A fast-track course on Cartan geometries

Meudon - December

Jack Borthwick

11 - 14 December 2022

These notes[1] are based on a short series of lectures I gave at the "Mathematical aspects of Black hole theory workshop" held at the Observatory of Paris on the topic of Cartan geometries. I hope they may also be useful to a wider audience. During the course of writing this document, I have also added a number of details and complements that I did not have time to cover during the lectures.

I assume the reader has followed a first course on General Relativity.

Notations: If $f : M \to N$ is a smooth map between smooth manifolds, then at each $p \in M$, f_{*p} denotes the tangent map at p. For maps between open sets of finite dimensional vector spaces, we will also use the notation $f'(x)$.

In these lectures I live in a smooth world and all manifolds and maps will be smooth. Manifolds are assumed finite dimensional, Hausdorff and second-countable.

I. Lecture I: Introduction and prerequisites

In 1872, F. Klein, in his now famous "Erlangen program", proposed a unified way of thinking about the "new" geometries that had appeared since the realisation of the fact that the so-called "Parallel postulate" of Euclidean geometry was logically independent from the other "more obvious" axioms. In modern mathematical language, Klein pointed out the following common feature. In each situation there was a configuration space X and a distinguished transformation group G, acting transitively on X. Fixing a given point $x_0 \in X$ and considering the isotropy subgroup (or stabiliser) H of x_0; i.e.

$$H = \{g \in G, g \cdot x_0 = x_0\},$$

[1] Due to unpopular demand

it is easy to see that there is a bijection with $X = G/H$. For example, in Euclidean geometry in n-dimensions, $G = \mathbb{R}^n \rtimes O(n)$, and $H = O(n)$.

This provides a natural class of generalisations of Euclidean geometry, that we shall refer to as *Klein geometries*, where G is a Lie group and H a closed subgroup. Another known generalisation of Euclidean geometry is Riemannian geometry that doesn't quite fit into this framework. *Cartan geometries*, that we shall introduce in these notes, are curved generalisations of Klein geometries and generalise also Riemannian geometry.

I.A. Lie groups

A key notion of Cartan geometries is the idea that they are based on a homogenous model G/H, where G is a Lie group and H a closed subgroup, our first aim will be to understand these. To begin with, we recall a few basic notions about Lie groups.

Definition I.1

A Lie group is a group $(G, \cdot)$ equipped with a smooth structure that is compatible with its group operations. In other words a smooth structure such that the maps:

$$\mu : \begin{array}{ccc} G \times G & \longrightarrow & G \\ (g_1, g_2) & \longmapsto & g_1 g_2 \end{array}, \qquad \iota : \begin{array}{ccc} G & \longrightarrow & G \\ g & \longmapsto & g^{-1} \end{array},$$

are smooth. In these notes we assume $\dim G < +\infty$ and that the topology of G satisfies the usual assumptions: Hausdorff[a] and second-countable.[b] Under these assumptions G is also *locally compact*. Let e denote the neutral element of G.

[a] This means that we can separate distinct points by open sets $x \neq y \Rightarrow \exists U, V$ open $U \cap V = \emptyset, x \in U, u \in V$.

[b] The topology has a countable base.

Example I.1. $(\mathbb{R}, +), (\mathbb{R}^*, \cdot), GL_n(\mathbb{R}), SL_n(\mathbb{R}), \ldots$

To warm up, lets recall a few facts about the topology of these groups, in what follows G is a Lie group but many of the results are true for all topological groups (when we only really use continuity). Let $g \in G$, call $L_g : x \mapsto gx$; it is a smooth diffeomorphism of G and $L_g{}^{-1} = L_{g^{-1}}$

Proposition 1. *Let H be a subgroup of G such that $\mathring{H} \neq \emptyset$ then H is closed.*

Proof. Let $g \in \overline{H}$, $x \in \mathring{H}$. Then $x^{-1}\mathring{H} = L_{x^{-1}}\mathring{H}$ is an open neighbourhood of e and $gx^{-1}\mathring{H}$ is an open neighbourhood of g. It follows that $gx^{-1}\mathring{H} \cap H \neq \emptyset$ so that $\exists y \in H$ such that $g = xy \in H$. □

Remark I.1. Note that if $\mathring{H} \neq 0$ then H is open, indeed, let $x \in H_0$ then for any $g \in H$, $U = gx^{-1}H_0 \subset H$ is neighbourhood of g hence H is a neighbourhood of each of its points and is open.

Corollary I.1. *Suppose that G is connected and let V be a neighbourhood of e, then:*

$$G = \bigcup_{n \in \mathbb{N}} V^n,$$

where V^n is defined recursively for $n \geq 2$, by $V^1 = V$, $V^n = \mu(V \times V^{n-1}) = \{v_1 v_2, v_1 \in V, v_2 \in V^{n-1}\}$.

Proof. Let $W = V \cap V^{-1}$, where $\iota(V) = V^{-1}$, then W is a neighbourhood of e such that $W^{-1} = W$ (we say that W is symmetric). Set $H = \bigcup_{n \in \mathbb{N}} W^n$, note that $H \subset \bigcup_{n \in \mathbb{N}} V^n$ and H has non-empty interior. Furthermore, H is the subgroup of G generated by W, it follows then that it is open and closed, hence $H = G$. □

Proposition 2. *For any group G the connected component G_0 of the identity is a closed subgroup.*

Proof. Let $g \in G_0$, then gG_0 is a connected since L_g is continuous. Furthermore, $e \in G_0$, $g \in gG_0$, therefore $gG_0 \subset G_0$ So $g_1 g_2 \in G_0$ for all $g_1, g_2 \in G_0$. Similarly G_0^{-1} is connected and contains the identity so $G_0^{-1} \subset G_0$. Finally, connected components are closed so G_0 is closed. □

Having refreshed our memory a bit, we will now quote two properties that are more specific to Lie groups.

Theorem I.1: Cartan - Von Neumann

A closed subgroup of a Lie group is also a submanifold and therefore a Lie subgroup.

Remark I.2. This is also known as the closed subgroup theorem. Its proof is quite beautiful but a bit off topic for these notes.

Corollary I.2. *The connected component G_0 of e is a Lie subgroup.*

I.B. The Lie algebra of a Lie group

Recall that every smooth manifold M has a natural vector bundle $\pi : TM \to M$ called the tangent bundle, whose fibre at $p \in M$ is:

$$\pi^{-1}(\{p\}) = T_p M.$$

Intuitively, it glues together all the tangent spaces in a way that is compatible with the local charts of M. The construction is described as follows, set[2] $TM = \coprod_{p\in M} T_pM$, and define the projection map by $\pi(X) = p$ if $X \in T_pM$. Now for any chart (x, U) on M construct a map $\tilde{x} : \pi^{-1}(U) \to x(U) \times \mathbb{R}^n$ such that for $X \in T_pM, p \in U$: $\tilde{x}(X) = (x(p), \tilde{x}_{*p}X)$; these bijective maps constitute natural candidates for charts on TM. Endow TM with the *coarsest* topology such that π, and all the $\tilde{x}$ are continuous. A prebasis for this topology is:

$$\mathscr{P} = \{\tilde{x}^{-1}(V),\ \text{V open in } x(U) \times \mathbb{R}^n,\ (x, U) \text{ local chart on } M\}$$

A basis is obtained by taking finite intersections of these sets and a set is open in this topology if and only if it is the union of finite intersections of elements in M This is the unique topology for which each $\tilde{x}$ is in fact a homeomorphism. Indeed, $\pi^{-1}(U) = \tilde{x}^{-1}(x(U) \times \mathbb{R}^n)$ is open in TM so any open set V of $\pi^{-1}(U)$ is open in TM. It is sufficient to check that $\tilde{x}(O)$ is open whenever $O = \tilde{y}^{-1}(V \times W)$ where $V \subset x(U)$ is open, W is open in $\mathbb{R}^n$ and $(y, \tilde{V})$ is another chart on M with $\tilde{V} \cap U \neq \emptyset$. We have:

$$\tilde{x}(O) = \{\left((x \circ y^{-1})(q), (x \circ y^{-1})'(q) \cdot v\right), q \in V \cap y(\tilde{V} \cap U), v \in W\}$$

which is open as the image of open set under a homeomorphism. The change of chart $\tilde{x} \circ \tilde{y}^{-1}$ (which also appears above) are easily seen to be smooth on their domain of definition and so define a smooth structure on TM. We leave it as an exercise for the reader to check that TM is Hausdorff and second-countable if M is.

