A short course on differential geometry

Besançon - Autumn 2022

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December 9, 2022

Avant-propos

Ces notes ont été rédigées dans le cadre d'un cours doctoral de 10h donné à l'École Doctorale Carnot-Pasteur. Elles s'adressent avant tout aux doctorants désireux d'avoir une introduction rapide aux variétés différentielles mais également aux étudiants de master qui n'ont jamais fait de géométrie et qui n'ont que des notions de topologie de niveau licence.

J'aimerais remercier L. Jeanjean pour sa lecture détaillée de ces notes ainsi que ses commentaires qui m'ont permis de les améliorer.

I. Manifolds and submanifolds

I.A. *The* Definition

Definition I.1

A topological manifold M modeled on a Banach space $(\mathbf{E}, \|\cdot\|)$, or $\mathbf{E}$-manifold, is a Hausdorff topological space such that every point $p \in M$ admits an open neighbourhood that is homeomorphic to an open subset of $\mathbf{E}$.

- If $\mathbf{E} = \mathbb{R}^n$, we say that M is of dimension n.

- In the first version of these notes I had assumed "paracompactness" in order to treat in a unified way infinite dimensional and finite-dimensional manifolds. However, upon further reflexion it appealed to me that this was not the ideal way to develop the basic theory as whilst it is a desired feature of the topology of manifolds, paracompactness does not behave well with respect to the usual set-theoretic operations. For instance, subsets of paracompact spaces, or products of paracompact spaces may not be paracompact. Instead, one should work with sufficient conditions for paracompactness that are better behaved. For *finite-dimensional* manifolds, it is custom to assume that the manifold is *second-countable*; and we will assume this. This means that we suppose that the topology has a countable basis. Since finite dimensional manifolds are automatically locally compact and [locally compact + second-countable] $\Rightarrow$ [paracompact]. Subspaces of second-countable spaces are second-countable; metric spaces are second-countable if and only if they are separable. In the infinite dimensional case, a sufficient condition would be to suppose that the topology is regular and second-countable.

- A *local chart* is a couple (U, ϕ) where U is an open subset of M and $\phi : U \to \phi(U)$ a homeomorphism onto an open subset $\phi(U) \subset \mathbf{E}$.

- An *atlas* is a collection of charts that cover the manifold.

Example I.1 (The stupid one). Open subsets of Banach spaces.

Example I.2 (Projective space). Let $\mathbb{R}^*$ act on $\mathbb{R}^{n+1} \setminus \{0\}$ by left multiplication and consider the quotient space $\mathbb{R}\mathrm{P}^n = \mathbb{R}^{n+1}/\mathbb{R}^*$, equipped with the *quotient topology*. This is intuitively the space of all lines in $\mathbb{R}^{n+1}$; we shall show that it is a n-dimensional topological manifold.

Recall that the quotient topology is the finest topology such that the canonical projection: $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}\mathrm{P}^n$ is continuous; it is described as follows:

$$\boxed{U \text{ is open in } \quad \mathbb{R}\mathrm{P}^n \Leftrightarrow \pi^{-1}(U) \text{ is open in } \mathbb{R}^{n+1} \setminus \{0\}.}$$

It satisfies the following universal property: let $f : \mathbb{R}^{n+1} \to X$ be a continuous map, such that $\pi(x) = \pi(y) \Rightarrow f(x) = f(y)$, then there is a unique continuous map $\tilde{f}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{R}^{n+1} \setminus \{0\} \arrow[r, "\pi"] \arrow[d, "f"] & \mathbb{R}\mathrm{P}^n \arrow[dl, dashed, "\tilde{f}"] \\
X &
\end{tikzcd}$$

The first step is to show that the quotient topology is Hausdorff and second-countable; neither are guaranteed for general quotients of topological spaces having these properties. However, our quotient comes from a continuous action of a topological group; the following are general arguments we record here for future reference:

Proposition I.1

Let $(G, \cdot)$ be a topological group, X a topological space and $\phi : G \times X \to X; (g, x) \mapsto g \cdot x$ a continuous group action. Let X/G be the quotient set equipped with the quotient topology, then the canonical projection $\pi : X \longrightarrow G/X$ is an open map.

Proof. Let U be open in X. In order to show that $\pi(U)$ is open in X/G one must show that $\pi^{-1}(\pi(U))$ is open in X. One has:

$$\pi^{-1}(\pi(U)) = \{x \in X, \exists g \in G, g \cdot x \in U\} = \bigcup_{g \in G} gU.$$

For any $g \in G$, $L_g : x \mapsto g \cdot x$ is a homeomorphism, and in particular, an open map. Therefore, for any open set U in X, and any $g \in G$, $gU \equiv L_g(U)$ is open. Hence, $\pi(U)$ is open. $\square$

One can now appeal to:

Corollary I.1. *If X is second-countable, X/G is second-countable.*

Proof. Let $(V_n)_{n \in \mathbb{N}}$ be a countable basis for the topology of X, the claim is that $(\pi(V_n))_{n \in \mathbb{N}}$ is a basis for the topology of X/G.
Let $[x] = \pi(x) \in X/G$ and U an open neighbourhood of $[x]$. By definition, $\pi^{-1}(U)$ is an open neighbourhood of $x \in X$, therefore one can find $n_0 \in \mathbb{N}$ such that $x \in V_{n_0} \subset \pi^{-1}(U)$, it follows that: $[x] = \pi(x) \in \pi(V_{n_0}) \subset U$. $\square$

As an immediate consequence:

$\mathbb{R}\mathrm{P}^n$ is second-countable.

We will now give a general argument that guarantees separation; it is slightly more involved and requires an additional condition on the group action.

Definition I.2

A continuous action of a topological group G on a locally compact topological space X is said to be ***proper***, if for any compact subset $K \subset X$ the set:

$$\{g \in G, gK \cap K \neq \emptyset\}$$

is compact.

The following proposition justifies this definition.

Proposition I.2

Let $\phi : G \times X \to X$ be a proper group action of a topological group on a locally compact[a] topological space X, then the quotient space X/G is Hausdorff.

[a] Some authors do not assume that locally compact spaces are Hausdorff, for us this hypothesis will always be implicit.

Proof. We shall divide the proof into two steps; first make the following observation, since the canonical projection $\pi : X \mapsto X/G$ is an open map, X/G is Hausdorff it is sufficient to show that $\mathcal{R} = \{(x, g \cdot x), x \in X, g \in G\}$ is closed in $X \times X$.

