{}_{d_j} \xi_{d_1 \dots d_{j-1} c d_{j+1} \dots d_{i-1} d_{i+1} \dots d_{k+1}} Z^{d_2}_{A_1} Z^{d_3}_{A_2} \dots Z^{d_{k+1}}_{A_k},$$

In the second case: $s(j-1)=1$, and:

$$a(\sigma,i,j)=(-1)^{i-1}\zeta^{d_1b}W_{d_ib}{}^{c}{}_{d_j}\xi_{d_1\dots d_{i-1}d_{i+1}\dots d_{j-1}cd_{j+1}\dots d_{k+1}}Z^{d_2}_{A_1}Z^{d_3}_{A_2}\dots Z^{d_{k+1}}_{A_k}.$$

Overall:

$$\begin{aligned}\frac{1}{(k-1)!}T_2^i=&\sum_{j=2}^{i-1}(-1)^{i-1}\zeta^{d_1b}W_{d_ib}{}^{c}{}_{d_j}\xi_{d_1\dots d_{j-1}cd_{j+1}\dots d_{i-1}d_{i+1}\dots d_{k+1}}Z^{d_2}_{A_1}Z^{d_3}_{A_2}\dots Z^{d_{k+1}}_{A_k}\\&+\sum_{j=i+1}^{k+1}(-1)^{i-1}\zeta^{d_1b}W_{d_ib}{}^{c}{}_{d_j}\xi_{d_1\dots d_{i-1}d_{i+1}\dots d_{j-1}cd_{j+1}\dots d_{k+1}}Z^{d_2}_{A_1}Z^{d_3}_{A_2}\dots Z^{d_{k+1}}_{A_k}.\end{aligned}$$

Reindex now as follows:

$$\begin{aligned}&\sum_{j=2}^{i-1}(-1)^{i-1}\zeta^{ab}W_{d_ib}{}^{c}{}_{d_{j-1}}\xi_{ad_1\dots d_{j-2}cd_j\dots d_{i-2}d_i\dots d_k}Z^{d_1}_{A_1}Z^{d_2}_{A_2}\dots Z^{d_k}_{A_k},\\&\qquad=\sum_{j=1}^{i-2}(-1)^{i-1}\zeta^{ab}W_{d_ib}{}^{c}{}_{d_j}\xi_{ad_1\dots d_{j-1}cd_{j+1}\dots d_{i-2}d_i\dots d_k}Z^{d_1}_{A_1}Z^{d_2}_{A_2}\dots Z^{d_k}_{A_k},\end{aligned}$$

and similarly for the second sum so that:

$$\begin{aligned}\frac{1}{(k-1)!}\sum_{i=2}^{k+1}T_2^i&=\frac{1}{(k-1)!}\sum_{i=1}^{k}T_2^{i+1},\\&=\sum_{i=1}^{k}\sum_{\substack{j=1\\j\neq i}}^{k}\zeta^{ab}W_{d_ib}{}^{c}{}_{d_j}(-1)^i\xi_{ad_1\dots d_{j-1}cd_{j+1}\dots d_{i-1}d_{i+1}\dots d_k}Z^{d_1}_{A_1}Z^{d_2}_{A_2}\dots Z^{d_k}_{A_k},\\&=\sum_{i=1}^{k}\sum_{\substack{j=1\\j\neq i}}^{k}\zeta^{ab}W_{d_ib}{}^{c}{}_{d_j}\xi_{d_1\dots d_{j-1}cd_{j+1}\dots d_{i-1}ad_{i+1}\dots d_k}Z^{d_1}_{A_1}Z^{d_2}_{A_2}\dots Z^{d_k}_{A_k}.\end{aligned}$$

This proves the result for the bottom slot. We briefly outline the proof for the top slot, it is simpler to work directly with $F_{A_1\dots A_k}=k\mu_{a_2\dots a_k}Y_{[A_1}Z^{a_2}_{A_2}\cdots Z^{a_k}_{A_k]}$,

$$\begin{aligned}F_{A_1\dots A_k}&=\frac{1}{(k-1)!}\sum_{\sigma\in\mathfrak{S}_k}\varepsilon(\sigma)\mu_{a_2\dots a_k}Y_{A_{\sigma(1)}}Z^{a_2}_{A_{\sigma(2)}}\cdots Z^{a_k}_{A_{\sigma(k)}}\\&=\sum_{i=1}^{k}(-1)^{i-1}\mu_{a_1\dots a_{i-1}a_{i+1}\dots a_k}Y_{A_i}Z^{a_1}_{A_1}\cdots Z^{a_{i-1}}_{A_{i-1}}Z^{a_{i+1}}_{A_{i+1}}\cdots Z^{a_k}_{A_k}.\end{aligned}$$

In this case:

$$T = \sum_{\substack{i=2 \\ \sigma\in\mathfrak{S}_{k+1}}}^{k} (-1)^{i-1}\varepsilon(\sigma) W^A_a W_{d_1 b}{}^{c}{}_{d_2}\mu_{cd_3\dots d_i d_{i+2}\dots d_k} Z^{d_1}_{D_{\sigma(1)}} Z^{d_2}_{D_{\sigma(2)}} Z^{d_3}_{D_{\sigma(3)}} \cdots Z^{d_i}_{D_{\sigma(i)}} Y_{D_{\sigma(i+1)}} Z^{d_{i+2}}_{D_{\sigma(i+2)}} \cdots Z^{d_{k+1}}_{D_{\sigma(k+1)}}.$$

Note that the index i starts at 2 since the first term vanishes as $Y_C W^C_c = 0$. We then apply the same method of computation as before to transform the sum over $\mathfrak{S}_{k+1}$.

4.7.5 Operator algebra

The projectively invariant operators $\mathscr{D}, \mathscr{D}^*$ and σ are the beginnings of an operator algebra that we will seek to exploit to write down a tractor version of the Proca equation. The commutators $[\mathscr{D}, \sigma]$ and $[\mathscr{D}^*, \sigma]$ are directly related to the weight 1 tractor: $I_A = D_A\sigma$, as follows:

Lemma 4.7.4. *Define the operators:* $\mathscr{I} : \mathcal{E}_{[A_1,\dots A_k]}(\omega) \to \mathcal{E}_{[A_1,\dots,A_{k+1}]}(\omega+1)$ *and* $\mathscr{I}^* : \mathcal{E}_{[A_1,\dots,A_k]}(\omega) \to \mathcal{E}_{[A_1,\dots,A_{k-1}]}(\omega+1)$ *by:*

$$\mathscr{I}F = I\wedge F, \text{ where } I_A = D_A\sigma,$$
$$\mathscr{I}^* = \varepsilon s(-1)^{(k+1)(n+1)+1} \star \mathscr{I} \star .$$

Then, in the scale ∇_g *on* M*, for* $F_{A_1\dots A_k} \overset{\nabla_g}{=} \begin{pmatrix}\mu \\ \xi\end{pmatrix} \in \mathcal{E}_{[A_1,\dots,A_k]}(\omega)$*;*

$$\mathscr{I}\begin{pmatrix}\mu \\ \xi\end{pmatrix} = \begin{pmatrix}2\sigma\xi \\ 0\end{pmatrix} \text{ and } \mathscr{I}^*\begin{pmatrix}\mu \\ \xi\end{pmatrix} = \begin{pmatrix}0 \\ -\frac{f}{2}\mu\end{pmatrix} = -I\lrcorner F.$$

Furthermore:

$$\begin{gathered}\{\mathscr{I}, \mathscr{I}^*\} = -f\sigma, \\ [\mathscr{D}, \sigma] = \mathscr{I}, \quad [\mathscr{D}^*, \sigma] = \mathscr{I}^*, \\ \mathscr{I}^2 = \mathscr{I}^{*2} = 0.\end{gathered} \tag{4.66}$$

$\frac{1}{f}\mathscr{I}$ and $\frac{1}{f}\mathscr{I}^*$ play an analogous role to $\frac{1}{f}\mathscr{D}, \frac{1}{f}\mathscr{D}^*$ with respect $\tilde{y}$. In the case of *normal* solutions to the Metrisability equation, we can push things a little further:

Lemma 4.7.5. *In the case that* H^{AB} *is a normal solution to the metrisability equation then:*

$$\{\mathscr{D}^*, \mathscr{I}\} = -\frac{f}{2}(\boldsymbol{\omega} + n + 1 - \boldsymbol{k}), \quad \{\mathscr{D}, \mathscr{I}^*\} = -\frac{f}{2}(\boldsymbol{\omega} + \boldsymbol{k}), \tag{4.67}$$

where $\boldsymbol{\omega}$ and $\boldsymbol{k}$ are respectively the weight and degree operators. Furthermore:

$$\{\mathscr{D}^*, \mathscr{I}^*\} + \{\mathscr{D}, \mathscr{I}^*\} = -fh,$$

with: $h = \boldsymbol{\omega} + \frac{n+1}{2}$.

In particular, we have the following statement that generalises Lemma 4.6.1 to forms.

Corollary 4.7.1. *Suppose that H^{AB} is a normal solution to the metrisability equation and set:*

$$\begin{cases} x = \sigma, \\ \tilde{y} = \frac{1}{f}\left(\mathscr{D}\mathscr{D}^* + \mathscr{D}^*\mathscr{D}\right) \\ h = \boldsymbol{\omega} + \frac{n+1}{2} \end{cases},$$

then $(x, \tilde{y}, h)$ is an $\mathfrak{sl}_2$-triple.