The above scheme is very classical when constructing bundles associated with a given manifold.

Recall that smooth vector fields X are smooth sections of TM, i.e. smooth maps $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$. They form a $C^\infty(M)$ module denoted by $\Gamma(TM)$, it is common to denote $X(p)$ by X_p. Note that for any chart (x, U), by construction of TU we have n-smooth vector fields on U, written $\frac{\partial}{\partial x_i}$, defined by:

$$\left(\frac{\partial}{\partial x_i}\right)_p = x^{-1}_{*x(p)} \cdot e_i,$$

where $e_1, \dots, e_n$ is the canonical basis of $\mathbb{R}^n$.

$\Gamma(TM)$ is also a Lie algebra with bracket: $[,]$ defined by:

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

[2] $\coprod_{p\in M} T_pM$ denotes the disjoint union (or coproduct) of sets T_pM. It comes with a family of injective maps $i_p : T_pM \to TM$ and satisfies the following universal property: Let S be an arbitrary set and suppose we are given a family of maps $f_p : T_pM \to S$, then there is a unique map $f : \coprod_p T_pM$ such that $f_p = f \circ i_p$ for every $p \in M$. This is a convoluted way of saying that to define a map on TM we only need to specify how it acts in each T_pM.

(Vector fields on a finite dimensional manifold are assimilated with derivations of the algebra of smooth functions $C^\infty(M)$). In a chart (x, U) this gives on U:

$$[X, Y] = \sum_{i,j=1} \left(X^j \frac{\partial Y^i}{\partial x_j} - Y^j \frac{\partial X^i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

Where $X = X^i \frac{\partial}{\partial x_i}$ and similarly for Y. It can also be defined using the local flow of the vector field X:

$$[X, Y]_p = \frac{d}{dt} (\phi^X_{-t})_{*\phi^X_t(p)} \cdot Y_{\phi^X_t(p)} \bigg|_{t=0}$$

where: ϕ^X_t solves the Cauchy problem:

$$\begin{cases} \frac{d}{dt}\phi^X_t(p) = X(\phi^X_t(p)) \\ \phi^X_0(p) = p \end{cases}$$

Definition I.2

Let G be a Lie group a vector field $X \in \Gamma(TG)$ is said to be left-invariant if for every $g \in G$:

$$L_{g_*} X = X,$$

i.e.

$$\forall g \in G, p \in G, L_{g_{*p}} X_p = X_{gp}.$$

Left invariant vector fields are completely determined by a single value at any given point on G and in particular by their value at e. The map $X_e \mapsto X$ where X is the vector field defined by $X_p = L_{p_{*e}} X_e$ is vector space isomorphism.

Proposition 3. *The vector space of left-invariant vector fields on G is a Lie subalgebra of $\Gamma(TG)$ called the Lie algebra of G and written $\mathfrak{g}$.*

As vector spaces is isomorphic to T_eG and it is customary to transfer the Lie algebra structure of $\mathfrak{g}$ to T_eG via this bijection and identify these two descriptions of $\mathfrak{g}$.

I.C. The exponential map

Let $X \in \mathfrak{g}$ be a left-invariant vector field, consider the ODE:

$$\begin{cases} \dot\gamma(t) = X(\gamma(t)), \\ \gamma(0) = e, \end{cases}$$

where $\gamma : I \to G$ is a curve.

Suppose $\gamma : I \to G$ is a solution, and fix $s \in I$, then: $\gamma(s+t) = \gamma(s)\gamma(t)$ for any $t \in (I - s) \cap I = J$ Indeed, consider: $c(t) = \gamma(s)^{-1}\gamma(s+t), t \in J$ then $c(0) = 0$ and:

$$\dot{c}(t) = L_{\gamma(s)^{-1}}{}_{*\gamma(s+t)}\dot{\gamma}(t+s) = L_{\gamma(s)^{-1}}{}_{*\gamma(s+t)}X(\gamma(s+t)) = X(c(t)).$$

So c and γ satisfy the same Cauchy problem on the connected set J, and so $c = \gamma$ on J. Using this, we can show that any maximal solution to the Cauchy problem is necessarily defined on all of $\mathbb{R}$ and $\gamma : \mathbb{R} \to G$ is a group homomorphism. We define:

$$\exp(X) = \gamma(1).$$

It follows that:

$$\gamma(t) = \exp(Xt).$$

The map: $\exp : \mathfrak{g} \to G$ is called, as notation suggests, the exponential map of G. The derivative at 0 of exp is easily seen to be the identity map, hence by the local inversion theorem, exp is a local diffeomorphism near 0.

Example I.2. Let $G = (\mathbb{R}_+^*, \cdot)$ then $\mathfrak{g} = (\mathbb{R}, +)$ and, $\exp : \mathbb{R} \to \mathbb{R}_+^*$ is given by $x \mapsto e^x$.

Example I.3. Let $G = GL_n(\mathbb{R})$, as an open subset of the vector space $M_n(\mathbb{R})$ its Lie algebra is immediately assimilated with $M_n(\mathbb{R}) \equiv \mathfrak{gl}_n(\mathbb{R})$. Under this identification, left multiplication being the restriction to G of a linear map we have for any p:

$$L_{g*p} : \begin{array}{ccc} T_pG \cong M_n(\mathbb{R}) & \longrightarrow & T_{pg}G \cong M_n(\mathbb{R}) \\ M & \longmapsto & gM \end{array}.$$

So the above system can be written for any $A \in M_n(\mathbb{R}) = \mathfrak{gl}(V)$,

$$\begin{cases} \dot{\gamma}(t) = \gamma(t)A, \\ \gamma(0) = I_n, \end{cases}$$

the solution being given by:

$$\exp(At) = \sum_{t=0}^{+\infty} \frac{(At)^n}{n!}.$$

Using the flow definition of the bracket of vector fields, the induced bracket on $\mathfrak{gl}_n(\mathbb{R})$ is now seen to be:

$$[A, B] = \frac{d}{dt}\left(\frac{d}{ds}\exp(tA)\exp(sB)\exp(-tA)\bigg|_{s=0}\right)\bigg|_{t=0} = AB - BA.$$

So no nasty surprises !

To conclude this section let us note that if H is a Lie subgroup of a Lie group G then the inclusion map $i : H \hookrightarrow G$ gives a natural identification of the Lie algebra of H, $\mathfrak{h}$, with a Lie subalgebra of $\mathfrak{g}$ (X in $\mathfrak{g}$ such that $\exp(tX) \in H$ for all $t \in \mathbb{R}$.)

I.D. Lie group homomorphisms

Definition I.3

A Lie group homomorphism is smooth group homomorphism.

Let $f : G \to G'$ be a Lie group homomorphism, then the tangent map at the identity gives a linear map $\mathfrak{f} : \mathfrak{g} \to \mathfrak{g}'$. This map also preserves the Lie algebra structure, i.e.

$$\mathfrak{f}([X,Y]) = [\mathfrak{f}(X), \mathfrak{f}(Y)].$$

Differentiating the relation at the identity: $f \circ L_g = L_{f(g)} \circ f$, we get:

$$f_{*g} \circ L_{g*e} = L_{f(g)_{*e}} \circ \mathfrak{f}.$$

Hence, for any vector $X_e \in T_eG$ the left-invariant vector field $\mathfrak{f}(X)$ generated by $\mathfrak{f}(X_e)$ on G' is f-related to the left-invariant vector field X generated by X_e, therefore:

$$f_{*e}[X_e, Y_e] = f_{*e}[X,Y]_e = (f_*[X,Y])_e = [\mathfrak{f}(X), \mathfrak{f}(Y)]_e.$$

It follows in particular that any linear representation $\rho : G \mapsto GL(V)$, induces a Lie algebra representation, $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$.