To see this, assume that $\mathcal{R}$ is closed and suppose that $\pi(x) \neq \pi(y)$ for some $x, y \in X$; then $(x, y) \in (X \times X) \setminus \mathcal{R}$. This set is open, therefore one can find neighbourhoods V_x and V_y of x and y respectively such that $V_x \times V_y \subset (X \times X) \setminus \mathcal{R}$, but then: $\pi(V_x)$ and $\pi(V_y)$ are open neighbourhoods of $\pi(x)$ and $\pi(y)$ such that $\pi(V_x) \cap \pi(V_y) = \emptyset$. Indeed: if $\pi(z) \in \pi(V_x) \cap \pi(V_y) \neq 0$, then $\exists g_1, g_2 \in G$ such that $(g_1 z, g_2 z) \in V_x \times V_y$ but $(g_1 z, g_2 z) = (g_1 z, g_2 g_1^{-1}(g_1 z)) \in \mathcal{R}$ and $\mathcal{R} \cap (V_x \times V_y) = \emptyset$.

We shall now prove that $\mathcal{R}$ is closed; let $(x, y) \in (X \times X) \setminus \mathcal{R}$. Using that X is locally compact, choose relatively compact neighbourhoods V_x and V_y of x and y respectively; then $V_x \times V_y$ is a relatively compact neighbourhood of (x, y). Introduce the map:

$$\tilde{\phi} : \begin{array}{ccc} G \times X & \longrightarrow & X \times X, \\ (g, x) & \longmapsto & (x, g \cdot x). \end{array}$$

Note that $\tilde{\phi}(G \times X) = \mathcal{R}$. Moreover, the image under $\tilde{\phi}$ of $\tilde{\phi}^{-1}(\overline{V_x \times V_y})$ is the set of points in $\overline{V_x \times V_y}$ that belong to $\mathcal{R}$ and that we want to avoid. If we can show that $\tilde{\phi}^{-1}(\overline{V_x \times V_y})$ is "small", i.e. compact, then the image will be compact too and easy to avoid.

Set $K_1 = \overline{V_x \times V_y}$ and $K_2 = \pi_1(K_1) \cup \pi_2(K_1)$ where π_1, π_2 are respectively projections onto the first and second components of $X \times X$. K_2 is a compact subset of X and so, since the action is proper, $G_0 = \{g \in G, gK_2 \cap K_2 \neq \emptyset\}$ is compact in G.

Now:

$$\tilde{\phi}^{-1}(K_1) \subset \tilde{\phi}^{-1}(K_2 \times K_2) = \{(g,x) \in G \times X, x \in K_2, g \cdot x \in K_2 \quad \} \subset G_0 \times K_2.$$

So, $\tilde{\phi}^{-1}(K_1)$ is a closed subset of the compact $G_0 \times K_2$, hence compact. $\tilde{\phi}$ being continuous it follows that $\tilde{\phi}(\tilde{\phi}^{-1}(K_1))$ is a compact subset of $\overline{V_x \times V_y}$. Now $V_x \times V_y$ and $(X \times X) \setminus \tilde{\phi}(\tilde{\phi}^{-1}(K_1))$ are open neighbourhoods of (x,y) so $(V_x \times V_y) \cap (X \times X) \setminus \tilde{\phi}(\tilde{\phi}^{-1}(K_1))$ is an open neighbourhood of (x,y) contained in $(X \times X) \setminus \mathcal{R}$. This proves that $\mathcal{R}$ is closed and consequently that X/G is Hausdorff. $\square$

The following sequential characterisation of properness is often useful.

Proposition I.3

Let X be a topological space, G a topological group, assume that both are second-countable and X locally compact. Suppose that G acts continuously on X, then the action is proper if and only if whenever $(g_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ are sequences of G and X respectively such that $(g_n \cdot x_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ converge then $(g_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

Proof. Let us prove $(\Leftarrow)$, let K be compact, consider $G_0 = \{g \in G, gK \cap K \neq \emptyset\}$. Since G is second-countable, we only need to show that G_0 is sequentially compact[1]. Choose a sequence $(g_n)_{n\in\mathbb{N}}$ in G_0, then for each $n \in \mathbb{N}$ one can find $x_n \in K$ such that $g_n \cdot x_n \in K$ since K is compact, extracting subsequences if necessary, we can assume that $(x_n)_{n\in\mathbb{N}}$ and $(g_n \cdot x_n)_{n\in\mathbb{N}}$ converge to $x \in K$ and $y \in K$ respectively. By assumption, it follows that $(g_n)_{n\in\mathbb{N}}$ has a convergent subsequence. Let g be the limit of this subsequence, by continuity one has: $g \cdot x = y \in K$ so that $g \in G_0$. It follows that G_0 is sequentially compact.

To prove $(\Rightarrow)$, let us assume that $G(K) = \{g \in G, gK \cap K \neq \emptyset\}$ is compact for every compact K. Suppose that $(g_n \cdot x_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ converge respectively to y and x for some sequences $(g_n)_{n\in\mathbb{N}}$,$(x_n)_{n\in\mathbb{N}}$. Since X is locally compact let V_x and V_y be relatively compact neighbourhoods of x and y. For large enough n, $g_n \cdot x_n \in V_y$ and $x_n \in V_x$, so that, for large enough $n \in \mathbb{N}$, $g_n \in G(K)$ where $K = \overline{V_1} \cup \overline{V_2}$. By assumption, $G(K)$ is compact and so $(g_n)_{n\in\mathbb{N}}$ has a convergent subsequence. $\square$

We can now apply Proposition I.2, to show that:

$\mathbb{R}\mathrm{P}^n$ is Hausdorff.

$\mathbb{R}^{n+1} \setminus \{0\}$ is certainly locally compact and second-countable and $\mathbb{R}^*$ second-countable, so we can use the sequential characterisation. Suppose that $(\lambda_n)_{n\in\mathbb{N}} \in$

[1]See Appendix A

$(\mathbb{R}^*)^{\mathbb{N}}, (x_n)_{n\in\mathbb{N}} \in (\mathbb{R}^{n+1}\setminus\{0\})^{\mathbb{N}}$ are sequences such that $\lambda_n x_n \to y \in \mathbb{R}^{n+1}\setminus\{0\}$ and $x_n \to x \in \mathbb{R}^{n+1}\setminus\{0\}$. Convergent sequences are bounded so there is $M > 0$ such that for all $n \in \mathbb{N}$, $|\lambda_n x_n| \leq M$. Futhermore, for large enough n, $||x_n|| > \frac{||x||}{2}$, therefore, for large enough n, $|\lambda_n| \quad \leq \frac{2M}{||x||}$. $(\lambda_n)_{n\in\mathbb{N}}$ is therefore a bounded sequence in $\mathbb{R}$ and has a convergent subsequence by the Bolzano-Weirestrass theorem. 0 is not a limit-point of the sequence since $|\lambda_n x_n| > \frac{||y||}{2}$ for large enough n; this shows that the subsequence converges in $\mathbb{R}^{n+1}\setminus\{0\}$.