Proof. x increases weight by 2 and $\tilde{y}$ decreases weight by 2, hence: $[h, x] = 2x$ and $[h, \tilde{y}] = -2\tilde{y}$, lastly, using the above results:

$$\begin{aligned} [x, \tilde{y}] &= \frac{1}{f}\left([\sigma, \mathscr{D}\mathscr{D}^*] + [\sigma, \mathscr{D}^*\mathscr{D}]\right), \\ &= \frac{1}{f}\left([\sigma, \mathscr{D}]\mathscr{D}^* + \mathscr{D}[\sigma, \mathscr{D}^*] + [\sigma, \mathscr{D}^*]\mathscr{D} + \mathscr{D}^*[\sigma, \mathscr{D}]\right), \\ &= -\frac{1}{f}\left(\{\mathscr{I}, \mathscr{D}^*\} + \{\mathscr{D}, \mathscr{I}^*\}\right) \\ &= h. \end{aligned}$$

Remark 4.7.2. The commutator also follows from the Weitzenbock identity in Proposition 4.7.4 and S. Porath's $\mathfrak{sl}_2$ in Proposition 4.6.1.

4.8 Asymptotic analysis of the Proca equation

We have now developed enough tools in order to write down a Maxwell type system for general k-cotractor forms.

$$\begin{cases} \mathscr{D}F = 0, \\ \mathscr{D}^*F = 0. \end{cases}$$

However, we have not yet reaped all the benefits of Equation (4.67), which, in fact, contains important information on the cohomology spaces of the co-chain complex defined by $\mathscr{D}$.

Note first that: $[\mathscr{D}, \boldsymbol{\omega}+\boldsymbol{k}] = 0 = [\mathscr{I}^*, \boldsymbol{\omega}+\boldsymbol{k}]$. Therefore, if $\omega+k \neq 0$ and $F \in \mathcal{E}_{[A_1,\dots,a_k]}(\omega)$ satisfies $\mathscr{D}F = 0$, then, according to Equation (4.67):

$$\mathscr{D}\left(-\frac{2}{f(\boldsymbol{\omega}+\boldsymbol{k})}\mathscr{I}^* F\right) = F.$$

In other words:

Proposition 4.8.1. *Let $\omega \neq 0$, then the cohomology spaces of the following co-chain complex are trivial:*

$$\Gamma(\mathcal{E}(\omega)) \overset{\mathscr{D}=D_{A_1}}{\longrightarrow} \mathcal{E}_{A_1}(\omega-1) \overset{\mathscr{D}}{\longrightarrow} \dots \overset{\mathscr{D}}{\longrightarrow} \mathcal{E}_{[A_1\dots A_{n+1}]}(\omega-(n+1)).$$

Proof. The case $k>0$ has already been treated. The case $k=0$ is easily seen as follows. In any scale ∇ in the projective class:

$$0 = D_A f \overset{\nabla}{=} \begin{pmatrix} \omega f \\ \nabla_a f \end{pmatrix} \Rightarrow f = 0,$$

because $\omega \neq 0$.

The above Proposition simply means that as long as $\omega \neq -k$ there is always a tractor potential ! Thus, in this case, $\mathscr{D}F = 0 \Leftrightarrow F = \mathscr{D}A$ and the co-tractor Maxwell system is completely equivalent to:

$$\begin{cases} F = \mathscr{D}A, \\ \mathscr{D}^*\mathscr{D}A = 0. \end{cases} \tag{4.68}$$

The potential formulation has a manifest gauge symmetry and if one works in a Lorenz type gauge: $\mathscr{D}^*A = 0$, the second equation becomes:

$$\tilde{y}A = 0.$$

Let us study what the equations $\mathscr{D}^*\mathscr{D}A = 0$ and $\mathscr{D}^*A = 0$ mean for the components of A *in the Levi-Civita* scale. Given our hypotheses, from Equation (4.62) we see that the gauge condition is:

$$\mathscr{D}^*A \overset{\nabla_g}{=} \begin{pmatrix} -\delta\mu \\ \delta\xi - (\omega+n+1-k)\frac{f\sigma^{-1}}{4}\mu \end{pmatrix} = 0,$$

from which we deduce that:

$$\mu = \frac{4\sigma}{f(\omega + n + 1 - k)} \delta\xi,$$

provided that $\omega + n + 1 - k \neq 0$. Moreover, since:

$$\mathscr{D}^* \mathscr{D} \begin{pmatrix} \mu \\ \xi \end{pmatrix} \overset{\nabla_g}{=} \begin{pmatrix} \delta d\mu - (\omega + k)\delta\xi \\ \delta d\xi + \frac{f\sigma^{-1}}{4}(\omega - 1 + n - k)d\mu - \frac{f\sigma^{-1}}{4}(\omega - 1 + n - k)(\omega + k)\xi \end{pmatrix}, \tag{4.69}$$

we see that the component ξ satisfies a Proca equation with source where the mass is defined by:

$$m^2 = (\omega - 1 + n - k)(\omega + k).$$

Now, let $\phi_{a_1 \dots a_k}$ be a k-form on M. We can construct a weight $(\omega + k)$, k-form in a natural manner by setting:

$$\xi_{a_1 \dots a_k} = \phi_{a_1 \dots a_k} \sigma^{\frac{\omega + k}{2}}.$$

ξ is then easily transformed into a weight ω co-tractor k-form via the map:

$$\xi_{a_1 \dots a_k} \longmapsto \xi_{a_1 \dots a_k} Z^{a_1}_{A_1} \dots Z^{a_k}_{A_k}.$$

Setting $A = \xi_{a_1 \dots a_k} Z^{a_1}_{A_1} \dots Z^{a_k}_{A_k}$ we see that the equation $\mathscr{D}^* \mathscr{D} A = 0$ expressed in the Levi-Civita scale implies the gauge condition $\mathscr{D}^* A = 0$ and implements on ξ the Proca equation with mass defined above in the Lorenz gauge. The tractor formalism we have developed can therefore be used to study the asymptotics of ξ, through the study of A and the equation $\tilde{y}A = 0$.

Since, according to Corollary 4.7.1, $(x, \tilde{y}, \sigma)$ form satisfy the same formal relations as the triplet (x, y, σ) we studied in Section 4.6.2. One can repeat the steps carried out in Paragraph 4.6.4 and produce a formal solution operator for $\tilde{y}$.

4.9 Conclusion

In this chapter, we have established, on a class of projectively compact manifolds, results that are parallel to those available in the case of conformally compact manifolds. In particular, we have constructed an exterior tractor calculus on order 2 projectively compact manifolds. It is hoped that this will constitute a basis for a geometric approach to the asymptotic analysis of classical fields on such backgrounds that would be an alternative to microlocal analysis. There are still some outstanding questions that we have not been able to touch upon. In particular, it is not yet clear how to give a clear-cut analytical meaning to the formal solution operators we obtain, and the question of how to treat the asymptotically flat case, in which the structure at the basis of the formal construction becomes trivial, remains open. This will be the object of work in the near future.

Appendix to *Maximal Kerr-de Sitter spacetimes*

A.1 Connection forms

$$\begin{aligned}
{\omega^0}_1 &= F\omega^0 - \frac{\varepsilon ar}{\rho^3}\sqrt{\Delta_\theta}\sin\theta\omega^3,\\
{\omega^0}_2 &= -\frac{\sqrt{\Delta_\theta}a^2\sin\theta\cos\theta}{\rho^3}\omega^0 - \frac{\sqrt{\varepsilon\Delta_r}a\cos\theta}{\rho^3}\omega^3,\\
{\omega^0}_3 &= \frac{\sqrt{\varepsilon\Delta_r}a\cos\theta}{\rho^3}\omega^2 - \frac{\varepsilon ar\sqrt{\Delta_\theta}\sin\theta}{\rho^3}\omega^1,\\
{\omega^1}_2 &= -\frac{a^2\sin\theta\cos\theta\sqrt{\Delta_\theta}}{\rho^3}\omega^1 - \varepsilon r\frac{\sqrt{\varepsilon\Delta_r}}{\rho^3}\omega^2,\\
{\omega^1}_3 &= -\varepsilon ar\sin\theta\frac{\sqrt{\Delta_\theta}}{\rho^3}\omega^0 - \varepsilon r\frac{\sqrt{\varepsilon\Delta_r}}{\rho^3}\omega^3,\\
{\omega^2}_3 &= -a\ \cos\theta\varepsilon\frac{\sqrt{\varepsilon\Delta_r}}{\rho^3}\omega^0 - \left(\mathrm{cotan}\theta(r^2+a^2)\frac{\sqrt{\Delta_\theta}}{\rho^3} + \frac{G}{\rho}\right)\omega^3.
\end{aligned} \tag{A.1}$$

where: $F = \frac{\partial}{\partial r}\left(\frac{\sqrt{\varepsilon\Delta_r}}{\rho}\right)$ and $G = \frac{\partial}{\partial\theta}\ \left(\sqrt{\Delta_\theta}\right)$.