Example I.4. Let G be a Lie group and consider for each g, the map:

$$\mathcal{A}d(g) : x \mapsto gxg^{-1}.$$

Then the derivative at the identity of this map gives an isomorphism denoted by $Ad(g) \in GL(\mathfrak{g})$. This defines a representation of G known as the ***adjoint representation*** and plays an important role in what follows. It actually takes its values in $Aut(\mathfrak{g})$ and the induced Lie algebra representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ takes it values in $Der(\mathfrak{g})$ the set of derivations[3] of the Lie algebra $\mathfrak{g}$. One can show that $ad(X)(Y) = [X,Y]$.

I.E. Principal bundles

Let P and M be two smooth manifolds, G a Lie group and $\pi : P \to M$ a smooth map. Suppose that G acts smoothly on P on the right and the map $(p,g) \mapsto pg$ is smooth. Suppose that:

1. $\pi(rg) = \pi(r)$ for all $r \in P, g \in G$,
2. For every $p \in M$, there is a neighbourhood U of p in M and a smooth diffeomorphism: $\phi : \pi^{-1}(U) \to U \times G$, such that the following diagram commutes:

[3] A derivation δ is a linear map on $\mathfrak{g}$ such that $\delta[x,y] = [\delta x, y] + [x, \delta y]$

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times G \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \pi_U} & \\ U & & \end{array}$$

and

$$\phi(rg) = \phi(r) \cdot g,$$

where G acts on $U \times G$ by right multiplication on the second factor. Such a ϕ is a *local trivialisation* or *bundle chart*.

The second condition means that P looks locally like a product equipped with the standard right action. We say that P is the total space, M the base and G is the structure group.

Remark I.3.

- The action of G on P is necessarily *free* i.e.

$$\forall g \in G, r \in P, rg = r \Rightarrow g = e,$$

- The orbits of the action are the fibres:

$$\pi^{-1}(\pi(r)) = \{rg, g \in G\}.$$

Proposition 4. *Local sections of P – smooth maps $\sigma : U \to P$ such that $\pi \circ \sigma = id_U$ – are equivalent to local trivialisations.*

Proof. If $\phi : \pi^{-1}(U) = U \times G$ is a local trivialisation then $x \mapsto \phi^{-1}(x, e)$ is a local section. Conversely, consider:

$$\psi : \begin{array}{ccc} U \times G & \longrightarrow & \pi^{-1}(U) \\ (p, g) & \longmapsto & \sigma(p)g \end{array}.$$

This is a smooth bijective map and

$$\psi_{*(p,g)}(X, Y) = \lambda_{\sigma(p)_{*g}}(Y) + R_{g_{*\sigma(p)}}\sigma_{*p}(X), X \in T_pM, Y \in T_gG$$

with $\lambda_r : G \mapsto P$ defined by $g \mapsto rg$. Suppose that $\psi_{*(p,g)}(X, Y) = 0$, then composing on the right with $\pi_{*\sigma(p)g}$ gives immediately $X = 0$, so the equation reduces to: $\lambda_{\sigma(p)_{*g}}(Y) = 0$. Let $\tilde{Y}$ denote the unique left-invariant vector field such that $\tilde{Y}_g = Y$, note that for any $s \in \mathbb{R}$

$$(R_{\exp(s\tilde{Y})} \circ \lambda_{\sigma(p)})_{*g}(Y) = R_{\exp(s\tilde{Y})_{*\sigma(p)g}} \lambda_{\sigma(p)_{*g}}(Y) = 0,$$

but

$$0 = (R_{\exp(s\tilde{Y})} \circ \lambda_{\sigma(p)})_{*g}(Y) = \frac{d}{dt}(\sigma(p)g\exp(t\tilde{Y})\exp(s\tilde{Y})|_{t=0}$$
$$= \frac{d}{dt}(\sigma(p)g\exp((t+s)Y)|_{t=0}$$
$$= \frac{d}{dt}(\sigma(p)g\exp(t\tilde{Y})|_{t=s}$$

Hence for any $s \in \mathbb{R}$:

$$\sigma(p)g\exp(s\tilde{Y}) = \sigma(p)g.$$

The action of G on P is free therefore:

$$\forall s \in \mathbb{R}, \exp(s\tilde{Y}) = e.$$

By local injectivity of exp on a neighbourhood of $0 \in \mathfrak{g}$, it follows that for small enough $s \neq 0$, we must have $s\tilde{Y} = 0$ which implies $\tilde{Y} = 0$ and therefore $Y = 0$. ψ is therefore a diffeomorphism and ϕ^{-1} is a bundle chart. □

Remark I.4. It follows from the above that if P has a global section then P is diffeomorphic to the trivial bundle $M \times G$.

Remark I.5. In the above proof, we showed that for any $r \in P$, the application λ_r is an immersion, therefore inducing a linear injection $\lambda_{r*g}L_{g^{-1}}{}_{*e} : \mathfrak{g} \to T_rP$ for each $r \in P$. Note that: $\text{im}\lambda_{r*g}\lambda_{g^{-1}}{}_{*e} \subset \ker \pi_{*r}$, however:

$$\dim \ker \pi_{*r} = \dim P - \dim M = \dim G = \dim \mathfrak{g},$$

hence: $\ker \pi_{*r} = \text{im}\lambda_{r*g}\lambda_{g^{-1}}{}_{*e}$.

Definition I.4: Vertical vectors and fundamental vector fields

Vectors in $\ker \pi_{*r}$ are called *vertical vectors*. For every $X \in \mathfrak{g}$ we can define a smooth vertical vector X^* on P field known as the fundamental vector field generated by X given by:

$$X^*_p = \frac{d}{dt} p\exp(tX)\bigg|_{t=0}.$$

I.F. Complement: The Lie group $G^2(n)$

Let U and V be two open neighbourhoods of 0 in $\mathbb{R}^n$, and $f : U \to V$ a smooth diffeomorphism such that $f(0) = 0$, two such diffeomorphisms are said to define the same 2-jet at 0 if they have the same partial derivatives up to order 2 at 0. This defines an equivalence relation and the equivalence class of f will be written $j^2_0 f$.

Define $G^2(n)$ to be the set of all 2-jets of diffeomorphisms between open neighbourhoods of 0. It is naturally a group, where the operation is defined by:

$$j_0^2 f \cdot j_0^2 g = j_0^2(f \circ g).$$

$G^2(n)$ has a natural coordinate system:

$$\omega^i(\frac{\partial f}{\partial x_j}(0)) = u_j^i, \omega^k(\frac{\partial f}{\partial x_l, x_m}(0)) = u_{lm}^k,$$

where u_{lm}^k is symmetric in the indices l and m and ω^i denotes the dual basis of the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$. This defines a natural smooth structure on $G^2(n)$. The group law is then given by (using the Einstein summation convention):

$$(u_j^i, u_{lm}^k)(v_j^i, v_{lm}^k) = (u_k^i v_j^k, v_{lm}^i u_i^k + v_l^i v_m^j u_{ij}^k).$$

II. Lecture 2 : Cartan geometries

II.A. Examples of Principal Bundles

The definition of Principal Bundle we saw at the end of Lecture 1 is very theoretical. We shall now consider some important examples to motivate its study and that should help us understand what the definition is trying to capture.

Example II.1 (The Frame Bundle). Let M be a smooth n-dimensional manifold, and TM its tangent bundle. There is a natural $GL_n(\mathbb{R})$-principal bundle with base M, interpreted as the set of linear frames of the tangent spaces. It is constructed as follows.