At this point, we have worked hard[2] to show that the quotient topology on $\mathbb{R}\mathrm{P}^n$ satisfies the first two (global) conditions of a topological manifold. We will now check the final – and most important ! – assumption: that we have local charts. For every $i \in [\![1, n+1]\!]$ let:

$$U_i = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}\setminus\{0\}, x_i \neq 0\}$$

clearly $\bigcup_{i=1}^{n+1} U_i = \mathbb{R}^{n+1}\setminus\{0\}$. Now set:

$$\phi_i : \begin{array}{ccc} U_i & \longrightarrow & \mathbb{R}^n \\ (x_1,\ldots,x_{n+1}) & \longmapsto & (\frac{x_1}{x_i},\ldots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\ldots,\frac{x_{n+1}}{x_i}). \end{array}$$

The result only depends on the class of $\pi(x_1,\ldots,x_{n+1})$, so that the map factors to a continuous map $\tilde{\phi}_i : \pi(U_i) \to \mathbb{R}^n$; this is easily seen to be a homeomorphism as the inverse map can be given explicitly:

$$\tilde{\phi}_i^{-1} : \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \pi(U_i) \\ (y_1,\ldots,y_n) & \longmapsto & \pi((y_1,\ldots,y_{i-1},1,y_i,\ldots,y_n)). \end{array}$$

$\mathcal{A} = \{(\pi(U_i), \tilde{\phi}_i), i \in [\![1, n+1]\!]\}$ is then an atlas for $\mathbb{R}\mathrm{P}^n$. Hence, $\mathbb{R}\mathrm{P}^n$ is a topological manifold of dimension n.

I.B. C^k manifolds

Definition I.3

Let M be a topological $\mathbf{E}$-manifold and $\mathcal{A}$ an atlas on M, $k \in \mathbb{N}, k \in [1,\infty]$; $\mathcal{A}$ will be said to be of class C^k if for any choice of charts (U,ϕ), $(V,\psi) \in \mathcal{A}$ such that $U \cap V \neq \emptyset$ the transition map: $\phi \circ \psi^{-1}$ is of class C^k.

In general, a full atlas will not be known explicitly and given a new chart (ϕ, U) there is no way of knowing if it is in the atlas or not. We should therefore work with atlases that contain all "reasonable" charts in the following sense:

[2]More direct proofs certainly exists, but do not show the general principle.

Definition I.4

Let $\mathcal{A}$ be a C^k-atlas on a manifold M and (U, ϕ) an arbitrary chart. We shall say that (U, ϕ) is compatible with $\mathcal{A}$ if its transition map with any chart in $\mathcal{A}$ is of class C^k.

Definition I.5

A C^k-atlas will be said to be maximal if it contains all compatible charts.

This definition is useful since:

Proposition I.4

Every C^k atlas $\mathcal{A}$ is contained a unique maximal C^k-atlas; that we will write $\hat{\mathcal{A}}$.

Proof. Define an equivalence relation on atlases: $\mathcal{A} \sim \mathcal{B}$ if and only if every chart in $\mathcal{B}$ is compatible with $\mathcal{A}$, this means that $\mathcal{A} \cup \mathcal{B}$ is an atlas. Taking the union of all atlases in an equivalence class of atlases yields the desired maximal atlas. $\square$

We can now define:

Definition I.6

A C^k manifold is a manifold equipped with a C^k-atlas. A maximal C^k-atlas is also said to be a C^k-differentiable structure on M. A C^∞-manifold is said to be smooth.

It should be understood from the discussion in this section that to define a differentiable structure on a manifold M, we only need to specify an atlas, and then consider the unique maximal atlas it is contained in. For instance:

Example I.3. $\mathcal{A} = \{(E, id_E)\}$, E a Banach space, is a smooth atlas on E and the (canonical) smooth structure on E is that of the maximal atlas $\hat{\mathcal{A}}$ containing $\mathcal{A}$.

Example I.4. On $\mathbb{R}\mathrm{P}^n$, consider the atlas $\mathcal{A} = \{(U_i, \tilde{\phi}_i), i \in [\![1, n+1]\!]\}$. The transition map: $\tilde{\phi}_j \circ \tilde{\phi}_i^{-1}$ is given on $\tilde{\phi}_i(U_i \cap U_j)$ by:

$$\tilde{\phi}_j \circ \tilde{\phi}_i^{-1}(y_1, \ldots, y_n) = \left(\frac{y_1}{y_j}, \ldots, \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \frac{y_{i+1}}{y_j}, \ldots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j}, \ldots, \frac{y_n}{y_j}\right).$$

This is smooth, so $\mathcal{A}$ is a smooth atlas for $\mathbb{R}\mathrm{P}^n$ and the maximal atlas $\hat{\mathcal{A}}$ makes $\mathbb{R}\mathrm{P}^n$ a smooth manifold.

I.C. Submanifolds

Already several pages of theory and only one non-trivial example... to remedy this let us define *submanifolds* and prove a classical theorem that produces many examples. The notion of submanifold is trickier than that of a subspace in a topology and there are, in fact, several non-equivalent definitions (that allow for different phenomena); we refer to [Sha97, Chapter 1] for an interesting discussion on this topic. Our definition corresponds to "regular" submanifolds.

Definition I.7

Let M be a smooth manifold, a subset $N \subset M$ is a smooth submanifold of M if for every $q \in N$ one can find a chart (U, ψ) near q such that:

$$\psi(U) = V_1 \times V_2, \qquad \psi(U \cap N) = V_1 \times \{0\},$$

V_1, V_2 open subsets of Banach spaces $\mathbf{E}_1, \mathbf{E}_2$ (that can be thought of as closed subspaces of E).

A smooth submanifold N has a natural atlas given by the submanifold charts. Let $q \in N$ and (U, ψ) where $\psi : U \to V_1 \times V_2$ is a homeomorphism, then restricting to $U \cap N$ and projecting onto the first factor we get a homeomorphism: $\tilde{\psi} : U \cap N \to V_1$. The transition functions are easily seen to be smooth so these charts yield a smooth structure on N that is compatible with the subspace topology.

In this definition, V_1 is an open subset of a Banach space $\mathbf{E}_1$ and one may wonder if $\mathbf{E}_1$ could change from point to point. (This was excluded in Definition I.1, but could in principle be allowed). In fact, since the transition maps between two charts are smooth[3] diffeomorphisms, we can see that all the possible $\mathbf{E}_1$ are topolinear isomorphic (i.e. there is a continuous vector space isomorphism between them) at points in the same connected component. So we get a manifold structure according to our definition on each connected component of the submanifold.

Remark I.1. In the finite dimensional case, this is true on topological manifolds due to the *Brouwer invariance of domain theorem*, which implies the topological invariance of dimension; this second statement is harder to prove than one might think naively ! There are nice proofs of this in homology theory and one can find a clear discussion in [Nab10]; although it is discussed in most standard books in algebraic topology.

Example I.5. Open subsets of manifolds are submanifolds.

We prove now a (sub-optimal) version of:

[3] C^1 is enough.