A.2 Geodesic equations "à la Cartan"

Let $\gamma : I \longrightarrow KdS$, be a curve on one of the Boyer-Lindquist blocks of Kerr-de Sitter spacetime. Decomposing on the orthonormal frame one has at each point $t \in I$, $\dot\gamma(t) = \Gamma^i(t)E_i(\gamma(t)) \equiv \Gamma^i(t)E_i(t)$, so:

$$\begin{aligned}
\frac{D}{dt}\dot\gamma(t) = (\nabla_{\dot\gamma}\dot\gamma)_{\gamma(t)} &= \dot\Gamma^i(t)E_i(t) + \Gamma^i(t)\Gamma^j(t)(\nabla_{E_i}E_j)_{\gamma(t)},\\
&= \dot\Gamma^i(t)E_i(t) + \Gamma^k(t)\Gamma^j(t)({\omega^i}_j)_{\gamma(t)}(E_k(t))E_i(t).
\end{aligned}$$

If γ is a geodesic, using (A.1) we find that the components satisfy the following system of differential equations:

$$\begin{aligned}
&\dot{\Gamma^0} + F\Gamma^0\Gamma^1 - 2\varepsilon ar\sin\theta\frac{\sqrt{\Delta_\theta}}{\rho^3}\Gamma^1\Gamma^3 - a^2\sin\theta\cos\theta\frac{\sqrt{\Delta_\theta}}{\rho^3}\Gamma^0\Gamma^2 = 0,\\
&\dot{\Gamma^1} + F(\Gamma^0)^2 - 2\varepsilon ar\sin\theta\frac{\sqrt{\Delta_\theta}}{\rho^3}\Gamma^0\Gamma^3 - a^2\sin\theta\cos\theta\frac{\sqrt{\Delta_\theta}}{\rho^3}\Gamma^2\Gamma^1\\
&\qquad\qquad -\varepsilon r\frac{\sqrt{\varepsilon\ \Delta_r}}{\rho^3}(\Gamma^2)^2 - \varepsilon r\frac{\sqrt{\varepsilon\Delta_r}}{\rho^3}(\Gamma^3)^2 = 0,\\
&\dot{\Gamma^2} - \varepsilon a^2\sin\theta\cos\theta\frac{\sqrt{\Delta_\theta}}{\rho^3}(\Gamma^0)^2 - 2\varepsilon a\cos\theta\frac{\sqrt{\varepsilon\ \Delta_r}}{\rho^3}\Gamma^0\Gamma^3 + \varepsilon a^2\sin\theta\cos\ \theta\frac{\sqrt{\Delta_\theta}}{\rho^3}(\Gamma^1)^2\\
&\qquad\qquad +r\frac{\sqrt{\varepsilon\Delta_r}}{\rho^3}\Gamma^1\Gamma^2 - \left(\text{cotan}\theta(r^2+a^2)\frac{\sqrt{\Delta_\theta}}{\rho^3} + \frac{G}{\rho}\right)(\Gamma^3)^2 = 0,\\
&\dot{\Gamma^3} + r\frac{\sqrt{\varepsilon\Delta_r}}{\rho^3}\Gamma^1\Gamma^3 + \left(\text{cotan}\theta(r^2+a^2)\frac{\sqrt{\Delta_\theta}}{\rho^3} + \frac{G}{\rho}\right)\Gamma^2\Gamma^3 = 0.
\end{aligned}$$

A.3 Resultant

Let k be a field, and $k[X]$ denote the ring of polynomials with coefficients in k. If $n \in \mathbb{N}^*$, $k_n[X]$ will denote the subspace of $k[X]$ of polynomials with degree at most n.

Let $P, Q \in k[X]$, $n = \deg P$, $m = \deg Q$. We suppose $n > 0$ and $m > 0$ so that neither P nor Q is zero. Consider the equation:

$$UP + VQ = 0, \tag{A.2}$$

where U et V are two elements of $k[X]$.

(A.2) is clearly equivalent to $UP = -VQ$. Let D denote the pgcd of P and Q then $P = DP'$ and $Q = DQ'$ where $\text{pgcd}(P', Q') = 1$.

With these notations (A.2) is equivalent to $UP' = -VQ'$, but, as $\text{pgcd}(P', Q') = 1$ and $k[X]$ is principal, then this implies that P' divides V. There is therefore a polynomial $C \in k[X]$ such that $V = P'C$, and so $U = -Q'C$. The set of solutions to (A.2) is hence:

$$\left\{\left(-\frac{Q}{D}C, \frac{P}{D}C\right), C \in k[X]\right\}$$

. From this, we deduce that there is a solution $(U, V) \in k_{m-1}[X] \times k_{n-1}[X]$ if and only if pgcd$(P, Q) \neq 1$. We can also express this in another way. Define a linear map $\phi_{P,Q}$ by:

$$\phi_{P,Q} : \begin{array}{ccc} k_{m-1}[X] \times k_{n-1}[X] & \longrightarrow & k_{n+m-1}[X] \\ (U, V) & \longmapsto & UP + VQ \end{array} \tag{A.3}$$

According to the preceding discussion we see that, $\phi_{P,Q}$ is injective if and only if pgcd$(P, Q) = 1$.

The transpose of the matrix of $\phi_{P,Q}$ expressed in the bases

$$\big((X^{m-1}, 0), \dots, (1, 0), (0, X^{n-1}), \dots, (0, 1)\big)$$

$$(X^{m+n-1}, X^{m+n-2}, \dots, X, 1)$$

of $k_m[X] \times k_n[X]$ and $k_{m+n-1}[X]$ respectively is called Sylvester's matrix $S(P, Q)$ and its determinant, denoted by $R(P, Q)$, (and thus the determinant of the endomorphism $\phi_{P,Q}$) is called the resultant of P and Q.

Proposition A.3.1. *Let $P = \sum_{i=0}^{n} a_i X^i, Q = \sum_{j=0}^{m} b_j X^j$ be two polynomials with coefficients in k then the Sylvester matrix $S(P, Q)$ is given by:*

$$S(P,Q) = \begin{pmatrix} a_n & \dots & \dots & \dots & a_0 & 0 & \dots & \dots & 0 \\ 0 & a_n & \dots & \dots & \dots & a_0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & a_n & \dots & \dots & \dots & a_0 \\ b_m & \dots & \dots & b_0 & 0 & \dots & \dots & \dots & 0 \\ 0 & b_m & \dots & \dots & b_0 & 0 & \dots & 0 & \vdots \\ \vdots & \ddots & \ddots & \dots & \dots & \ddots & \ddots & \dots & \vdots \\ 0 & \dots & \dots & 0 & b_m & \dots & \dots & b_0 & 0 \\ 0 & \dots & \dots & \dots & 0 & b_m & \dots & \dots & b_0 \end{pmatrix} \tag{A.4}$$

From our previous discussion we have:

$$R(P, Q) = 0 \Leftrightarrow \text{pgcd}(P, Q) \neq 1$$

If we move instead to an extension L of K containing all the roots of P and Q, then this condition is equivalent to the fact that P and Q have a common root in L. We recall the following result regarding the resultant:

Proposition A.3.2. *Let $P, Q \in k[X]$, $\deg P = n$, $\deg Q = m$. Let L be a splitting field of P and $\alpha_1, \dots \alpha_n$ be the (not necessarily distinct) roots of P, then:*

$$R(P,Q) = a_n^m \prod_i Q(\alpha_i).$$

In this formula, a_n is the coefficient of X^n in P.

Definition A.3.1. When $\deg P' = n-1$ (which is always the case when the characteristic of k is 0), the discriminant of P is defined by:

$$\Delta(P) = \frac{(-1)^{\frac{n(n-1)}{2}}}{a_n} R(P, P')$$

From Proposition A.3.2 we deduce:

Proposition A.3.3. *Let $P \in k[X]$ and suppose that P' is of degree $n-1$ then, in a splitting field of P:*

$$\Delta(P) = a_n^{2n-1} \prod_{i<k} (\alpha_i - \alpha_k)^2$$

Where $\alpha_1, \dots, \alpha_n$ are the (not necessarily distinct) roots of P.

A.4 Diverse useful formulae in Boyer-Lindquist like coordinates

Lemma A.4.1.

$$g_{\phi\phi} g_{tt} - g_{\phi t}^2 = -\frac{\Delta_r \Delta_\theta \sin^2\theta}{\Xi^4}$$

Lemma A.4.2.

$$(g^{ij}) = \begin{pmatrix} -\frac{g_{\phi\phi}\Xi^4}{\sin^2\theta\Delta_\theta\Delta_r} & 0 & 0 & \frac{\Xi^4 g_{\phi t}}{\sin^2\theta\Delta_r\Delta_\theta} \\ 0 & \frac{1}{g_{rr}} & 0 & 0 \\ 0 & 0 & \frac{1}{g_{\theta\theta}} & 0 \\ \frac{\Xi^4 g_{\phi t}}{\sin^2\theta\Delta_r\Delta_\theta} & 0 & 0 & -\frac{g_{tt}\ \Xi^4}{\sin^2\theta\Delta_r\Delta_\theta} \end{pmatrix}$$

Lemma A.4.3. *The metric-dual of dt is given by:*

$$\nabla t = \frac{\Xi^4}{\sin^2\theta \Delta_\theta \Delta_r}(-g_{\phi\phi}\partial_t + g_{\phi t}\partial_\phi)$$

Lemma A.4.4. *In Boyer-Lindquist-like coordinates one can write:*

$$g_{tt} = \frac{1}{\Xi^2}\left(-1 + \frac{2Mr}{\rho^2} + l^2(r^2 + a^2\sin^2\theta)\right)$$
$$g_{\phi t} = -\frac{a\sin^2\theta}{\Xi^2}\left(l^2(r^2+a^2) + \frac{2Mr}{\rho^2}\right)$$

A.5 Gluing topological spaces

Let X and Y be two topological spaces, U and V be open subsets of X and Y respectively and ϕ be a homeomorphism of U onto V. We outline here the construction of a new topological space containing both X and Y and where U and V have been identified. In a sense, we will have glued X to Y along U and V. Let $X \coprod Y$ denote their coproduct (or disjoint union) and $i : X \longrightarrow X \coprod Y, j : Y :\longrightarrow X \coprod Y$ the canonical injections. Define an equivalence relation on $X \coprod Y$ by:

$$p \sim q \Leftrightarrow ([p = q] \text{ or } [p = i(x), q = j(\phi(x)), x \in U] \text{ or } [q = i(x), p = j(\phi(x)), x \in U]) \tag{A.5}$$

Denote by $X \coprod_\phi Y$ the quotient space of $X \coprod Y$ by this equivalence relation and $\pi : X \coprod Y \longrightarrow X \coprod_\phi Y$ the canonical projection. We quote without proof the following results:

Proposition A.5.1.
1. *$\bar{j} = \pi \circ j, \bar{i} = \pi \circ i$ are continuous injective and open maps. X and Y can then be identified with the open subsets $\bar{i}(X)$ and $\bar{j}(Y)$ of $X \coprod_\phi Y$.*
2. *$\bar{i}(X) \cap \bar{j}(Y) = \bar{i}(U) = \bar{j}(V)$*
3. *If F is an arbitrary topological space, $f : X \coprod_\phi Y \to F$ is continuous if and only if the maps $f \circ \bar{i}$ et $f \circ \bar{j}$ are.*
4. *π is an open map*

Points 2 and 3 can be useful for constructing maps on $X \coprod_\phi Y$ from maps f, g defined on X and Y separately. In fact, it suffices that they satisfy $f(x) = g(\phi(x))$ for every

$x \in U$ for them to piece together to form a well-defined continuous map on $X \coprod_\phi Y$. This is sometimes called the mapping lemma; it has natural generalisations to maps and manifolds with more regularity. The above proposition also serves to prove the following results:

Proposition A.5.2. *1. If X and Y are both locally Euclidean, then $X \coprod_\phi Y$ is too.*

2. If X and Y are both second-countable, $X \coprod_\phi Y$ is too.

It is well known that separation properties of a quotient are relatively independent of the separation properties of the original space, however since the canonical projection map is open one has the following result:

Lemma A.5.1. *$X \coprod_\phi Y$ is Hausdorff if and only if $R = \{(p,q) \in (X \coprod Y)^2, p \sim q\}$ is closed in $(X \coprod Y)^2$*

With this result we can prove a technical criterion that will guarantee separation in all cases of interest in the text:

Lemma A.5.2. *Suppose that X and Y are Hausdorff and first countable then if there is no sequence $(x_n)_{n\in\mathbb{N}}$ of points in U converging to a point in $\bar{U} \setminus U$ and such that $\phi(x_n)_{n\in\mathbb{N}}$ converges to a point in $\bar{V} \setminus V$, $X \coprod_\phi Y$ is Hausdorff.*

Proof. By Lemma A.5.1 it suffices to show that $R = \{(p,q) \in (X \coprod Y)^2, p \sim q\}$ is closed in $(X \coprod Y)^2$. Furthermore, as X and Y are first countable, it suffices to show that if two sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ of points in $X \coprod Y$ are such that $\forall n \in \mathbb{N}, p_n \sim q_n$ and $p_n \underset{n\to\infty}{\longrightarrow} p, q_n \underset{n\to\infty}{\longrightarrow} q$ then $p \sim q$.

Let $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ be two such sequences. We can restrict ourselves to the case where $p \in i(X)$ and $q \in j(Y)$ as p and q play symmetric roles and if $p \in i(X)$ (resp. $j(Y)$) then $q \in i(X)$ (resp. $j(Y)$) then for all large enough n, $p_n \in i(X)$ and $q_n \in i(X)$, as $i(X)$ is open in $X \coprod Y$, hence:

$$\exists N \in \mathbb{N}, \forall n \geq N, p_n = q_n \Rightarrow p = q.$$

Assume now that $p \in i(X)$ and $q \in j(Y)$, we distinguish 3 cases:

Case 1: $p \in i(X) \setminus \overline{i(U)}$, then there is $N \in \mathbb{N}$ such that $\forall n \geq N, p_n \in i(X) \setminus \overline{i(U)}$, but as $q_n \sim p_n$ for every $n \in \mathbb{N}$ it follows that for all $n \geq N, p_n = q_n$ so $p = q$. Which is excluded as $i(X) \cap j(Y) = \emptyset$

Case 2: $p \in i(U)$, then again, there is $N \in \mathbb{N}$ such that $\forall n \geq N, p_n \in i(U)$. Since $q \in j(Y)$ there is also $N' \in \mathbb{N}$ such that $\forall n \geq N', q_n \in j(Y)$. Moreover, as for every $n \in \mathbb{N}, p_n \sim q_n$ it follows from (A.5) that:

$$\forall n \geq \max(N, N'), \begin{cases} q_n = j(y_n), y_n \in V \\ p_n = i(x_n), x_n \in U \\ y_n = \phi(x_n) \end{cases}$$

As i and j are homeomorphisms onto their ranges, the sequences (x_n) and (y_n) converge to points $x \in X$ and $y \in Y$ respectively. Furthermore, ϕ being continuous, one must have $y = \phi(x)$ so: $p \sim q$.

Case 3: $p \in \overline{i(U)} \setminus i(U)$, if only a finite number of points of the sequence lie in $i(U)$ then there is a rank N above which $q_n = p_n$ so $q = p$ which is excluded as $q \in j(Y)$. Thus, we can assume that one can extract a subsequence $(p_{\varphi(n)})_{n \in \mathbb{N}}$ of $(p_n)_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$, $p_{\varphi(n)} \in i(U)$. Necessarily, $q \in \overline{j(V)}$, but $q \notin j(V)$ as this would imply $p \in i(U)$, so $q \in \overline{j(V)} \setminus j(V)$. However, as $\forall n \in \mathbb{N}, q_n \sim p_n$ there must exist sequences (x_n) and (y_n) of points of X and Y respectively such that (x_n) converges to a point in $\bar{U} \setminus U$, (y_n) to a point in $\bar{V} \setminus V$ and $y_n = \phi(x_n)$ for sufficiently large n, but this contradicts our hypothesis. Hence $p \sim q$ and R is closed.

Appendix to *Scattering theory for Dirac fields near an Extreme Kerr-de Sitter black hole*

B.1 The Helffer-Sjöstrand formula

At several points in the text the Helffer-Sjöstrand formula is used quite liberally to establish results about commutators. In this appendix, the reader will find some more details about this formula.

B.1.1 Almost-analytic extensions

Let $f \in C^\infty(\mathbb{R})$, one can extend f to $\mathbb{C}$ in the following manner: let $n \geq 1$ and $\tau \in C^\infty(\mathbb{R})$ be a smooth cut-off function satisfying: $\tau(s) = 1$ for $|s| < 1$ and $\tau(s) = 0$ for $|s| > 2$, then we set for $z \in \mathbb{C}, z = x + iy, (x, y) \in \mathbb{R}^2$:

$$\begin{aligned} \tilde{f}(z) &= \sigma(x,y) \sum_{r=0}^{n} \frac{f^{(r)}(x)}{r!}(iy)^r, \\ \sigma(x,y) &= \tau\left(\frac{y}{\langle x \rangle}\right). \end{aligned} \tag{B.1}$$

$\tilde{f}$ is $\mathbb{R}$-smooth and:

$$\begin{aligned} \partial_{\bar{z}} \tilde{f} = \frac{1}{2}\{\partial_x \tilde{f} + i\partial_y \tilde{f}\} =& \frac{1}{2} \sum_{r=0} \left(\frac{f^{(r)}(x)}{r!}(iy)^r \right) (\partial_x \sigma + i\partial_y \sigma) \\ &+ \frac{1}{2}\sigma(x,y) \frac{f^{n+1}(x)}{n!}(iy)^n. \end{aligned} \tag{B.2}$$

Since $(\partial_x \sigma + i\partial_y \sigma) \neq 0$ only if $\langle x \rangle \leq y \leq 2\langle x \rangle$ then if x is fixed and $y \to 0$, the expression in (B.2) implies that $|\partial_{\bar{z}} \tilde{f}(z)| \leq O(|y|^n)$ when $y \to 0$; in particular, it is 0 if $z \in \mathbb{R}$.

B.1.2 The formula

The Helffer-Sjöstrand formula gives a convenient form of the functional calculus for a class of symbols f. In [Dav95], it is used to construct the entire functional calculus, but it was originally proved assuming the usual function calculus in [HS87]. The formula makes sense for symbols f for which there is some $\beta \in \mathbb{R}_+$ such that, for all $n \in \mathbb{N}$:

$$\sup_{x \in \mathbb{R}} |f^{(n)}(x)\langle x \rangle^{n+\beta}| < +\infty.$$

Following [Dav95], let us denote this set $\mathscr{A}$, examples of elements of $\mathscr{A}$ are elements in $\boldsymbol{S}^{1,1}$. The result can be stated as:

Theorem B.1.1. *Let $f \in \mathscr{A}$, then if A is a self-adjoint operator on a separable Hilbert space $\mathscr{H}$:*

$$f(A) = \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(z)(A - z)^{-1} dz \wedge d\bar{z}. \tag{B.3}$$

The integral converges in the operator norm topology and is independent of the choices of n and σ in the almost-analytic extension.

The integral above can be interpreted as a Bochner integral and its convergences follows from the following estimate on $\mathbb{C} \setminus \mathbb{R}$ of the norm of the integrand:

$$||\partial_{\bar{z}} \tilde{f}(z)(A - z)^{-1}|| \leq c \sum_{r=0}^{n} |f^{(r)}(x)| \langle x \rangle^{r-2} \mathbf{1}_U(x, y) + c f^{n+1}(x)|y|^{n-1} \mathbf{1}_V(x, y), \tag{B.4}$$

for some $c \in \mathbb{R}_+^*$; $U = \{(x, y) \in \mathbb{R}^2, \langle x \rangle < y < 2\langle x \rangle\}$ and $V = \{(x, y) \in \mathbb{R}^2, 0 < y < 2\langle x \rangle\}$.