- For each $p \in M$, let $L(TM)_p = GL(\mathbb{R}^n, T_pM)$. Each $u_p \in L(TM)_p$ can be interpreted as a choice of basis for T_pM.
- Set $L(TM) = \coprod_{p \in M} L(TM)_p$ and define the projection π so that $u_p \mapsto p$ if $u_p \in L(TM)_p$.
- $GL_n(\mathbb{R}) \cong GL(\mathbb{R}^n)$ acts in a natural way on the right on $L(TM)$ by setting $u_p \cdot g = u_p \circ g$ when $u_p \in L(TM)_p$ and $g \in GL_n(\mathbb{R})$. Clearly: $\pi(u_p g) = \pi(u_p)$
- We now need to make this smooth in a way that is compatible with the smooth structure of M. Let (x, U) be a local chart on M we have a natural map: $\hat{x} : \pi^{-1}(U) \to x(U) \times GL_n(\mathbb{R})$ defined for $u_p \in L(TM)_p, p \in U$ by:

$$\hat{x}(u_p) = (p, x_{*p} \circ u_p).$$

- Endow $L(TM)$ with the coarsest topology such that π and each $\hat{x}$ is smooth for every chart (x, U) on M; the $\hat{x}$ are then homeomorphisms. If M is Hausdorff and second countable $L(TM)$ is too. The maps $\hat{x}$ determine a smooth structure on M since of (y, V) is another chart then: $\hat{y} \circ \hat{x}^{-1} : x(U \cap V) \times G \to y(U \cap V) \times G$ is given by the smooth map

$$(q, g) \mapsto ((y \circ x^{-1})(q), (y \circ x^{-1})'(q) \circ g).$$

- For each (x, U) if we compose $\hat{x}$ with the map $(q, g) \mapsto (x^{-1}(q), g)$ defined on $x(U) \times G$ we get a bundle chart $\phi : \pi^{-1}(U) \to U \times G$ and by construction:

$$\phi(u_p g) = \phi(u_p) g.$$

  Then $\pi : L(TM) \to M$ is a G-principal bundle with total space $L(TM)$ and base M.

Local sections $\sigma : U \to L(TM)$ of the frame bundle $L(TM)$ formalise the idea of a smooth choice of basis at each point in U; if there is a global section, then clearly M is parallelisable.

A principal bundle should therefore be thought of as a generalised "frame bundle" with change of frame being achieved by an element of the structure G.

For our next example let us quote without proof the following theorem:

Theorem II.1

Let P be a smooth manifold and H a Lie group that acts smoothly, freely and properly on the right on P. Let $\pi : P \to P/H$ be the canonical projection, then:

1. Endowed with the ***quotient topology*** P/H is a topological manifold. This topology is given by:

   $$U \text{ is open in } P/H \text{ if and only if } \pi^{-1}(U) \text{ is open in } P.$$

   (Note that π is an open map)

2. P/H has a unique smooth structure such that $\pi : P \to P/H$ is a smooth submersion (the tangent map π_* is surjective at each point).

3. $\pi : P \to P/H$ is a principal H-bundle.

Remark II.1. The action of a Lie group H on a manifold P is said to be proper if for any compact set $K \subset P$ the set:

$$\{h \in H, Kh \cap K \neq \emptyset\},$$

is itself compact. Since P and H are finite dimensional manifolds they are second countable and locally compact, it follows then that we have a sequential characterisation of proper actions:

The action is proper if and only if whenever two sequences $(h_n) \in H^{\mathbb{N}}$, $(r_n) \in P^{\mathbb{N}}$ are such that $(r_n \cdot h_n)$ and (r_n) converge, then (h_n) has a convergent subsequence.

Example II.2. Let G be a Lie group and H a closed subgroup, let H act on G by right multiplication. This action is smooth, free and proper (use the above sequential characterisation for instance), so according to the previous theorem G/H endowed with its quotient topology is a topological manifold and has a unique smooth structure such that $\pi : G \to G/H$ is a smooth submersion and is a principle H-bundle.

To develop an intuition for what this means let us specialise to a more familiar setting:

Example II.3. Let $G = \mathbb{R}^n \rtimes GL_n(\mathbb{R})$ be the affine group acting on affine space, identified with $\mathbb{R}^n$ and its canonical affine structure. Recall that G is a Lie group as the product manifold $\mathbb{R}^n \times GL_n(\mathbb{R})$ (open subset of the vector space $\mathbb{R}^n \times M_n(\mathbb{R})$) equipped with the group law:

$$(x, A) \times (y, B) = (Ay + x, AB), (x, A)^{-1} = (-A^{-1}x, A^{-1}), e = (0, I_n).$$

These are nice and smooth. It acts on $\mathbb{R}^n$ according to $(x, A)p = Ap + x, p \in \mathbb{R}^n$ and the action is clearly transitive. The isotropy subgroup of the origin p is given by $(0, A), A \in GL_n(\mathbb{R})$ which we naturally identify with $GL_n(\mathbb{R}) = H$. Consider the smooth map, $G \to \mathbb{R}^n$ that projects onto the first factor of $\mathbb{R}^n \rtimes H$ then this induces a smooth diffeomorphism between G/H (with its topology given by the theorem) and $\mathbb{R}^n$, so we shall assimilate G/H with affine space itself and $\pi : G \to G/H$ with the projection onto the first factor.

In this case – which is mostly tautological – we have a smooth global section $\sigma : \mathbb{R}^n \to G$ given by $\sigma(p) = (p, I_n)$, we think of this as attaching to each point p the canonical basis of $\mathbb{R}^n$ (therefore obtaining an affine frame with origin p). Indeed, the linear frame bundle of $\mathbb{R}^n$ is easily seen to be the trivial bundle $\mathbb{R}^n \times H$ and so σ defines a principal bundle homomorphism between G and $L(T\mathbb{R}^n)$.

To clarify the interpretation here and carry it over to the more abstract case of a Klein geometry G/H, take $n = 2$ and suppose that we want to understand the movement of a triangle through affine space. Configurations of the triangle are related by an element of the affine group (keep in mind that for affine geometry only relative lengths make sense so it is a vague notion of triangle that may stretch as it moves.) To describe this movement we can first fix a point q on the triangle and study the curve $\gamma : I \to \mathbb{R}^n$ it traces through affine space. To account for the movement of the other points, having chosen at each point in affine space an element of g (a reference configuration or frame, this is the role of the *section*), we only need a curve $h : I \to H = GL_n(\mathbb{R})$ that at each point tells us how to go from the reference configuration $g(t)$ and the actual configuration at t of X. This is tentatively illustrated in Figure 1.

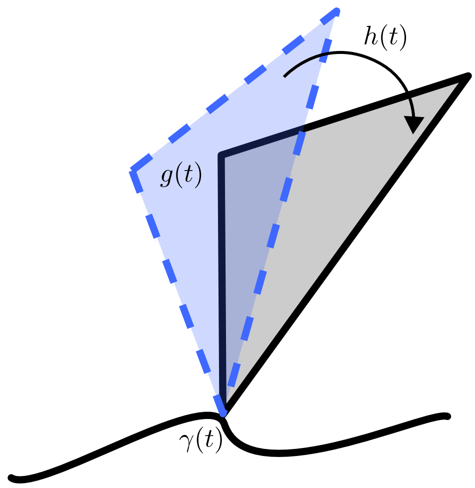

Figure 1: Frames

II.B. The Maurer-Cartan form

Up to now, we have done no geometry; what we are lacking is a notion of connection. For instance in the above, thinking of a frame as $(O, e_1, \ldots, e_n)$, we have not identified the infinitesimal structure that enables us to compare frames at distinct points.

More generally in the group picture we are developing, there is another canonical structure on G that we have ignored; left multiplication by an element of G. Of course, the reader might have already realised that to generalise the structures we have here Lie groups will be substituted for principal bundles and there will be no natural left multiplication, but instead we may be able to generalise its infinitesimal version.

Let us denote for each $g \in G$, L_g the map $r \mapsto gr$, it is a smooth diffeomorphism of G and induces a linear isomorphism between tangent spaces. In particular, we have a linear isomorphism ${L_{g^{-1}}}_{*g}$ between T_gG and $\mathfrak{g}$; this yields a natural map: $\omega : TG \to \mathfrak{g}$ that we interpret as a $\mathfrak{g}$-valued differential form.[4].