Theorem I.1: Implicit function theorem

Let $\mathbf{E}$ be a Banach space, $f : U \subset \mathbf{E} \longrightarrow \mathbb{R}^m$, a smooth function defined on an open subset U of $\mathbf{E}$ such that for every $x \in f^{-1}(\{0\})$, $f'(x)$ is surjective; then $f^{-1}(\{0\})$ is a submanifold of $\mathbf{E}$.

Proof. Let $x_0 \in f^{-1}(\{0\})$, and set $\mathbf{F} = \ker f'(x_0)$. $\mathbf{E}/F \cong \mathbb{R}^m$ so has a basis $(e_1, \ldots, e_m)$. For each $i \in [\![1, m]\!]$, choose $\tilde{e}_i$ in $\mathbf{E}$ such that $\pi(\tilde{e}_i) = e_i$ where $\pi : \mathbf{E} \to \mathbf{E}/\mathbf{F}$ is the canonical projection. Set $\mathbf{G} = \text{span}(e_1, \ldots, e_m)$ and note that we have the topological direct sum decomposition $\mathbf{E} = \mathbf{F} \oplus \mathbf{G}$; in particular the projections are continuous.

Indeed, $\mathbf{F}$ and $\mathbf{G}$ are closed subspaces of $\mathbf{E}$ which automatically implies that the projections $p_\mathbf{F}$ and $p_\mathbf{G}$ are continuous. This follows from the closed graph theorem: suppose that $(x_n)_{n\in\mathbb{N}}$ converges to x and $(p_\mathbf{F}(x_n))_{n\in\mathbb{N}}$ converges to y, since $\mathbf{F}$ is closed it follows that $y \in \mathbf{F}$, similarly, since $\mathbf{G}$ is closed and $(x_n - p_\mathbf{F}(x_n))$ is a sequence in $\mathbf{G}$ that converges to $x - y$: $x - y \in \mathbf{G}$. But, $x = x - y + y$ and this decomposition is unique, thus $y = p_\mathbf{F}(x)$ and the graph is closed. Therefore, $p_\mathbf{F}$ and $p_\mathbf{G} = Id_\mathbf{E} - p_\mathbf{F}$ are continuous.

Define now $\tilde{f} : U \to \mathbf{F} \times \mathbf{G}$, by: $\tilde{f}(x) = (p_\mathbf{F}(x), f(x))$. We claim that $\tilde{f}'(x_0) \in GL(\mathbf{E}, \mathbf{F} \times \mathbf{G})$. In virtue of the open mapping theorem, it is sufficient to show that $\tilde{f}'(x_0)$ is bijective.

- If $\tilde{f}'(x_0)h = 0$, then $p_\mathbf{F}(h) = 0$ and $f'(x_0)h = 0$. So, $h \in \mathbf{F}$ and $h = p_\mathbf{F}(h) = 0$; this implies that $\tilde{f}'(x_0)$ is injective.
- Let $(f, g) \in \mathbf{F}\times\mathbf{G}$, since $f'(x_0)$ is surjective one can find h_0 such that $f'(x_0)h_0 = g$. Set $h = h_0 - \pi_\mathbf{F}(h_0) + f$, then $\tilde{f}'(x_0)h = (f, g)$. This proves surjectivity.

We can now apply the inverse function theorem to $\tilde{f}$ which provides neighbourhoods V and $V_1 \times V_2 \subset \mathbf{F} \times \mathbf{G}$ of x_0 and $(\pi_\mathbf{F}(x_0), 0)$ respectively, such that $\tilde{f}|_V : V \to V_1 \times V_2$ is a diffeomorphism. This is the desired submanifold chart near x_0. □

Remark I.2. The theorem extends to maps with values in Banach spaces, but we need to strengthen the hypothesis on the derivative of points in $x \in f^{-1}(\{0\})$. The correct hypothesis is that: $f'(x)$ is surjective and the short exact sequence

$$0 \longrightarrow \ker f'(x) \xrightarrow{i} \mathbf{E} \xrightarrow{f'(x)} \mathbf{F} \longrightarrow 0$$

splits; this means that $\mathbf{E} \cong \ker f'(x) \oplus \mathbf{F}$. As we have seen, this is automatic if $\mathbf{F}$ is finite dimensional. A differentiable map that satisfies these conditions is called a *submersion*.

We now have some more examples of manifolds that arise as submanifolds:

Example I.6. $S^n = \{x \in \mathbb{R}^{n+1}, ||x||_2^2 = 1\}$.

Example I.7. $SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}), \det A = 1\}$.

Example I.8. $O_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}), A^t A = I_n\}$.

Let us conclude this section by quoting without proof the following theorem:

Theorem I.2: The strong Whitney embedding theorem

Every n-dimensional manifold can be realised as a submanifold of $\mathbb{R}^{2n}$.

Remark I.3. $2n$ is optimal, but on specific examples one can often do better.

I.D. Appendix: The stereographic projection on the n-sphere

It feels that no course on differential geometry would be complete without mentioning the stereographic projection as a way to define an atlas on the n-sphere. Let us consider $S^n = \{x \in \mathbb{R}^{n+1}, ||x||_2^2 = \sum_{i=1}^{n+1} x_i^2 = 1\}$ equipped with the subspace topology. Set $N = (0, \ldots, 0, 1)$, and $U_N = S^n \setminus \{N\}$.

Let H be the hyperplane of equation $x_{n+1} = 0$ and define a map $p : U_N \to H$ such that $p(x)$ is the intersection of the line passing through x and N. A short computation gives:

$$p(x) = \frac{1}{1 - x_{n+1}} (x_1, \ldots, x_n, 0), \qquad x = (x_1, \ldots, x_{n+1}) \in U_N.$$

Identifying H with $\mathbb{R}^n$ in a natural way p gives a chart: $\phi_N : U_N \to \mathbb{R}^n$, $\phi_N(x) = \frac{1}{1-x_{n+1}}(x_1, \ldots, x_n)$; continuous on U_N. It is in fact a homeomorphism, as:

$$\phi_N^{-1}(y_1, \ldots, y_n) = \left(\frac{2y_1}{1 + ||y||_2^2}, \ldots, \frac{2y_n}{1 + ||y||_2^2}, \frac{||y||_2^2 - 1}{1 + ||y||_2^2} \right).$$

Setting $S = (0, \ldots, 0, -1)$ and $U_S = S^n \setminus \{S\}$ one can define a chart ϕ_S in a similar fashion. Then:

$$\mathcal{A} = \{(U_N, \phi_N), (U_S, \phi_S)\},$$

is an atlas on S^n. Let us prove that it is smooth by determining: $\phi_S \circ \phi_N^{-1}$ on $\mathbb{R}^n \setminus \{0\}$, one finds:

$$(\phi_S \circ \phi_N)^{-1}(y_1, \ldots, y_n) = \frac{1}{||y||_2^2} (y_1, \ldots, y_n).$$

This is smooth on $\mathbb{R}^n \setminus \{0\}$ so $\hat{\mathcal{A}}$ is a smooth structure on S^n.