B.2 The Faà di Bruno formula

Let $f, g \in C^\infty(\mathbb{R})$, then for any $n \geq 1$:

$$(f \circ g)^{(n)} = \sum_{(m_1,\dots,m_n) \in I_n} \frac{n! f^{(m_1+\dots+m_n)} \circ g}{m_1! 1!^{m_1} m_2! 2!^{m_2} \dots m_n! n!^{m_n}} \prod_{j=1}^{n} \left(g^{(j)}\right)^{m_j},$$

$$I_n = \{(m_1, \dots, m_n) \in \mathbb{N}^n, \sum_{j=1}^{n} j m_j = n\}.$$

Appendix C

Vector valued differential forms

In the literature on principal connections, we are inevitably confronted with the notion of vector valued differential forms. In this appendix, for completeness, we recall some of the definitions and vocabulary useful for the main text and that one comes across in textbooks on the topic. We will mainly follow [Ble05], who adopts a point of view close to that predominant in Physics literature. We will also discuss equivalent ways to think about so-called « basic » or « tensorial » forms and forms with values in a vector bundle V over M.

C.1 Definitions

Definition C.1.1. Let V be a fixed finite dimensional vector space and P a smooth manifold, a V-valued differential form is a section of the vector bundle :

$$\Lambda^k(P,V) = \Lambda^k(T^*P) \otimes (P \times V).$$

If (P, π, M) is a G-principal fibre bundle and (V, ρ) is a representation of G we will say that $\alpha \in \Gamma(\Lambda^k(P,V))$ is:

- *equivariant* if $R_g^*\alpha = \rho(g)^{-1}\alpha$,
- *horizontal* if for all vector fields on P $(X_1, \dots, X_k)$, at least one of which vertical, then $\alpha(X_1, \dots, X_k) = 0$.

A form that is both horizontal and equivariant is said to be *tensorial.*

Proposition C.1.1. *Let (P, π, M) be a G-principal fibre bundle, and (V, ρ) a finite dimensional representation of G. A tensorial V-valued differential form on P is equivalent to a section of the vector bundle over M:*

$$\Lambda^k(T^*M) \otimes (P \times_G V).$$

Proof. Let us first point out that the base of the bundle $\Lambda^k(T^*M)\otimes(P\times_G V)$ is indeed M and not P. The equivalence is quite clear in one direction, if we allow ourselves to choose a connection on P. In this case, the projection π of P induces a vector space isomorphism:

$$\mathrm{d}\pi_p : H_p \subset T_pP \to T_{\pi(p)}M,$$

which enables us to horizontally lift vector fields over M to vector fields over P. If X is a vector field over M, let us call $\tilde{X}$ its horizontal lift. We will also denote by p the canonical projection mapping $P\times V$ onto $P\times_G V$ (cf. Paragraph 1.4). With these notations, if α is a tensorial k-form over P then for any vector fields $X_1,\dots,X_k$ over M, $x\in M$ and $p\in\pi^{-1}(\{x\})$, the equation :

$$\alpha_{M,x}(X_1,\dots,X_k)=\mathsf{p}(\alpha_p(\tilde{X}_1,\dots,\tilde{X}_k)),$$

is independent of the choice of p and the horizontal lift and thus defines a section of $\Lambda^k(T^*P)\otimes(P\times_G V)$.

The other direction is slightly more subtle. Let α_M be a section of $\Lambda^k(T^*M)\otimes(P\times_G V)$. Of course, the idea is to consider the pullback of α by the projection π, $\pi^*\alpha_M$, however, $\pi^*\alpha_M$ is *a priori* only a section of $\Lambda^k(T^*P)\otimes\pi^*(P\times_G V)$, where : $\pi^*(P\times_G V)$ is the pullback bundle of $P\times_G V$ by π defined by :

$$\pi^*(P\times_G V)=\left\{(p,v),\in P\times(P\times_G V),\tilde{\tilde{\pi}}(v)=\pi(p)\right\}\cong\coprod_{p\in P}\left\{(p,v),v\in(P\times_G V)_{\pi(p)}\right\}.$$

In the above, we have borrowed notation from Paragraph 1.4 and have introduced the projection $P\times_G V\to M$, $\tilde{\tilde{\pi}}$. The remainder of the proof is dedicated to showing that the bundle $\pi^*(P\times_G V)$ is in fact trivial. Let $r\in P$ and set :

$$\tilde{\rho}(r):\begin{array}{ccc} V & \longrightarrow & \tilde{\tilde{\pi}}^{-1}(\{\pi(r)\}) \\ v & \longmapsto & \mathsf{p}((r,v)). \end{array}$$

As for any $v\in V, g\in G$, $\mathsf{p}((rg,v))=\mathsf{p}((r,\rho(g)v))$, one has :

$$\tilde{\rho}(rg)=\tilde{\rho}(r)\rho(g).$$

One can then define a map from $\pi^*(P \times_G V)$ into $P \times V$ by :

$$(p, v) \mapsto (p, \tilde{\rho}(p)^{-1} v). \tag{C.1}$$

We leave the proof of the smoothness of our maps to the reader. Minus this detail, the above shows that $\pi^*(P \times_G V)$ is actually parallelisable, because the vector bundle morphism defined by (C.1) is a vector space isomorphism on each of the fibres. The explicit form of the map shows that, after untangling the pullback bundle, $\pi^* \alpha_M$ is indeed a tensorial form on P.

C.2 A few usual operations

C.2.1 The case of a Lie algebra

Let $\mathfrak{g}$ be a Lie algebra, with bracket $[\ ,\ ]$. For $\mathfrak{g}$ valued differential forms, there is a natural generalisation of the wedge product. Let α and β be respectively $\mathfrak{g}$ valued k-form and l-form, one defines :

$$[\alpha \wedge \beta](X_1, \dots, X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma\ \in \mathfrak{S}_{k+l}} \varepsilon(\sigma)[\alpha(X_{\sigma(1)}, \dots, X_{\sigma(k)}), \beta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)})].$$

In the special case that $\mathfrak{g}$ is a Lie subalgebra of the usual matrix algebra $\mathfrak{gl}_n(\mathbb{R})$ then in fact :

$$[\alpha \wedge \beta] = \alpha \wedge \beta - (-1)^{kl} \beta \wedge \alpha,$$

where $\alpha \wedge \beta$ is matrix product where component-wise multiplication is replaced by the usual wedge product.

C.2.2 Exterior covariant derivative

Let (P, π, M) be a G-principal bundle, V a finite dimensional vector space, α a V-valued k-form on P, and, (E_i) an arbitrary basis of V. Writing $\alpha = \alpha^i E_i$, one sets:

$$\mathrm{d}\alpha = \mathrm{d}\alpha^i E_i.$$

This definition is, in fact, independent of the choice of basis and inherits all the usual properties of the exterior derivative on forms. When there is a connection, ω, on P we can define a covariant exterior derivative that sends tensorial forms to tensorial forms. Indeed, call the horizontal component of an arbitrary vector X, X^H and set :

$$\mathrm{d}^\omega \alpha(X_1, \ldots, X_n) = \mathrm{d}\alpha(X_1^H, \ldots, X_n^H).$$

$\mathrm{d}^\omega \alpha$ is clearly horizontal and one can check equivariance as follows :

$$R_g^* \mathrm{d}\alpha = \mathrm{d}R_g^* \alpha = \rho(g^{-1}) \mathrm{d}\alpha.$$

If ρ is a representation and ρ_* is the induced Lie-algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$ then for a V-valued tensorial k-form α, and an arbitrary $\mathfrak{g}$ valued l-form, β one can define:

$$\beta \dot{\wedge} \alpha(X_1, \ldots, X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \varepsilon(\sigma) \rho_*(\beta(X_{\sigma(1)}, \ldots, X_{\sigma(l)})) \cdot \alpha(X_{\sigma(l+1)}, \ldots, X_{\sigma(l+k)}).$$

This leads to a useful formula for the exterior covariant derivative:

Lemma C.2.1. *Let (P, π, M) be a G-principal bundle, ω a principal connection on P, V a finite dimensional vector space, $\rho : G \to GL(V)$ a representation and α a V-valued tensorial k-form on P, then :*

$$d^\omega \alpha = d\alpha + \omega \dot{\wedge} \alpha. \tag{C.2}$$

Appendix D

The transformation law for $(\Omega^i_j(\cdot, e_i))$

We follow the notation introduced in paragraph 4.2.3. Recall that in the equation:

$$\Omega_U = d\Pi_U + \Pi_U \wedge \Pi_U,$$

$\wedge$ is the matrix product where component-wise multiplication is replaced with the exterior product of differential forms and $d\Pi$ is the matrix $(d\Pi)^i_j = d(\Pi^i_j)$. We recall also that the transformation law for Π_U is given by (4.15) :

$$\Pi_V = g^{-1}dg + g^{-1}\Pi_U g - \frac{1}{n+1}\mathrm{tr}(g^{-1}dg)I_n - \frac{1}{n+1}g^{-1}Ag, \tag{D.1}$$

where $A^i_j = \mathrm{tr}(g^{-1}dg(e_j))\omega^i$ and the local sections are related by $\sigma_V = \sigma_U g$ for some $g : U \cap V \to GL_n(\mathbb{R})$.