Pullback of $\mathfrak{g}$-valued differential forms can be defined in the same was as for usual differential forms. In order to generalise the exterior product we couple the usual definition with the Lie bracket and obtain for α, β respectively $\mathfrak{g}$-valued k and l

[4]This can also be thought of as a smooth section of the vector bundle $T^*G \otimes (G \times \mathfrak{g})$

forms

$$[\alpha, \beta](X_1, \dots, X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \varepsilon(\sigma)[\alpha(X_{\sigma(1)}, \dots, X_{\sigma(k)})\beta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)})].$$

Example II.4. Let α, β be 1-forms then:

$$[\alpha, \beta](X, Y) = [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)].$$

In particular if $\beta = \alpha$ we get:

$$[\alpha, \alpha] = 2[\alpha(X), \alpha(Y)].$$

The exterior derivative d also extends to $\mathfrak{g}$-valued forms in the same way as usual one-forms by imposing on $\mathfrak{g}$ functions that for any $l \in \mathfrak{g}^*$, vector field X we have $l(X(f)) = X(l(f))$. This is way of saying that fixing a basis of $\mathfrak{g}$, d acts on each component separately and the result is independent of the chosen basis. In this way, the usual invariant formula extends to $\mathfrak{g}$-valued differential forms. (In fact this also works when $\mathfrak{g}$ is replaced by an arbitrary finite dimensional vector space V)

Example II.5. Let α be a $\mathfrak{g}$-valued one form, then for any vector fields (X, Y):

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]).$$

We can now state the important properties of the Maurer-Cartan form:

Theorem II.2

Let G be a Lie group, H a closed subgroup and ω its Maurer-Cartan form, then:

1. For every $g \in G$, $\omega_g : T_gG \to \mathfrak{g}$ is a linear isomorphism,
2. For every $h \in H$, $R_h^*\omega = Ad(h^{-1})\omega$
3. For every $X \in \mathfrak{h}$ the fundamental vector field X^* on G satisfies: $\omega(X^*) = X$.

Additionally it satisfies the following structure equation:

$$d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

Proof. 1. By definition.

2. Let $X_g \in TgG$, $h \in H$, let us calculate $R_h^* \omega_g(X_g)$, we have:

$$R_h^* \omega_g(X_g) = \omega_{gh}(R_{h*g}X_g) = L_{h^{-1}g^{-1}}{}_{*gh} R_{h*g} X_g$$
$$\text{[chain rule]} = (L_{h^{-1}g^{-1}} \circ R_h)_{*g} X_g.$$

Now the map: $L_{h^{-1}g^{-1}} \circ R_h$ acts on $p \in G$ as follows:

$$L_{h^{-1}g^{-1}} \circ R_h(g) = h^{-1}g^{-1}ph = \mathcal{A}d(h^{-1}) \circ L_{g^{-1}}(p).$$

Therefore:

$$R_h^* \omega_g(X_g) = Ad(h^{-1})(L_{g^{-1}}{}_{*g} X_g = Ad(h^{-1})\omega_g(X_g).$$

3. Let $X \in \mathfrak{h}$ then $X_g^* = L_{g*e}X$ for any $g \in G$, so that:

$$\omega_g(X_g^*) = L_{g^{-1}}{}_{*g} L_{g*e} X = X,$$

by the chain rule.

4. To prove the structure equation, recall that the value of $\omega(X)$ for any vector field at any point g only depends on the value X_g. Hence we can work with left-invariant vector fields X and Y, in this case: $\omega(X) = X_e$ and $\omega(Y) = Y_e$ are constant and so:
$$d\omega(X, Y) = -\omega([X, Y]).$$
But $[X, Y]$ is a left invariant vector field, so: $\omega([X, Y]) = [X_e, Y_e] = [\omega(X), \omega(Y)] = \frac{1}{2}[\omega, \omega](X, Y)$, the structure equation follows.

□

Let us look at the Maurer-Cartan form in affine space to get a feel for its role in our picture. Let us consider the affine plane $\mathbb{R}^2$ in this case: $G = \mathbb{R}^2 \rtimes GL_2(\mathbb{R})$, $H = GL_2(\mathbb{R})$, the Lie algebra $\mathfrak{g}$ is seen to be: $\mathfrak{g} = \mathbb{R}^2 \oplus \mathfrak{gl}_2(\mathbb{R})$ equipped with the semi-direct product law:

$$[(a, M), (b, M')] = (Mb - M'a, [M, M']).$$

Let $g = (x, A) \in G$ and note that $L_g : r = (y, B) \mapsto (Ay + x, AB)$, then we find:

$$\begin{array}{rccc} L_{g*r}: & T_y\mathbb{R} \cong \mathbb{R} \times T_B G \cong M_n(\mathbb{R}) & \longrightarrow & T_{Ay+x}\mathbb{R} \cong \mathbb{R} \times T_{AB} G \cong M_n(\mathbb{R}) \\ & (a, M) & \longmapsto & (Aa, AM) \end{array}$$

Hence the Maurer-Cartan form can be written:

$$\omega(a, M) \mapsto (A^{-1}a, A^{-1}M).$$

Let us consider an open subset $U \subset \mathbb{R}^2$ where we can use polar coordinates (r, θ) defined by $x = r\cos\theta, y = r\sin\theta$. Recall that:

$$\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}.$$

Define the frame $e_r = \frac{\partial}{\partial r}$, $e_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}$ at each point $p \in U$. With the canonical identifications: $\mathbb{R}^2 \times H = L(T\mathbb{R}^2) = G$, this corresponds to a section σ:

$$\begin{array}{rcl} \sigma: \ U & \longrightarrow & G = \mathbb{R}^2 \rtimes GL_2(\mathbb{R}) \\ p & \longmapsto & \left(\begin{pmatrix} x(p) \\ y(p) \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right) \end{array}.$$

To extract information on ω, lets pull it back to M using the section σ, to have a $\mathfrak{g}$-valued 1-form on M.

$$\begin{aligned} (\sigma^*\omega)_p = \omega_{\sigma(p)} \circ \sigma_{*p} &= \left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}, \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -\sin\theta d\theta & -\cos\theta d\theta \\ \cos\theta d\theta & -\sin\theta d\theta \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} \cos\theta dx + \sin\theta dy \\ -\sin\theta dx + \cos\theta dy \end{pmatrix}, \begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix} \right) \end{aligned}$$

Now:

$$\cos\theta dx + \sin\theta dy = dr, \quad -\sin\theta dx + \cos\theta dy = r d\theta.$$

Observe that $(\omega^1 = dr, \omega^2 = r d\theta)$ is in fact the dual basis to (e_r, e_θ) so in the first component we have a $\mathbb{R}^2$-valued one form that maps any vector in $T_p\mathbb{R}^2$ to its coordinates in the basis (e_r, e_θ). This tells us how the basis is point is changing in the coordinates (e_r, e_θ) To interpret the second component, we calculate $\nabla e_1, \nabla e_2$ with the canonical flat connection ∇ on $\mathbb{R}^2$, in particular the global basis vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial x}$ are parallel; so that:

$$\begin{aligned} \nabla e_r &= -\sin\theta d\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y} = d\theta e_\theta \\ \nabla e_\theta &= -\cos\theta d\theta\frac{\partial}{\partial x} - \sin\theta\frac{\partial}{\partial y} = -d\theta e_r \end{aligned}.$$

Let us rewrite this:

$$\begin{pmatrix} \nabla e_r & \nabla e_\theta \end{pmatrix} = \begin{pmatrix} e_r & e_\theta \end{pmatrix} \begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix}.$$

Hence the second component is the infinitesimal change of basis. The Maurer-Cartan form appears to encode the same information as the flat (linear) connection in the usual sense.

Motivated by this we give the following definition:

Definition II.1: Cartan geometry

Let G be a Lie group, H a closed subgroup, and a manifold M. A Cartan geometry on M modeled on the Klein geometry G/H consists of the following data:

- A principal bundle $\pi : P \to M$ with structure group H.
- A Cartan connection on P, i.e. a $\mathfrak{g}$-valued one-form ω on P such that:
  1. $\forall h \in H, R_h^* \omega = Ad(h^{-1})\omega$,
  2. $\forall X \in \mathfrak{h}, \omega(X^*) = X$,
  3. For each $r \in P$, $\omega_r : T_r P \to \mathfrak{g}$, is an *isomorphism* of vector spaces.