II. The tangent bundle

For simplicity, from now on, all manifolds will be smooth.

II.A. Smooth functions and tangent vectors

Definition II.1

Let M and N be smooth manifolds and $f : M \to N$, we will say that f is smooth if for any $p \in M$ one can find charts (U, ϕ), (V, ψ), $p \in U$, $f(U) \subset V$ such that: $\psi \circ f \circ \phi^{-1}$ is C^∞ in the usual sense.

This definition does not depend on the charts (U, ϕ),(V, ψ). Indeed, suppose, for instance, that: $(\tilde{U}, \tilde{\phi})$ is another chart such that $p \in U$, then (on an appropriate intersection of the domains):

$$\psi \circ f \circ \tilde{\phi}^{-1} = (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \tilde{\phi}^{-1}),$$

transition functions are smooth by definition of the smooth atlas, so we can see that $\psi \circ f \circ \tilde{\phi}^{-1}$ is C^∞.

Example II.1. Let $N \subset M$ be a submanifold of M, then the inclusion map $i : N \to M$ is smooth. To see this let $q \in N$ and (V, ψ), $\psi : V \to V_1 \times V_2$ a submanifold chart; by definition of its atlas there is a corresponding chart $(V_1, \tilde{\psi})$, now:

$$\psi \circ i \circ \tilde{\psi}^{-1} : V_1 \to V_1 \times V_2,$$

is given by:

$$(\psi \circ i \circ \tilde{\psi}^{-1})(v) = (v, 0),$$

which is C^∞.

Example II.2. The composition of smooth maps is smooth, in particular if $f : M \to N$ is smooth and X is a submanifold of M, then the restriction of f to X is smooth.

Example II.3. Let M be a smooth $\mathbf{E}$-manifold and (U, ψ) a chart, then $\psi : U \to \mathbf{E}$ is smooth; indeed: $id_{\mathbf{E}} \circ \psi \circ \psi^{-1}$ is smooth.

We would now like to extend the notion of differential between functions on Banach spaces to manifolds. One could suggest, as above, that we take local charts and differentiate; but this would not be a chart invariant definition. Indeed let $f : M \to N$ be a smooth map, choose, near $p \in M$, charts: $(U_1, \phi_1), (U_2, \phi_2)$ such that $p \in U_1 \cap U_2$ and charts $(V_1, \psi_1), (V_2, \psi_2)$ such that $f(p) \in V_1 \cap V_2$. Let us study how the local representations of the function change:

$$(\psi_2 \circ f \circ \phi_2^{-1})'(x) = \underbrace{(\psi_2 \circ \psi_1^{-1})'(\psi_1 \circ f \circ \phi_2^{-1}(x))}_{Q^{-1}} \circ (\psi_1 \circ f \circ \phi_1^{-1})'(\phi_1 \circ \phi_2^{-1}(x)) \circ \underbrace{(\phi_1 \circ \phi_2^{-1})'(x)}_{P}.$$

We can see from this formula that the local chart derivative transforms under a change of chart like a linear map when we change basis. Inspired by this analogy, we introduce the following equivalence relation. Let $p \in M$ be fixed and consider triples (U, ϕ, v), (U, ϕ) is a chart and $v \in \mathbf{E}$ is a vector in the model space. We define an equivalence relation on (U, ϕ, v) by:

$$(U, \phi, v) \sim (V, \psi, w) \Leftrightarrow w = (\psi \circ \phi^{-1})'(p) \cdot v$$

Definition II.2: Tangent vectors and tangent space at a point $p \in M$

Let $p \in M$, a tangent vector at p is an equivalence class for the relation $\sim$. We denote by T_pM the set of all tangent vectors and refer to it as the tangent space at p; it inherits from a chart a Banachisable topological vector space structure that does not depend on the choice of chart.

Remark II.1. Let us make the statement of Definition II.2 more explicit. Fix $p \in M$ and let $[(\phi, U, v)]_p$ denote the tangent vector of which (ϕ, U, v) is a representative. We define a vector space structure on T_pM by:

$$[(\phi, U, v)]_p + [(\phi, U, w)]_p = [(\phi, U, v + w)]_p,$$

where (ϕ, U) is an arbitrary local chart such that $p \in U$. The definition makes sense since if (ψ, W) is another chart with $p \in W$, then: $[(\phi, U, v)]_p = [\psi, V, (\psi \circ \phi^{-1})'(\phi(p)) \cdot v]_p$ for any $v \in \mathbf{E}$, linearity of the derivative guarantees that:

$$[(\phi, U, v + w)]_p = [(\psi, V, (\psi \circ \phi^{-1})'(\phi(p)) \cdot v + (\psi \circ \phi^{-1})'(\phi(p))(v) \cdot w))]_p.$$

Given a choice of chart (ϕ, U) we have a (non-canonical) bijection between T_pM and $\mathbf{E}$, given by the maps:

$$\phi_{*p} : [(\phi, U, v)]_p \in T_pM \mapsto v \in \mathbf{E}, \qquad (\phi_{*p})^{-1} : v \in \mathbf{E} \mapsto [(\phi, U, v)]_p \in T_pM.$$

This bijection depends on the choice of chart (ϕ, U).

To get a topology on T_pM, we pull back the topology of $\mathbf{E}$ through such a bijection, i.e. we endow T_pM with the topology that makes the above bijection a linear homeomorphism; this topology is uniquely defined and has the basis:

$$\mathscr{B} = \{(\phi_{*p})^{-1}(U),\ U \text{ open in } \mathbf{E} \quad \}.$$

It may seem that the topology could depend on a choice of chart, but this is not the case as by definition:

$$\psi_{*p} \circ (\phi_{*p})^{-1} = (\psi \circ \phi^{-1})'(\phi(p)),$$

which is a continuous linear isomorphism $\mathbf{E} \to \mathbf{E}$, hence the *topology* is independent of the choice of chart. Note that, whilst one can use the bijection ϕ_{*p} to define a norm on T_pM, this norm *depends* on the choice of chart, which is why we only get a Banachisable topological vector space structure.

Definition II.3: The tangent map at a point $p \in M$

Let $f : M \to N$ be a smooth map, and $p \in M$. Suppose that M is an $\mathbf{E}_1$ manifold and N and $\mathbf{E}_2$ manifold. The *tangent map* at p, written f_{*p} or T_pf is the unique continuous linear map defined by the following commutative diagram:

$$\begin{array}{ccc} T_pM & \xrightarrow{f_{*p}} & T_{f(p)}N \\ \big\downarrow{\scriptstyle \phi_{*p}} & & \big\downarrow{\scriptstyle \psi_{*f(p)}} \\ \mathbf{E}_1 & \xrightarrow{(\psi\circ f\circ\phi^{-1})'(\phi(p))} & \mathbf{E}_2 \end{array}$$

where (U, ϕ) is a chart such that $p \in U$ and (V, ψ) is a chart such that $f(p) \in V$. In the diagram we denote (abusively) by $\phi_{*p} : T_pM \to E_1$ the canonical continuous linear isomorphism $[(U, \phi, v)] \to v$ and similarly for $\psi_{*f(p)}$.