To determine the transformation law for $(\Omega^i_j(\cdot, e_i))$ we first compute Ω_V in several steps. First of all, $d\Pi_V$:

$$\begin{aligned} d\Pi_V = &- \underline{g^{-1}dgg^{-1} \wedge dg} - \underline{g^{-1}dgg^{-1} \wedge \Pi_U g} + g^{-1}d\Pi_U g - \underline{g^{-1}\Pi \wedge dg} \\ &+ \frac{1}{n+1}\mathrm{tr}(g^{-1}dgg^{-1} \wedge dg)I + \underline{\frac{1}{n+1}g^{-1}dgg^{-1} \wedge Ag} - \frac{1}{n+1}g^{-1}dAg \\ &+ \underline{\frac{1}{n+1}g^{-1}A \wedge dg}. \end{aligned} \tag{D.2}$$

Followed by :

$$
\begin{aligned}
\Pi_V \wedge \Pi_V = g^{-1}\Pi_U \wedge \Pi_U g + \underline{g^{-1}\Pi_U \wedge dg} - \frac{1}{n+1} g^{-1}\Pi_U g \wedge \mathrm{tr}(g^{-1}dg)I \\
- \frac{1}{n+1} g^{-1}(\Pi_U \wedge A)g + \underline{g^{-1}dg \wedge g^{-1}\Pi_U g} + \underline{g^{-1}dg \wedge g^{-1}dg} \\
- \frac{1}{n+1} g^{-1}dg \wedge \mathrm{tr}(g^{-1}dg)I - \underline{\frac{1}{n+1} g^{-1}dgg^{-1}Ag} - \frac{1}{n+1}\mathrm{tr}(g^{-1}dg)I \wedge g^{-1}\Pi_U g \\
- \frac{1}{n+1}\mathrm{tr}(g^{-1}dg)I \wedge g^{-1}\mathrm{dg} + \frac{1}{(n+1)^2}\underbrace{\mathrm{tr}(g^{-1}dg) \wedge \mathrm{tr}(g^{-1}dg))}_{=0} I \\
+ \frac{1}{(n+1)^2}\mathrm{tr}(g^{-1}\mathrm{dg})I \wedge g^{-1}Ag - \frac{1}{n+1} g^{-1}A \wedge \Pi_U g - \underline{\frac{1}{n+1} g^{-1}A \wedge dg} \\
+ \frac{1}{(n+1)^2} g^{-1}(A \wedge A)g + \frac{1}{(n+1)^2} g^{-1}Ag \wedge \mathrm{tr}(g^{-1}dg)I
\end{aligned}
\tag{D.3}
$$

When we sum together (D.2) and (D.3), the terms in (D.3) underlined by a straight line cancel exactly those in (D.2). The terms in (D.3) underlined with a wavy line cancel between themselves by anti-symmetry of the usual exterior product because I commutes with any matrix. After simplification, we find that :

$$
\begin{aligned}
\Omega_V =& g^{-1}\Omega_U g + \frac{1}{n+1}\mathrm{tr}(g^{-1}dgg^{-1} \wedge dg)I - \frac{1}{n+1} g^{-1}dAg \\
&- \frac{1}{n+1} g^{-1}(\Pi_U \wedge A)g - \frac{1}{n+1} g^{-1}(A \wedge \Pi_U)g + \frac{1}{(n+1)^2} g^{-1}(A \wedge A)g.
\end{aligned}
\tag{D.4}
$$

Now, denote by $(\tilde{e}_i)$ the moving frame associated with σ_V, then, for each i, $\tilde{e}_i = g_i^k e_k$. Consequently, calling the components of Ω_V, $\tilde{\Omega}_j^i$:

$$
\tilde{\Omega}_j^i(\cdot, \tilde{e}_i) = g_i^k \tilde{\Omega}_j^i(\cdot, e_k) = (g\Omega_V)_j^k(\cdot, e_k).
$$

Therefore, we only need to multiply (D.4) by g and evaluate the trace in the basis (e_i).

Computing each term separately :

$$\begin{aligned}(A \wedge \Pi_U)^k_j(\cdot, e_k) &= \sum_k \sum_m \mathrm{tr}(g^{-1}dg(e_m))\omega^k \wedge \Pi^m_j(\cdot, e_k), \\ &= \sum_k \sum_m \mathrm{tr}(g^{-1}dg(e_m)) \sum_i \Pi^m_j(e_i)\omega^k \wedge \omega^i(\cdot, e_k), \\ &= \sum_k \sum_m \mathrm{tr}(g^{-1}dg(e_m)) \sum_i \Pi^m_j(e_i)(\delta^i_k \omega^k - \omega^i)), \\ &= -(n-1) \sum_m \Pi^m_j \mathrm{tr}(g^{-1}dg(e_m)), \\ &= -(n-1)(\gamma' \Pi_U)_j.\end{aligned} \tag{D.5}$$

Similarly :

$$\begin{aligned}(\Pi_U \wedge A)^k_j(\cdot, e_k) &= -\mathrm{tr}(g^{-1}dg(e_j)) \sum_m \sum_k \Pi^k_m(e_k)\omega^m, \\ (A \wedge A)^k_j(\cdot, e_k) &= -(n-1)\mathrm{tr}(g^{-1}dg(e_j))\mathrm{tr}(g^{-1}dg), \\ &= -(n-1)\mathrm{tr}(g^{-1}dg)\gamma'_j, \\ \mathrm{tr}(g^{-1}dgg^{-1} \wedge dg) &= 0.\end{aligned} \tag{D.6}$$

The term dA is slightly more complicated :

$$\begin{aligned}(dA)^i_j = d(A^i_j) = &- \mathrm{tr}(g^{-1}dgg^{-1}dg(e_j)) \wedge \omega^i + \mathrm{tr}(g^{-1}d(dg(e_j))) \wedge \omega^i \\ &+ \mathrm{tr}(g^{-1}dg(e_j))d\omega^i.\end{aligned} \tag{D.7}$$

Using the structure equation (4.14) the last term is seen to be:

$$\mathrm{tr}(g^{-1}dg(e_j))d\omega^i = -\mathrm{tr}(g^{-1}dg(e_j))(\Pi^i_k \wedge \omega^k).$$

Since,

$$\begin{aligned}\Pi^i_k \wedge \omega^k(\cdot, e_i) &= \Pi^i_k(e_m)\omega^m \wedge \omega^k(\cdot, e_i) \\ &= \Pi^i_k(e_m)(\omega^m \delta^k_i - \delta^m_i \omega^k) \\ &= -\Pi^i_k(e_i)\omega^k,\end{aligned} \tag{D.8}$$

we conclude that :

$$\mathrm{tr}(g^{-1}dg(e_j))d\omega^i(\cdot, e_i) = \mathrm{tr}(g^{-1}dg(e_j)) \sum_m \sum_k \Pi^k_m(e_k)\omega^m. \tag{D.9}$$

The remaining terms evaluate to the following expressions :

$$\begin{aligned}
\mathrm{tr}(g^{-1}d(dg(e_j))) \wedge \omega^i(\cdot, e_i) &= (n-1)\mathrm{tr}(g^{-1}d(dg(e_j))), \\
-\mathrm{tr}(g^{-1}dgg^{-1}dg(e_j)) \wedge \omega^i &= -(n-1)\mathrm{tr}(g^{-1}dgg^{-1}dg(e_j)), \\
-\mathrm{tr}(g^{-1}dgg^{-1}dg(e_j)) \wedge \omega^i + \mathrm{tr}(g^{-1}d(dg(e_j))) \wedge \omega^i(\cdot, e_i) &= (n-1)d\gamma'_j.
\end{aligned} \tag{D.10}$$

Putting together equations (D.5),(D.6),(D.9) and (D.10), it follows that:

$$\tilde{\Omega}^i_j(\cdot, \tilde{e}_i) = \Omega^i_k(\cdot, e_i)g^k_j - \frac{n-1}{n+1}d\gamma'_k g^k_j + \frac{n-1}{n+1}(\gamma'\Pi_U)_k g^k_j - \frac{n-1}{(n+1)^2}\mathrm{tr}(g^{-1})\gamma'_k g^k_j, \tag{D.11}$$

which is nothing more than $(n-1)$ times Equation (4.19).

Appendix E

Proofs that are not essential to the main text

E.1 Calculating $\nabla_a W_{cd}{}^a{}_f$

The computation is based on the second Bianchi identity :

$$\nabla_{[a} R_{bc]}{}^d{}_e = 0.$$

In this case of interest this leads to :

$$\begin{aligned} 0 &= \nabla_a R_{cd}{}^a{}_f + \nabla_c R_{da}{}^a{}_f + \nabla_d R_{ac}{}^a{}_f, \\ &= \nabla_a R_{cd}{}^a{}_f - \nabla_c R_{df} + \nabla_d R_{cf}, \\ &= \nabla_a W_{cd}{}^a{}_f + 2\nabla_a \delta^a_{[c} P_{d]f} + 3\nabla_{[c}\beta_{df]} - (n-1)Y_{cdf}, \\ &= \nabla_a W_{cd}{}^a{}_f + 3\nabla_{[c}\beta_{df]} - (n-2)Y_{cdf}. \end{aligned}$$

Hence :

$$\nabla_a W_{cd}{}^a{}_f = (n-2)Y_{cdf} - 3\nabla_{[c}\beta_{df]}. \tag{E.1}$$

E.2 Proof of Lemma 4.7.1

Let us prove the transformation law (4.52) for the components of a k-cotractor form when we change connection according to $\hat{\nabla} = \nabla + \Upsilon$. This can be done by induction. The case $k = 1$ is well known, but we prove it here for completeness. Consider:

$$F_A \stackrel{\nabla}{=} \begin{pmatrix} \sigma \\ \mu_a \end{pmatrix} \stackrel{\hat{\nabla}}{=} \begin{pmatrix} \hat{\sigma} \\ \hat{\mu}_a \end{pmatrix}$$

then for any tractor $T^A \overset{\nabla}{=} \rho Y_A + \nu^b W_b^A \overset{\hat{\nabla}}{=} \hat{\rho} Y_A + \hat{\nu}^b W_A^b$:

$$F_A T^A = \rho\sigma + \mu_b \nu^b = \hat{\rho}\hat{\sigma} + \hat{\mu}_b \hat{\nu}^b.$$

Using Equation (4.25), it follows that:

$$\rho\sigma + \mu_b \nu^b = (\rho - \Upsilon_b \nu^b)\hat{\sigma} + \hat{\mu}_b \nu^b.$$