Note that we do note impose that the Maurer-Cartan form satisfy the structural equation, since the proof illustrated that this should be understand as a very specific property of the homogeneous space. Instead we define:

Definition II.2: Curvature of the Cartan connection

Let (P, ω) be a Cartan geometry on M modeled on the homogenous space G/H, the curvature of ω is the $\mathfrak{g}$-valued 2-form:

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

In fact, the intuition is correct that $\Omega = 0$ *locally* characterises G/H. For a precise statement we refer to [Sha97, Theorem 5.1].

The curvature satisfies:

Proposition 5. *Bianchi identity*

$$d\Omega + [\omega, \Omega] = 0$$

Proof. This reduces down to the Jacobi identity of the Lie algebra:

$$d\Omega = \frac{1}{2}([d\omega, \omega] - [\omega, d\omega]) = -[\omega, d\omega] = -[\omega, \Omega] + \frac{1}{2}[\omega, [\omega, \omega]].$$

But

$$\begin{aligned}\frac{1}{2}[\omega, [\omega, \omega]](X, Y, Z) &= [\omega(X), [\omega(Y), \omega(Z)]] + [\omega(Y), [\omega(Z), \omega(X)]] + [\omega(Z), [\omega(X), \omega(Y)]] \\ &= 0.\end{aligned}$$

□

Cartan geometries are curved generalisations of Klein geometries: the principal bundle P generalises the Lie group G and ω the Maurer-Cartan form. The third condition on the Cartan geometry imposes a sort of tangency of the model at each point. Expressing things in terms of the base M, one can imagine that we have glued at each point a copy of the model G/H in such a way that it is tangent to M. The Cartan connection then tells how to move between two copies of the model attached to infinitesimally close points. The curvature of the Cartan connection measures in a certain sense how much it differs from the model.

The notion of model is central to the study of Cartan connections, in particular understanding the infinitesimal model $(\mathfrak{g}, \mathfrak{h})$, the adjoint representation of H, etc, provides useful geometric tools and informations for understanding the curved generalisations.

II.C. Complement: The tangent bundle of a Klein Geometry

Let G be a Lie group and H a closed subgroup, $\pi : G \to G/H$ the canonical projection. Let $p \in G/H$ and choose $g \in G$ such that $\pi(g) = p$ then the Maurer-Cartan form of G ω fits into the diagram: Each column in Figure 2 is a short exact sequence,

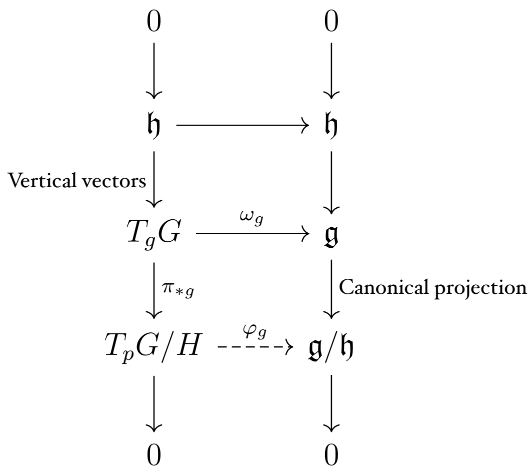

Figure 2: Construction of an isomorphism between T_pM and $\mathfrak{g}/\mathfrak{h}$

the image of each is the kernel of the map that follows, in particular π_{*g} is surjective. Horizontally, the Maurer-Cartan form is an isomorphism between T_gG and hence we have a natural isomorphism ϕ_g between T_pM and $\mathfrak{g}/\mathfrak{h}$ defined by:

$$\phi_g(\pi_{*g}(X)) = \mathrm{Pr}_{g \to \mathfrak{g}/\mathfrak{h}} \circ \omega_g(X)$$

This definition makes sense because every vector in $\tilde{X} \in T_p(G/H)$ can be written $\pi_{*g}(X)$ for some $X \in T_gG$. Two such representations differ by a vertical vector Y^*, but $\omega(Y^*) = Y \in \mathfrak{h}$. φ_{g*} is injective and since $\dim T_p(G/H) = \dim \mathfrak{g}/\mathfrak{h}$, is an isomorphism.

This isomorphism is non-canonical in the sense that it depends on the choice of g in the fibre of p. However, note that $\pi_{*gh}(R_h)_{*g} = (\pi \circ R_h)_{*g} = \pi_{*g}$. Hence, if $\tilde{X} \in T_p(G/H)$ and $X \in T_gG$ such that $\pi_{*g}X = \tilde{X}$ then $\pi_{*gh}(R_h)_{*g}X = \tilde{X}$. Thus:

$$\begin{aligned}\phi_{gh}(\tilde{X}) = \phi_{gh}(\pi_{*gh}R_{h*}gX) = \omega_{gh}(R_{h*g}X) = (R_h^*\omega)_g(X) &= Ad(h^{-1})\omega_g(X) \\ &= Ad(h^{-1})\phi_g(\tilde{X}).\end{aligned}$$

Hence, the isomorphism are the same up to the action of the (induced) adjoint representation on $\mathfrak{g}/\mathfrak{h}$.

In fact let us consider $P = G \times \mathfrak{g}/\mathfrak{h}$ and let H act on P on the right by:

$$(p, v)h = (gh, Ad(h^{-1})v).$$

This action is free and proper. So we can consider P/H with its quotient topology and smooth structure given by Theorem II.1. Denote by $[g, v]$ the image of (g, v) under the canonical projection: $q : P \mapsto P/H$. Then P/H can be given the structure of a smooth vector bundle with base G/H. Indeed, let $p : P/H \to G/H$ be defined by factorisation of the map:

$$\begin{array}{rccc}\tilde{\pi}: & P & \longrightarrow & G/H \\ & (g, v) & \longmapsto & \pi(g)\end{array}.$$

(i.e. $\tilde{\pi} = p \circ q$).

The fibres of p are then easily seen to be isomorphic as vector spaces to $\mathfrak{g}/\mathfrak{h}$. We now construct vector bundle charts, for this let $\phi : \pi^{-1}(U) \to U \times H$ be a bundle chart for $\pi : G \to G/H$, set:

$$\begin{array}{rccc}\Phi: & (p \circ q)^{-1}(U) & \longrightarrow & U \times V \\ & (g, v) & \longmapsto & (\pi(g), Ad((\mathrm{Pr}_H\phi(g))v)\end{array},$$

Factorisation of this map gives a smooth map: $\tilde{\phi} : p^{-1}(U) \to U \times V$. This vector bundle is called the *associated vector bundle* to the representation $(Ad, \mathfrak{g}/\mathfrak{h})$ of H, it is denoted by:

$$P = G \times_H \mathfrak{g}/\mathfrak{h}.$$

The maps ϕ_g constructed at the beginning of the section can then be used to show that we have the vector bundle isomorphism:

$$T(G/H) \cong G \times_H \mathfrak{g}/\mathfrak{h}.$$

This property extends to the Cartan geometry case, if $\pi : P \to M$, ω is a Cartan geometry with structure group H then:

$$TM \cong P \times_H \mathfrak{g}/\mathfrak{h}.$$

Once again, this is consistent with the idea of gluing a copy of G/H that is tangent to M at each point.

III. Lecture 3 : (Pseudo)-Riemannian geometry as a Cartan geometry

This final lecture will be devoted to a familiar model in order to get an idea of the tools, and we shall illustrate an instance of the equivalence problem, where a specific type of Cartan geometry is related to a set of data on the base manifold.

Set $G = \mathbb{R}^n \rtimes O(p,q)$ and $H = O(p,q)$. The homogenous space $G/H \cong \mathbb{R}^{p+q}$ is (pseudo)-Euclidean space. For definiteness, you can think of $p = n$, $q = 1$ for Minkowski space but the whole discussion is independent of this choice. Note here that since we choose $O(n,1)$ in the model as opposed to $SO(n,1)$ or $SO_o(n,1)$, we do not assume any time orientation or orientation.

Let us call a (pseudo)-Euclidean geometry on a manifold M a Cartan geometry with this model. That is a principal $O(p,q)$-bundle $\pi : P \to M$ and a Cartan connection $\omega : TP \to \mathfrak{g}$.