Remark II.2. When $\mathbf{E}$ is a Banach space, it is clear that for any $p \in \mathbf{E}$, there is a *canonical* identification $T_p\mathbf{E} = \mathbf{E}$ in the global chart given by the identity; we will systematically make this identification. This also justifies the notation ϕ_{*p} in Definition II.3, as one can easily verify that this is indeed the tangent map of the chart ϕ at p!

The tangent map behaves like the usual differential, in particular:

Proposition II.1: Chain rule

Let $f : M \to N$, $g : N \to S$ be smooth maps between smooth manifolds, then:

$$(g \circ f)_{*p} = g_{*f(p)} \circ f_{*p}.$$

Proof. Introduce local charts (U, ϕ), (V, ψ), (W, η) such that: $f(U) \subset V$, $g(V) \subset W$, and fix $p \in U$. By definition:

$$\eta_{*g(f(p))} \circ (g \circ f)_{*p} = (\eta \circ g \circ f \circ \phi^{-1})(\phi(p))' \circ \phi_{*p}.$$

Now:

$$\begin{aligned}(\eta \circ g \circ f \circ \phi^{-1})'(\phi(p)) &= (\eta \circ g \circ \psi^{-1} \circ \psi \circ f \circ \phi^{-1})'(\phi(p)),\\ &= (\eta \circ g \circ \psi^{-1})'(\psi(f(p))) \circ (\psi \circ f \circ \phi^{-1})'(\phi(p)),\\ &= (\eta \circ g \circ \psi^{-1} \circ \psi \circ f \circ \phi^{-1})'(\phi(p)),\\ &= (\eta \circ g \circ \psi^{-1})'(\psi(f(p))) \circ \psi_{*p} \circ (\psi_{*p})^{-1} \circ (\psi \circ f \circ \phi^{-1})'(\phi(p))\\ &= \eta_{*g(f(p))} g_{*f(p)} \circ f_{*p} \circ (\phi_{*p})^{-1}.\end{aligned}$$

It follows that:

$$\eta_{*g(f(p))} \circ (g \circ f)_{*p} = \eta_{*g(f(p))} g_{*f(p)} \circ f_{*p}.$$

The result follows. $\square$

Corollary II.1. *For any local chart* (ϕ, U)*:*

$$(\phi_{*p})^{-1} = \phi^{-1}_{*\phi(p)}.$$

II.B. Derivations of the algebra of germs of smooth functions

With their current definition, tangent vectors are not all that easy to manipulate. However, they have a very strong link with derivations on the algebra of germs of smooth functions on M that we shall explore in this section. First we define:

Definition II.4: Derivations of an algebra

Let A be a real algebra, M an A-bimodule then a derivation on A is an $\mathbb{R}$-linear map $D : A \to M$ such that:

$$\forall a, b \in A, D(ab) = (Da) \cdot b + a \cdot (Db).$$

Definition II.5: Germ of smooth function

Let $p \in M$, two smooth functions f, g on M are said to define *the same germ at* p, if there is an open neighbourhood U of p, such that $f|_U = g|_U$. This defines an equivalence class on smooth functions on M that we call the germ of f at p, written f_p. Let $C^\infty_p(M)$ denote the set of all germs at p, pointwise multiplication and addition gives $C^\infty_p(M)$ the structure of an $\mathbb{R}$-algebra.

Example II.4. $\mathbb{R}$ is a $C^\infty_p(M)$-bimodule by defining $f_p\lambda = \lambda f_p := f(p)\lambda$; $\lambda \in \mathbb{R}$, $f_p \in C^\infty_p(M)$. A derivation $D : C^\infty_p(M) \longrightarrow \mathbb{R}$ is an $\mathbb{R}$-linear map that satisfies the Leibniz rule:

$$D(f_p g_p) = (Df_p)g(p) + f(p)Dg_p.$$

In the above formula, $f(p)$ and $g(p)$ denote the value at p of any representative f (resp. g) of the germ f_p (resp. g_p); by definition it does not depend on this choice. Note that if f_p is the germ of a constant map, then $Df_p = 0$. Indeed, suppose f_p is the germ of a constant map equal to λ, then for any g_p, $f_p g_p = \lambda g_p$ so that: $\lambda D g_p = D(\lambda g_p) = D(f_p g_p) = D(f_p)g(p) + \lambda D g_p$. Thus $D(f_p)g(p) = 0$ for any function g, this implies $D(f_p) = 0$.

The important result is:

Proposition II.2

Let $p \in M$, $v_p \in T_pM$, $f : M \to \mathbb{R}$ a smooth function. Define $D_{v_p} f = f_{*p}(v_p)$, then $D_{v_p} f$ only depends on the germ of f at p and defines a derivation D_{v_p} at p. The correspondance $v_p \mapsto D_{v_p}$ is injective; if M is finite-dimensional it is a bijection (any derivation is of the form D_{v_p} for some $v_p \in T_pM$).

Proof. Choose a chart (U, ϕ) near p and note that $D_{v_p} f = (f \circ \phi^{-1})'(\phi(p))(\phi_{*p} v_p)$ This formula shows that $D_{v_p} f$ only depends on the germ of f at p so factors to a map on $C_p^\infty(M)$. Using the product rule,

$$\begin{aligned}
D_{v_p}(fg) &= ((fg) \circ \phi^{-1})'(\phi(p))(\phi_{*p} v_p) \\
&= (f \circ \phi^{-1} \cdot g \circ \phi^{-1})'(\phi(p))(\phi_{*p} v_p) \\
&= \left((f \circ \phi^{-1})'(\phi(p)) \cdot \phi_{*p} v_p\right) (g \circ \phi^{-1})(\phi(p)) + ((g \circ \phi^{-1})'(\phi(p)) \cdot \phi_{*p} v_p)(f \circ \phi^{-1})(\phi(p)) \\
&= (D_{v_p} f) g(p) + f(p) D_{v_p} g,
\end{aligned}$$

so that this map is a derivation. For the injectivity of the correspondence, suppose that for some $v_p \in T_pM$, $D_{v_p} f = 0$ for any f. Let $l \in \mathbf{E}'$ and consider $f = l \circ \phi$. Then for any $l \in \mathbf{E}'$, $l(\phi_{*p} v_p) = 0$ so $\phi_{*p} v_p = 0 \Rightarrow v_p = 0$.