Hence:

$$\rho(\sigma - \hat{\sigma}) + \nu^b(\mu_b + \Upsilon_b \hat{\sigma} - \hat{\mu}_b) = 0,$$

since this holds for arbitrary ρ, ν^b we conclude that:

$$\begin{cases} \hat{\sigma} = \sigma, \\ \hat{\mu}_b = \mu_b + \Upsilon_b \sigma. \end{cases}$$

Assume now that (4.52) is true for k-forms, we prove that it then holds for $k+1$ forms. Let $F_{A_1 \dots A_{k+1}} \overset{\nabla}{=} (k+1)\mu_{a_2 \dots a_{k+1}} Y_{[A_1} Z_{A_2}^{a_2} \cdots Z_{A_{k+1}]}^{a_{k+1}} + \xi_{a_1 \dots a_{k+1}} Z_{A_1}^{a_1} \cdots Z_{A_{k+1}}^{a_{k+1}}$. As in the case $k=1$, let $T^A \overset{\nabla}{=} \rho X^A + \nu^b W_b^A$ be an arbitrary tractor, we calculate the contraction:

$$\begin{aligned} F_{A_1 \dots A_{k+1}} T^{A_{k+1}} = ((-1)^k \rho \mu_{a_1 \dots a_k} + \xi_{a_1 \dots a_k b} \nu^b) Z_{A_1}^{a_1} \dots Z_{A_k}^{a_k} \\ + (k+1)\mu_{a_2 \dots a_{k+1}} Y_{[A_1} Z_{A_2}^{a_2} \cdots Z_{A_{k+1}]}^{a_{k+1}} \nu^b W_b^{A_{k+1}}. \end{aligned}$$

The final term requires special attention:

$$\begin{aligned} (k+1)\mu_{a_2 \dots a_{k+1}} Y_{[A_1} Z_{A_2}^{a_2} \cdots Z_{A_{k+1}]}^{a_{k+1}} \nu^b W_b^{A_{k+1}} = \\ \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_{k+1}} \varepsilon(\sigma) \mu_{a_2 \dots a_{k+1}} Y_{A_{\sigma(1)}} Z_{A_{\sigma(2)}}^{a_2} \cdots Z_{A_{\sigma(k+1)}}^{a_{k+1}} W_b^{A_{k+1}} \nu^b. \end{aligned}$$

If $\sigma(1) = k+1$, then the summand is zero, hence:

$$\begin{aligned} \sum_{\sigma \ \in \mathfrak{S}_k} \varepsilon(\sigma) \mu_{a_2 \dots a_{k+1}} Y_{A_{\sigma(1)}} Z_{A_{\sigma(2)}}^{a_2} \cdots Z_{A_{\sigma(k+1)}}^{a_{k+1}} W_b^{A_{k+1}} \nu^b = \\ \sum_{i=1}^{k} \sum_{\substack{\sigma \in \mathfrak{S}_{k+1} \\ \sigma(1) = i}} \varepsilon(\sigma) \mu_{a_2 \dots a_{k+1}} Y_{A_i} Z_{A_{\sigma(2)}}^{a_2} \cdots Z_{A_{\sigma(k+1)}}^{a_{k+1}} W_b^{A_{k+1}} \nu^b. \end{aligned}$$

Reorganising the terms in the product, we have that:

$$\begin{aligned}
\mu_{a_2\dots a_{k+1}} Y_{A_i} Z^{a_2}_{A_{\sigma(2)}} \cdots Z^{a_{k+1}}_{A_{\sigma(k+1)}} W^{A_{k+1}}_b \nu^b = & \\
&= \mu_{a_2\dots a_{k+1}} Y_{A_i} Z^{a_{\sigma^{-1}(1)}}_{A_1} \cdots Z^{a_{\sigma^{-1}(i-1)}}_{A_{i-1}} Z^{a_{\sigma^{-1}(i+1)}}_{A_{i+1}} \cdots Z^{a_{\sigma^{-1}(k)}}_{A_{k+1}} W^{A_{k+1}}_b \nu^b, \\
&= \mu_{a_{\sigma(2)}\dots a_{\sigma(k+1)}} \nu^{a_{k+1}} Y_{A_i} Z^{a_1}_{A_1} \dots Z^{a_{i-1}}_{A_{i-1}} Z^{a_{i+1}}_{A_{i+1}} \dots Z^{a_k}_{A_k}, \\
&= (-1)^{i-1} \varepsilon(\sigma) \mu_{a_1 a_2 \dots a_{i-1} a_{i+1} \dots a_{k+1}} \nu^{a_{k+1}} Y_{A_i} Z^{a_2}_{A_2} \dots Z^{a_{i-1}}_{A_{i-1}} Z^{a_{i+1}}_{A_{i+1}} \dots Z^{a_k}_{A_k}.
\end{aligned}$$

The final equation comes from the following observation. If we relabel:

$$\mu_{a_{\sigma(2)}\dots a_{\sigma(k+1)}} = \mu_{\bar{a}_1\dots\bar{a}_k};$$

then for any $s \in \mathfrak{S}_k$,

$$\mu_{\bar{a}_{s(1)}\bar{a}_{s(2)}\dots\bar{a}_{s(k)}} = \mu_{a_{\sigma(s(1)+1)}\dots a_{\sigma(s(k)+1)}}.$$

Since $\sigma(\{2,\dots k\}) = [\![1, k+1]\!] \setminus \{i\}$, we can reorder the indices such that we have $\mu_{a_1\dots a_{i-1}a_{i+1}\dots a_{k+1}}$ if we choose s such that :

$$s(j) = \begin{cases} \sigma^{-1}(j) - 1 & \text{if } 1 \le j < i, \\ \sigma^{-1}(j+1) - 1 & \text{if } i \le j \le k. \end{cases}$$

The signature of this permutation can be determined [1] to be $(-1)^{i-1}\varepsilon(\sigma)$. One can observe that in the quotient group $\mathbb{R}^*/\mathbb{R}^*_+$:

$$\begin{aligned}
\varepsilon(s) &= \prod_{1\le m<l\le k} s(l) - s(m), \\
&= \prod_{1\le m<l<i} \sigma^{-1}(l) - \sigma^{-1}(m) \prod_{1\le m<i\le l\le k} \sigma^{-1}(l+1) - \sigma^{-1}(m) \prod_{i\le m<l\le k} \sigma^{-1}(l+1) - \sigma^{-1}(m+1), \\
&= \prod_{1\le m<l<i} \sigma^{-1}(l) - \sigma^{-1}(m) \prod_{1\le m<i<l\le k+1} \sigma^{-1}(l) - \sigma^{-1}(m) \prod_{i+1\le m<l\le k+1} \sigma^{-1}(l) - \sigma^{-1}(m).
\end{aligned}$$

This differs from $\varepsilon(\sigma^{-1})$ by the sign of :

$$\prod_{1\le m<i} 1 - \sigma^{-1}(m) \equiv (-1)^{i-1} \mathrm{mod}\ \mathbb{R}^*_+.$$

1. in a rather tedious way

Overall, we find that :

$$F_{A_1\dots A_{k+1}}T^{A_1} = ((-1)^k\rho\mu_{a_1\dots a_k} + \nu^b\xi_{ba_1\dots a_k})Z_{A_1}^{a_1}\dots Z_{A_k}^{a_k} + k\mu_{a_2a_3\dots a_ka_{k+1}}\nu^{a_{k+1}}Y_{[A_1}Z_{A_2}^{a_2}\cdots Z_{A_k]}^{a_k}.$$

This is a k-cotractor, therefore, according to the induction hypothesis we must have:

$$\begin{cases} \mu_{a_2\dots a_kb}\nu^b = \hat{\mu}_{a_2\dots a_kb}\hat{\nu}^b, \\ (-1)^k\hat{\rho}\hat{\mu}_{a_1\dots a_k} + \hat{\nu}^b\hat{\xi}_{a_1\dots a_kb} = (-1)^k\rho\mu_{a_1\dots a_k} + \xi_{a_1\dots a_kb}\nu^b + k\Upsilon_{[a_1}\mu_{a_2\dots a_k]b}\nu^b. \end{cases}$$

Plugging the first equation into the second and using Equation (4.25), we have

$$\hat{\xi}_{a_1\dots a_kb}\nu^b = \nu^b\left(\xi_{a_2\dots a_kb} + \underbrace{k\Upsilon_{[a_1}\mu_{a_2\dots a_k]b} + (-1)^k\Upsilon_b\mu_{a_1\dots a_k}}_{k\Upsilon_{[a_1}\mu_{a_2\dots a_kb]}}\right).$$

The tractor T^A being arbitrary, it follows that :

$$\begin{cases} \mu = \hat{\mu}, \\ \hat{\xi} = \xi + \Upsilon\wedge\mu, \end{cases}$$

and the result follows by induction.