The Lie algebra of G is the semi-direct product of Lie algebras:

$$\mathfrak{g} = \mathbb{R}^n \oplus \mathfrak{o}(p,q),$$

writing elements of this Lie algebra as: $x + A$, $x \in \mathbb{R}^n$, $A \in \mathfrak{o}(p,q)$ the Lie bracket can be written:

$$[x + A, y + B] = Ay - Bx + [A,B].$$

$\mathfrak{o}(p,q)$ is identified with a Lie subalgebra of $\mathfrak{gl}_{p+q}(\mathbb{R})$ that we shall determine. Let η be the matrix in the canonical basis of the canonical (p,q) metric on $\mathbb{R}^{p+q}$. Then a curve $\gamma : I \to GL_{p+q}(\mathbb{R})$ such that $\gamma(0)$ has values in $O(p,q)$ if and only if for every $t \in I$,

$$(\gamma(t))^T \eta \gamma(t) = \eta.$$

Differentiating this identity at $t = 0$, yields:

$$\gamma'(0)^T \eta + \eta \gamma'(0) = 0.$$

However, if a matrix $X \in \mathfrak{gl}_{p+q}(\mathbb{R})$ satisfies the above condition then

$$\forall t \in \mathbb{R}, \exp(tX) \in O(p,q).$$

Indeed:

$$\exp(tX)^T \eta \exp(tX) \eta^{-1} = \exp(tX^T) \exp(t\eta X \eta^{-1}) = \exp(tX^T) \exp(-tX^T) = I_{p+q}.$$

Hence:

$$\mathfrak{o}(p,q) = \{X \in \mathfrak{gl}_{p+q}(\mathbb{R}), X^T \eta + \eta X = 0.\}$$

Let us consider the adjoint representation of $H = O(p,q)$ on $\mathfrak{g}$, which in these notations can be written:

$$Ad(h)(x + A) = hx + hAh^{-1}.$$

This formula shows that the natural semi-direct product structure of $\mathfrak{g}$ is actually a *H-module decomposition* in the sense that each component is invariant under $Ad(h)$ for all $h \in H$. Another way to see this is that the components are subrepresentations of H.

This is an example of what is known as a *reductive* Cartan geometry. In this case the Cartan connection splits into two parts:

$$\omega = \theta + \gamma.$$

There properties are as follows:

Proposition 6. *1. The $\mathbb{R}^n$ valued one form θ satisfies:*

- $\theta(X^*) = 0$ *for any fundamental vector field.*
- $R_h^*\theta = h^{-1}\theta$.

We say that θ is horizontal and equivariant.

2. The $\mathfrak{h}$- valued one form γ satisfies:

- $R_h^*\gamma = Ad(h^{-1})\gamma$,
- $\gamma(X^*) = X$*, if X^* is the fundamental vector field associated with $X \in \mathfrak{h}$*

γ is therefore a principal connection *on P.*

Proof. $\omega(X^*) = 0 + X$. □

θ is sometimes called a solder form. It can be interpreted here as being explicitly the part of the connection that in the final remarks of Section II.C glues the structure to the base manifold, by identifying the tangent bundle with an associated vector bundle of P.

The curvature form Ω also splits into two parts:

Proposition 7.

$$\Omega = \underbrace{d\theta + [\gamma,\theta]}_{T} \quad + \underbrace{d\gamma + \frac{1}{2}[\gamma,\gamma]}_{R},$$

where $[\gamma,\theta](X,Y) = \gamma(X)\theta(Y) - \gamma(Y)\theta(X)$.

In order to construct examples, we shall now show that pseudo-Riemannian manifolds are naturally equipped with a pseudo-Euclidean geometry.

For this let us first make the following observation:

Proposition 8. *Let $\pi : L(TM) \to M$ be the linear frame bundle (the structure group is $GL_n(\mathbb{R})$. There is a canonical choice of solder form θ given by:*

$$\theta_{u_p}(X) = u_p^{-1} \circ \pi_{*u_p}(X), \textit{ for any } u_p \in GL(\mathbb{R}^n, T_pM), X \in T_{u_p}L(TM).$$

Proof.

1. Since for any fundamental vector field $X^* \in \ker \pi_{*u_p}$ it is immediate that $\theta(X^*) = 0$.

2. Let us determine $R_h^*\theta$, pick $X \in T_{u_p}M$.

$$(R_h^*\theta)_{u_p}(X) = \theta_{u_ph}(R_{h*u_p}(X)) = h^{-1}u_p^{-1}\pi_{*u_ph}R_{h*u_p}(X).$$

But $\pi_{*u_ph} \circ R_{h*u_p} = (\pi \circ R_h)_{*u_p} = \pi_{u_p*}$. Hence:

$$(R_h^*\theta)_{u_p}(X) = h^{-1}\theta_{u_p}(X).$$

□

It follows from this that to specify an affine connection (Cartan connection with model $G = \mathbb{R}^n \rtimes GL_n(\mathbb{R}), H = GL_n(\mathbb{R})$) on M we only need to specify a principal connection γ on $L(TM)$.

The problem for us is that $L(TM)$ is not an $O(p,q)$-principal bundle. We need to specify a way to reduce the structure group $GL_n(\mathbb{R})$ to $O(p,q)$. Thinking in terms of frames the solution is clear: the relevant frames for a pseudo-Euclidean geometry are not linear frames but instead pseudo-orthonormal frames ! So what we need to do is to specify which frames are pseudo-orthonormal amongst our general frames. This is called a *reduction* of $L(TM)$.

A pseudo-Riemannian metric g on M does exactly this job. Indeed let us consider $\mathbb{R}^n$ equipped with its canonical signature (p,q) metric η and consider for each $p \in M$ the set $O(TM)_p$ of linear isometries between $(\mathbb{R}^n, \eta)$ and (T_pM, g_p). This is clearly seen to be equivalent to the choice of n vectors in T_pM that form an orthonormal basis of T_pM. Repeating essentially the construction of the frame bundle we did in Lecture 1[5], we get the bundle of orthonormal frames $O_{(p,q)}(TM) = \coprod_{p \in M} O_{(p,q)}(TM)_p$. It is clear that $O(TM)_p \subset L(TM)_p$ at each p and in fact we have a natural smooth map f that fits into the commutative diagram given in Figure 3. The map f is also such that for any $u \in O_{(p,q)}(TM)$, $f(uh) = f(u)i(h)$ where i is the inclusion map $O(p,q) \hookrightarrow GL_n(\mathbb{R})$.

One can also guess that given a $O(p,q)$ principal bundle P with a map $\tilde{f}$ that fit into a diagram obtained from Figure 3 by substituting P for $O_{(p,q)}(TM)$ and $\tilde{f}$ for

[5] There is a slight additional subtlety in that we need to show that from any coordinate basis, after perhaps restricting the chart to a smaller open subset, one can construct a pseudo-orthonormal frame

$$\begin{array}{ccc} O_{(p,q)}(TM) & \xrightarrow{f} & L(TM) \\ {\scriptstyle \pi_{O_{(p,q)}(TM)}}\big\downarrow & \swarrow {\scriptstyle \pi_{L(TM)}} & \\ M & & \end{array}$$

Figure 3: The pseudo-orthonormal frame bundle as a reduction of the frame bundle $L(TM)$.

f, one also gets a pseudo-Riemannian metric on M. Any point in the fibre of the bundle maps, through $\tilde{f}$, to a linear frame on T_pM. We hence get a basis $(E_1, \ldots, E_n)$ of T_pM. Now define g_p by $g_p(e_i, e_j) = \eta_{ij}$ where η_{ij} is the matrix of the canonical metric with signature (p, q) on $\mathbb{R}^n$.

The canonical solder form can be pulled back to $O_{(p,q)}(TM)$ and so one can construct a Cartan connection by specifying a principal connection on $O_{(p,q)}(TM)$. A pseudo-Riemannian metric also provides the solution to this thanks to its Levi Civita connection ∇ in the usual sense.