Now assume that M is of dimension n. We shall show that all derivations are of the form D_{v_p} for some $v_p \in T_pM$. Let D be a derivation and (U, ϕ) be a local chart near p. Let $f_p \in C_p^\infty(M)$ and f a representative of the germ, using a *bump function*, it can be chosen such that $\operatorname{supp} f \subset U$. For q in the support of f one can write:

$$f(q) = f(p) + \int_0^1 (f \circ \phi^{-1})'(\phi(p) + t(\phi(q) - \phi(p)) \cdot (\phi(q) - \phi(p)) \mathrm{d}t.$$

Decomposing $\phi(q) - \phi(p)$ onto the canonical basis of $\mathbb{R}^n$, we obtain:

$$f(q) = f(p) + \sum_{i=1}^n e_i^*(\phi(q) - \phi(p)) \int_0^1 (f \circ \phi^{-1})'(\phi(p) + t(\phi(q) - \phi(p)) \cdot e_i \mathrm{d}t.$$

Applying D in this formula we obtain, due to the Leibniz rule and the fact that it annihilates germs of constants:

$$\begin{aligned} Df_p = \sum_{i=1}^{n} D((e_i^* \circ \phi)_p)(f \circ \phi^{-1})'(\phi(p))(e_i) &= \sum_{i=1}^{n} D((e_i^* \circ \phi)_p) f_{*p} \cdot \phi_{*p}(e_i) \\ &= f_{*p} v_p, \end{aligned}$$

where: $v_p = \phi_{*p} \left(\sum_{i=1}^{n} D((e_i^* \circ \phi)_p) e_i \right).$ □

This justifies:

Definition II.6: Tangent spaces on finite dimensional manifolds

Let M be a finite dimensional manifold, then the tangent space at p, T_pM, is defined to be the set of derivations of germs of smooth functions at p.

We can now revisit the definition of the tangent map, **thinking of tangent vectors as derivations**:

Definition II.7: Tangent map at p

Let M, N be a finite dimensional manifolds, $\phi : M \to N$ a smooth map, then we define the tangent map at p, denoted by ϕ_{*p} by:

$$\underbrace{\phi_{*p}(X)}_{\text{A derivation at } \phi(p)} (f_{\phi(p)}) = \underbrace{X}_{\text{A derivation at } p}((f \circ \phi)_p), \qquad X \in T_pM$$

where $f_{\phi(p)} \in C^{\infty}_{\phi(p)}(N)$ and $(f \circ \phi)_p$ denotes the germ at p of $f \circ \phi$, where f is any representative of the germ.

Remark II.3. If $\phi : M \to N$, and $f : N \to \mathbb{R}$ then the germ of $f \circ \phi$ at p only depends on the germ of f at $\phi(p)$. Indeed let f and g coincide on an open neighbourhood U of $\phi(p)$, then by continuity of f, there is an open neighbourhood V of p such that $f(V) \subset U$, then $g \circ \phi = f \circ \phi$ on V.

Lastly let us introduce a common notation:

Definition II.8

Let M be a finite dimensional manifold, $p \in M$, (U, x) a local chart we define:

$$\left(\frac{\partial}{\partial x_i}\right)_p = x^{-1}_{*x(p)} e_i.$$

Any tangent vector at p can be written[a]: $X = X^i \left(\frac{\partial}{\partial x_i}\right)_p$. $X_i \in \mathbb{R}$

[a] See the Einstein's summation convention

To conclude this section we shall check that Definition II.7 and Definition II.3, coincide on finite dimensional manifolds. Let $\phi : M \to N$ be a smooth map between two finite dimensional manifolds $\dim M = n$, $\dim N = m$, choose $p \in M$, (x, U) a chart at p and (y, V) a chart on N such that $f(U) \subset V$.

First, let us make explicit II.8. In $\mathbb{R}^n$ the vector e_i is interpreted as the partial derivative operator ∂_{e_i}, so, using Definition II.7, $x^{-1}_{*x(p)} e_i$ should be interpreted as the derivation defined by:

$$\left(\frac{\partial}{\partial x_i}\right)_p (f) = \partial_{e_i}(f \circ x^{-1})(x(p)).$$

f is the germ of some smooth function at p and, the right hand side is the ith partial derivative (in the usual sense) of the function $f \circ x^{-1}$, defined on the open set $x(U) \subset \mathbb{R}^n$. Now, let us calculate $\phi_{*p} X_p$, where $X_p = X^i \left(\frac{\partial}{\partial x_i}\right)_p$ again using Definition II.7,

$$\begin{aligned}
(\phi_{*p} X_p)(f) = X_p(f \circ \phi) &= \sum_{i=1}^{n} X^i \left(\frac{\partial}{\partial x_i}\right)_p (f \circ \phi) \\
&= \sum_{i=1}^{n} X^i \partial_{e_i}(f \circ \phi \circ x^{-1})(x(p)) \\
&= \sum_{i=1}^{n} X^i \left(\frac{\partial}{\partial x_i}\right)_p (f \circ \phi) \\
&= \sum_{i=1}^{n} X^i \partial_{e_i}(f \circ y^{-1} \circ (y \circ \phi \circ x^{-1}))(x(p)) \\
\text{using the chain rule } &= \sum_{i=1}^{n} \sum_{j=1}^{m} X^i \partial_{e_i}(y \circ \phi \circ x^{-1})_j(x(p)) \partial_{e_j}(f \circ y^{-1})(y(\phi(p))) \\
&= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} X^i \partial_{e_i}(y \circ f \circ x^{-1})_j(x(p)) \right) \left(\frac{\partial}{\partial y_j}\right)_{\phi(p)} (f).
\end{aligned}$$

Where we have introduced the notation $(y \circ f \circ x^{-1})_j$ to denote the j-th component in the canonical basis of $y \circ f \circ x^{-1}$. Now we compare with Definition II.3, recall that one must have:

$$(y \circ \phi \circ x^{-1})'(x(p)) \circ x_{*p} = y_{*\phi(p)} \circ \phi_{*p} \Leftrightarrow \phi_{*p} = y^{-1}_{*y(\phi(p))} \circ (y \circ \phi \circ x^{-1})'(x(p)) \circ x_{*p}$$

Decomposing onto the canonical basis of $\mathbb{R}^m$:

$$(y \circ \phi \circ x^{-1})'(x(p)) = \sum_{j=1}^{m} e_j (y \circ \phi \circ x^{-1})'_j(x(p))$$

Moreover:

$$X = \sum_{i=1}^{n} X^i \left(\frac{\partial}{\partial x_i}\right)_p = x^{-1}_{*x(p)} \left(\sum_{i=1} X^i e_i\right),$$

hence:

$$\begin{aligned}(y \circ \phi \circ x^{-1})'(x(p)) \circ x_{*p}(X) &= \sum_{j=1}^{m}\sum_{i=1}^{n} e_j X^i (y \circ \phi \circ x^{-1})'_j(x(p)) \cdot e_i, \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} e_j X^i \partial_{e_i}(y \circ \phi \circ x^{-1})_j(x(p)).\end{aligned}$$

Since $y^{-1}_{*y(\phi(p))} e_j = \left(\frac{\partial}{\partial y_j}\right)_{\phi(p)}$, it follows that the two definitions are identical.