E.3 Hodge star of wedge product

Proposition E.3.1. *Let ξ and Υ be respectively a k-form and a 1-form on a pseudo-Riemannian manifold (M, g), then :*

$$\star(\Upsilon\wedge\xi) = (-1)^k\Upsilon^\sharp\lrcorner(\star\xi) \tag{E.2}$$

Proof. The reason is essentially the fact that contraction and wedge product are adjoint operators. Precisely, if ξ is k-form and α an arbitrary $(k+1)$-form then:

$$g(\Upsilon\wedge\xi, \alpha) = g(\xi, \Upsilon^\sharp\lrcorner\alpha). \tag{E.3}$$

Postponing for now the proof of (E.3), we prove Equation (E.2). For any $k+1$-form α:

$$\begin{aligned}\alpha \wedge \star(\Upsilon \wedge \xi) &= g(\alpha, \Upsilon \wedge \xi)\omega_g \\ &= g(\Upsilon^\sharp \lrcorner \alpha, \xi)\omega_g \\ &= (\Upsilon^\sharp \lrcorner \alpha) \wedge \star\xi.\end{aligned}$$

Since, $\Upsilon^\sharp \lrcorner \underbrace{(\alpha \wedge \star\xi)}_{=0} = (\Upsilon^\sharp \lrcorner \alpha) \wedge \star\xi - (-1)^k \alpha \wedge (\Upsilon^\sharp \lrcorner \star \xi)$ it follows that for any α:

$$\alpha \wedge \left((-1)^k \Upsilon^\sharp \lrcorner \star \xi\right) = g(\alpha, \star(\Upsilon \wedge \xi))\omega_g.$$

This property uniquely defines the Hodge star, therefore :

$$\star(\Upsilon \wedge \xi) = (-1)^k \Upsilon^\sharp \lrcorner \star \xi.$$

We prove now (E.3), for instance, using the abstract index notation:

$$\begin{aligned}g(\Upsilon \wedge \xi, \alpha) &= \frac{1}{(k+1)!} g^{a_1 b_1} \cdots g^{a_{k+1} b_{k+1}} (k+1) \Upsilon_{[a_1} \xi_{a_2 \dots a_{k+1}]} \alpha_{b_1 \dots b_{k+1}}, \\ &= \frac{1}{k!(k+1)!} \sum_{\sigma \in \mathfrak{S}_{k+1}} \varepsilon(\sigma) g^{a_1 b_1} \cdots g^{a_{k+1} b_{k+1}} \Upsilon_{a_{\sigma(1)}} \xi_{a_{\sigma(2)} \dots a_{\sigma(k+1)}} \alpha_{b_1 \dots b_{k+1}}, \\ &= \frac{1}{k!(k+1)!} \sum_{\sigma \in \mathfrak{S}_{k+1}} \varepsilon(\sigma) g^{a_1 b_{\sigma(1)}} \cdots g^{a_{k+1} b_{\sigma(k+1)}} \Upsilon_{a_1} \xi_{a_2 \dots a_{k+1}} \alpha_{b_1 \dots b_{k+1}}, \\ &= \frac{1}{k!(k+1)!} \sum_{\sigma \in \mathfrak{S}_{k+1}} \varepsilon(\sigma) g^{a_1 b_1} \cdots g^{a_{k+1} b_{k+1}} \Upsilon_{a_1} \xi_{a_2 \dots a_{k+1}} \alpha_{b_{\sigma^{-1}(1)} \dots b_{\sigma^{-1}(k+1)}}, \\ &= \frac{1}{k!(k+1)!} \sum_{\sigma \in \mathfrak{S}_{k+1}} g^{a_1 b_1} \cdots g^{a_{k+1} b_{k+1}} \Upsilon_{a_1} \xi_{a_2 \dots a_{k+1}} \alpha_{b_1 \dots b_{k+1}}, \\ &= \frac{1}{k!} g^{a_2 b_2} \cdots g^{a_{k+1} b_{k+1}} \xi_{a_2 \dots a_{k+1}} g^{a_1 b_1} \Upsilon_{a_1} \alpha_{b_1 \dots b_{k+1}}, \\ &= g(\xi, \Upsilon^\sharp \lrcorner \alpha).\end{aligned}$$

Appendix F

Useful formulae in de-Sitter spacetime

F.1 Connection forms

In appropriate coordinates $(1+n)$-dimensional de-Sitter space (dS, g) is the « warped » direct product of the pseudo-Riemannian manifolds $(\mathbb{R}_\psi, -d\psi^2)$ and $(S^n, d\sigma^n)$, where $d\sigma^n$ is the usual round metric in n dimensions. In other words, $dS = \mathbb{R} \times S^n$ but the metric is given by :

$$g = -d\psi^2 + f(\psi)d\sigma^n, f(\psi) = \cosh^2\psi.$$

The function f is responsible for the « warping » of the direct product. In well-chosen local frames of dS^n that are adapted to the direct sum decomposition of dS into $\mathbb{R} \times S^n$, it is possible to determine the local connection forms in terms of those of $d\sigma^n$ on S^n and $-d\psi^2$ on $\mathbb{R}$.

For our purposes we will work on a coordinate patch where ψ can be replaced by the boundary defining function $\rho = \frac{1}{2\cosh^2\psi}$. We recall that g is then given by :

$$g = -\frac{\mathrm{d}\rho^2}{4\rho^2(1-2\rho)} + \frac{1}{2\rho}\mathrm{d}\sigma^n.$$

Choosing a local orthonormal frame on S^n and writing θ^i_j for the local connection forms on S^n in this basis then the matrix-valued local connection form for the Levi-Civita connection of g is:

Lemma F.1.1.

$$(\omega^i{}_j)_{1\leq i,j\leq n+1} = \begin{pmatrix} \left(\frac{1}{1-2\rho} - \frac{1}{\rho}\right) d\rho & -(1-2\rho)\omega^j_\theta \\ -\frac{1}{2\rho}\omega^i_\theta & \theta^i_j - \frac{d\rho}{2\rho} \end{pmatrix}. \tag{F.1}$$

From this it follows that :

$$\Box\rho = g^{ab}\nabla_a\nabla_b\rho = 2\rho(n-2+2\rho(3-n)), \tag{F.2}$$

and the local connection form matrix in the same local-frame corresponding to the connection $\hat{\nabla} = \nabla + \frac{\mathrm{d}\rho}{2\rho}$ is:

Lemma F.1.2.

$$(\hat{\omega}^i{}_j)_{1\leq i,j\leq n+1} \begin{pmatrix} \frac{d\rho}{1-2\rho} & -(1-2\rho)\omega^j_\theta \\ 0 & \theta^i_j \end{pmatrix}. \tag{F.3}$$

Using this one can show by a direct computation that, acting on scalar fields:

Lemma F.1.3.

$$g^{ab}\hat{\nabla}_a\hat{\nabla}_b = -4\rho^2(1-2\rho)\partial_\rho^2 + 2\rho(2\rho(1-n)+n)\partial_\rho + 2\rho\Delta_{S^n}.$$

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UNIVERSITE BRETAGNE LOIRE / MATHSTIC

Université de Bretagne Occidentale

Titre : Scattering analytique et projectif sur des espaces-temps avec constante cosmologique positive.

Mot clés : analyse asymptotique, trous noirs Kerr-de Sitter extrême, équation de Dirac, compactifications projectives, équation de Proca, calcul extérieur pour les tracteurs projectifs

Résumé : La thèse comporte deux projets principaux. En premier lieu, la construction d'une théorie de scattering analytique pour des champs de Dirac (massifs ou non) à l'extérieur d'un trou noir de type de Sitter-Kerr extrême : un trou noir en rotation dont les horizons coïncident pour former un trou noir « double » ou extrême. Les effets conjugués de la rotation, la constante cosmologique, et l'horizon double, se traduisent dans l'expression de l'opérateur de Dirac par des potentiels de type Coulomb à l'horizon et une perturbation de l'opérateur de Dirac sur la sphère. Dans ces travaux, les méthodes de Nicolas–Häfner, s'appuyant sur la théorie de Mourre, sont adaptées pour montrer la complétude asymptotique. En particulier, une étude précise de la partie angulaire montre qu'il est possible de décomposer l'opérateur de façon à pouvoir appliquer les résultats de T. Daudé, développés pour traiter le cas d'un trou noir de Reissner-Nordström. Ce travail a également donné lieu à une classification complète de la famille des trous noirs de de Sitter-Kerr et à une étude et construction détaillée des extensions maximales. Le deuxième projet explore l'application des outils de la géométrie projective à l'étude du comportement asymptotique de champs massifs de spin entier (champs de « Proca »). Une théorie analogue à celle développée par A.R. Gover, E. Latini et A. Waldron dans le cas de variétés admettant une compactification conforme est obtenue dans le cas de variétés Einstein projectivement compactes et asymptotiquement de Sitter, dont en particulier, un calcul extérieur pour les tracteurs projectifs donnant lieu à un calcul à la frontière et des opérateur de solution formelle.

Title: Analytical and projective scattering on spacetimes with positive cosmological constant.

Keywords: asymptotic analysis, extreme Kerr-de Sitter black hole, Dirac equation, projective compactification, Proca equation, projective exterior tractor calculus

Abstract: The thesis is composed of two principal projects in the domain of asymptotic analysis in General Relativity. First of all, the thesis details the construction of an analytical scattering theory for Dirac fields (massive or not) outside an *extreme* de Sitter-Kerr black hole. This is a geometric model for a rotating black hole in a de Sitter type universe for which two of its horizons coincide, forming a so-called extreme horizon. The conjugated effects of the rotation, the cosmological constant and the extreme horizon translate into long-range potentials and a perturbation of the Dirac operator on the sphere in the expression of the global Dirac operator. In this work, the methods used by J.P. Nicolas and D. Häfner, based on Mourre's theory, are adapted to this situation to prove asymptotic completeness. In par-

ticular, a thorough inspection of the angular part of the operator shows that it is possible to reduce the problem to a spherically symmetric problem to which the results, developed by T. Daudé, for a Reissner-Nordström blackhole can be applied. This work also gives a complete classification of the de-Sitter Kerr family and a detailed study and construction of their maximal analytical extensions. The second project explores how to apply tools from projective differential geometry to the study of the asymptotic behavior of massive fields with integer spin (« Proca » fields). The thesis develops, for the case of projectively compact Einstein asymptotically de-Sitter manifolds, results that are parallel to those obtained by A.R. Gover, E. Latini and A. Waldron for conformally compact manifolds. It builds, in particular, an exterior tractor calculus that leads to a boundary calculus and formal asymptotic solution operators.