Proposition 9. *Let (M, g) be a pseudo-Riemannian manifold and ∇ its Levi-Civita connection. Consider $(U_i)_{i\in I}$ a covering of M by open sets and for each $i \in I$, a local section $\sigma_i : U_i \to O_{(p,q)}(TM)$. Let $i \in I$ be fixed and set $(E_k)_p = \sigma(p) \cdot e_k, k \in \{1, \ldots, n\}$ for each $p \in U_i$ where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$. These are smooth vector fields on U that at each point form a pseudo-orthonormal basis of T_pM. Let $(\omega^1, \ldots, \omega^n)$ denote the dual frame, i.e. $\omega^k(E_l) = \delta^k_l$.*

Define a matrix valued one-form $\Gamma(U_i)$ by:

$$\Gamma(U_i)^k_l = \omega^k(\nabla E_l).$$

Then there is a unique principal connection γ on $O_{(p,q)}(TM)$ such that for every i

$$\sigma_i^*\gamma = \Gamma(U_i).$$

Corollary III.1. *A pseudo-Riemannian metric induces a Cartan connection $\omega = \theta + \gamma$ where θ is the restriction to $O_{(p,q)}(TM)$ of the canonical solder form θ.*

Proof of Proposition.

1. Let us first verify that for every $i \in I$, $\Gamma(U_i) \in \mathfrak{o}(p, q)$. Consider: $g(E_k, E_l) = \delta^k_l$, then it follows from the definition of the Levi-Civita connection that:

$$\begin{aligned} 0 = g(\nabla E_k, E_l) + g(E_j, \nabla E_l) &= g(\Gamma(U_i)^m_k E_m, E_l) + g(E_k, \Gamma(U_i)^m_l E_m), \\ &= \Gamma(U_i)^m_k \eta_{ml} + \Gamma(U_i)^m_l \eta_{jm}. \end{aligned}.$$

2. Now suppose that γ is a $\mathfrak{o}(p, q)$ valued one form that solves the problem. Let $i, j \in I$ such that $V = U_i \cap U_j \neq \emptyset$. Note that, there is a smooth function

$h : V \to O(p,q)$ such that:

$$\sigma_j = \sigma_i h.$$

At each point $p \in V$, $h(p)$ is the change of basis matrix between the basis determined by σ_i and the one determined by σ_j. Let us determine a relation between $\sigma_j^*\gamma$ and $\sigma_i^*\gamma$. Let $X \in T_pM$:

$$(\sigma_j^*\gamma)_p = \gamma_{\sigma(p)}(\sigma_{j*p}X).$$

Now by the product rule:

$$\sigma_{j*p} = (R_h\sigma_i)_{*p} = R_{h(p)*\sigma_i(p)}\sigma_{i*p} + \lambda_{\sigma_i(p)*h(p)}h_{*p}.$$

Using the transformation rule of the principal connection, $R_h^*\gamma = Ad(h^{-1})\gamma$, we get

$$(\sigma_j^*\gamma)_p = Ad(h(p)^{-1})(\sigma_i^*\gamma)_p(X) + \gamma_{\sigma(p)}(\lambda_{\sigma_i(p)*h(p)}h_{*p}X).$$

Observe now that $\lambda_{\sigma_i(p)*h(p)}h_{*p}X$ is a vertical vector. In fact it is exactly equal to $[(\omega_H)_{*h(p)}(h_{*p}X)]^*$ where ω_H is the Maurer Cartan form of H. Overall we find that:

$$\sigma_j^*\gamma = Ad(h(p)^{-1})\sigma_i^*\gamma + h^*\omega_H.$$

Let us check that $\Gamma(U_i)$ and $\Gamma(U_j)$ are related in this way. Let $B = (E_1, \ldots, E_n)$ be the frame determined by σ_i and $\tilde{B} = (\tilde{E}_1, \ldots, \tilde{E}_n)$ by σ_j at each point p, $h(p)$ is the change of basis matrix from B to $\tilde{B}$. So, omitting dependance on p,

$$\tilde{E}_k = E_l h_k^l.$$

Introducing the dual frame $(\omega^1, \ldots, \omega^n)$, $(\tilde{\omega}^1, \ldots, \tilde{\omega}^n)$ we also have:

$$\tilde{\omega}^k = (h^{-1})_l^k \omega^l.$$

Now:

$$\begin{aligned} \Gamma(U_j)_l^k = \tilde{\omega}^k(\nabla E_l) &= (h^{-1})_m^k \omega^m(\nabla(E_s h_l^s)), \\ &= (h^{-1}\Gamma(U_i)h)_l^k + \delta_s^m (h^{-1})_m^k \nabla h_l^s. \end{aligned}$$

Hence:

$$\Gamma(U_j) = h^{-1}\Gamma(U_i)h + h^{-1}\nabla h = Ad(h^{-1})\Gamma(U_i) + h^*\omega_H.$$

3. Observe that every section σ_i determines a splitting

$$T_{\sigma_i(x)}O_{(p,q)}(TM) = T_xM \oplus \mathfrak{h},$$

for every $x \in U_i$. Where $\mathfrak{h} \simeq \ker \pi_{*\sigma_i(p)}$ using fundamental vectors and $T_x M \simeq \sigma_{i*x}(TM)$. So we can define a form at $\gamma_{\sigma(x)}$ by:

$$\gamma_{\sigma_i(x)}(\sigma_{i*x}(X) + A^*) = \Gamma(U_i)_x(X) + A.$$

This can be extended to the fibre of x by imposing the required transformation rule: $R_h^*\gamma = Ad(h^{-1})\omega$.

Let $j \neq i$ and write $\sigma_j = \sigma_i h$ on $U_i \cap U_j$. Now if $x \in U_i \cap U_j$ we have defined $\gamma_{\sigma_j(x)}$ in two ways, by the formula:

$$\gamma_{\sigma_j(x)}(\sigma_{j_{*x}}(X) + A^*) = \Gamma(U_i)(X) + A,$$

but it also determined by $(R_{h(x)}^*\gamma)_{\sigma_i(x)} = Ad(h^{-1})\gamma_{\sigma_i(x)}$, we need to ensure that these definitions are coherent. Now recall that:

$$\begin{aligned} \sigma_{j_{*x}} = (R_h\sigma_i)_{*x} &= R_{h(x)_{*\sigma_i(x)}}\sigma_{i*x} + \lambda_{\sigma_i(x)_{*h(x)}}h_{*x} \\ &= R_{h(x)_{*\sigma_i(x)}}\sigma_{i*x} + ((h^*\omega_H)_x)^* \end{aligned}$$

Hence let us calculate using the second definition:

$$\begin{aligned} \gamma_{\sigma_j(x)}(\sigma_{j_{*x}}(X) + A^*) &= \gamma_{\sigma_j(x)}(R_{h(x)_{*\sigma_i(x)}}\sigma_{i*x}(X) + ((h^*\omega_H)_x(X))^* + A^*) \\ &= \underbrace{\gamma_{\sigma_j(x)}(R_{h(x)_{*\sigma_i(x)}}\sigma_{i*x}(X))}_{(R_{h(x)}^*\gamma)_{\sigma_i(x)}\sigma_{i*x}(X) = Ad(h^{-1})\gamma_{\sigma_i(x)}(\sigma_i^*(X))} \quad +(h^*\omega_H)_x(X) + A \\ &= Ad(h^{-1})\Gamma(U_i)_x(X) + (h^*\omega_H)_x(X) + A \\ &= \Gamma(U_j)_x(X) + A. \end{aligned}$$

This shows that the separate forms glue together to form a form γ that by construction satisfies the conditions of a principal connection.

□

We have shown that we have that a pseudo-Riemannian metric on M yields a Cartan geometry (P, ω) modeled on G/H with $G = \mathbb{R}^n \rtimes O(p, q)$ and $H = O(p, q)$, which we identify with pseudo-Euclidean space. One may wonder in what sense the converse may be considered true? This question is studied (and solved) in the Riemannian case for example in [Sha97, Chapter 6], the answer is, up to a constant factor, yes. Similar problems are studied in projective, conformal, c-projective [DN20] geometry and for light-like manifolds [Pal21].

In my opinion it is equally interesting to pose this question broadly. This, I believe, is also a relevant question for physicists who identify a number of interesting structures and symmetries in their theories, generally described on the base manifold by distinguished vector fields and tensors. Can some of these structures be understood as Cartan geometries? Can the general machinery built in [AJ09] be applied to these situations to provide a better understanding of the symmetries and structures, new invariants, or generalisations ?

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