One can also use Definition II.7 to give a much more elegant (and coordinate free) proof of the chain rule in finite dimensions.

Alternative proof of the chain rule in finite dimensions. Let $\phi : M \to N$, $\psi : N \to S$, let f be the germ of a smooth function at $(\psi \circ \phi)(p)$ for fixed $p \in M$. Let $X_p \in T_pM$, thought of as a derivation at p, then:

$$\begin{aligned}(\phi \circ \psi)_{*p}(X_p)(f) &= X_p((f \circ \phi) \circ \psi), \\ &= (\psi_{*p}(X_p))(f \circ \phi) \\ &= (\phi_{*\psi(p)} \circ \psi_{*p})(X_p)(f).\end{aligned}$$

□

II.C. The tangent bundle

We would now like generalise the notion of vector fields to manifolds and make sense of their smoothness. For this we are going to construct out of M a new manifold TM, known as the *tangent bundle*. It is an object obtained by gluing together all of the information contained in the tangent spaces at different points. It is the first example that we will encounter of a *fibre bundle*. Furthermore, here the fibres will also have a topological vector space structure, and hence it is an instance of a *vector bundle*. We give the general definition:

Definition II.9: Vector bundles

Let E, M be smooth manifolds, $\mathbf{E}$ a Banach space, and $\pi : E \to M$ a smooth map; suppose that there is a covering $\{U_i\}$ of M with open neighbourhoods and a family of smooth maps $\tau_i : \pi^{-1}(U_i) \to U_i \times \mathbf{E}$, known as bundle or trivialisation charts, such that:

1. π commutes with the projection on U_i, i.e. the following diagram commutes:
$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\tau_i"] \arrow[d, "\pi"] & U_i \times \mathbf{E} \arrow[dl, "\pi_{U_i}"] \\ U_i & \end{tikzcd}$$

2. On each *fibre*, i.e. $\pi^{-1}(\{p\}) \equiv E_p$, $p \in M$, there is a topological vector space structure and for each $p \in U$, the restriction: $\tau_{i,p} : \pi^{-1}(\{p\}) \to \mathbf{E}$ is a toplinear[a] isomorphism.

3. For every $(U_i, \tau_i), (U_j, \tau_j)$ with $U_i \cap U_j \neq \emptyset$, the *transition* map: $p \in U_i \cap U_j \mapsto (\tau_{j,p} \circ \tau_{i,p}^{-1}) \in L(\mathbf{E})$ is smooth.

The family $\mathcal{B} = \{(U_i, \tau_i)\}$ is said to be a vector bundle atlas. In the exact same way as for atlases on manifolds, a vector bundle atlas determines a unique maximal vector bundle atlas that we call a vector bundle structure on π (or E). We say that E is the total space and M the base manifold.

[a] Fancy way of saying continuous linear.

Remark II.4. If M and E are finite-dimensional manifolds and $\mathbf{E}$ a finite dimensional vector space then the third condition is implied by the others.

The transition maps $\tau_{ji,p} = \tau_{j,p} \circ \tau_{i,p}^{-1}$ satisfy a cocycle condition:

$$\tau_{kj,p} \circ \tau_{ji,p} = \tau_{ki,p}.$$

They contain in fact contain all the essential information of the bundle: given a covering $\{U_i\}$ and a family of transition maps $\{\tau_{ji,p}\}$ satisfying the cocycle condition, it is possible to reconstruct E and π.

Let us now describe how to construct TM, suppose that M is an $\mathbf{E}$-manifold. Consider first the coproduct of sets (i.e. disjoint union) $TM = \coprod_{p \in M} T_pM$. Specify

$\pi : TM \to M$ on each T_pM to be the constant map $v_p \in T_pM \mapsto p$.[4] For every chart (U, ϕ) consider a map: $\hat{\phi} : \pi^{-1}(U) \to \phi(U) \times \mathbf{E}$ defined by[5]:

$$v_p \in T_pM \hookrightarrow TM \mapsto (\phi(p), \phi_{*p}v_p).$$

Now, equip TM with the *coarsest* topology[6] that makes π and all the $\hat{\phi}$ continuous; with this topology the $\hat{\phi}$ are homeomorphisms. The transition map between any two charts is:

$$(\hat{\psi} \circ \hat{\phi}^{-1})(p, v) \mapsto ((\psi \circ \phi^{-1})(p), (\psi \circ \phi^{-1})'(p)v),$$

and hence is smooth. The charts $\{(\pi^{-1}(U), \hat{\phi}), (U, \phi) \text{ chart on } M\}$ therefore determine a smooth structure on TM. Define now: $\tilde{\phi} : \pi^{-1}(U) \to U \times \mathbf{E}$ by $\phi(v_p) = (p, \phi_{*p}v_p), v_p \in T_pM$. and observe that these are smooth bundle charts; the transition maps are easily seen to be given by: $p \mapsto (\psi \circ \phi^{-1})'(p)$ and are smooth, which shows that π is a vector bundle.

Remark II.5. Although TM is a *local* product, in general it is not globally diffeomorphic to a product of manifolds; when this is possible the manifold is said to be parallelisable.

We quote a few examples that we can describe explicitly:

Example II.5. If U is an open subset of a Banach space $\mathbf{E}$, $TU = U \times \mathbf{E}$.

Example II.6. Let $S^n \subset \mathbb{R}^{n+1}$ be the Euclidean sphere, then:

$$TS^n = \{(x, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, \, ||x||^2 = 1, \langle x, v \rangle = 0\}.$$

Proposition II.3

If $\phi : M \to N$, one can define a $\phi_* : TM \to TN$ by setting for any $p \in M, v_p \in T_pM \hookrightarrow TM$, $\phi_* v_p = \phi_{*p}v_p \in T_{\phi(p)}N \hookrightarrow TN$. The construction of the topologies and smooth structures of TM and TN makes this map smooth; we will refer to it as the tangent map of ϕ.

[4] We apply here the universal property of coproducts.

[5] As before it is sufficient to define the function on each T_pM to define a function on $TM = \coprod_{p \in M} T_pM$

[6] Let $(Y_i)_{i \in I}$ be a family of topological spaces and $f_i : X \to Y_i$ a family of maps. The initial topology of $(f_i)_{i \in I}$ is the finest topology on X such that all the maps are continuous. A basis for this topology are sets of the form $\cap_{j \in J} f_j^{-1}(U_j)$ where $J \subset I$ is finite and $U_j \subset Y_j$ is open.

II.D. Operations on vector bundles

Certain functors of subcategories of Banach spaces extend to functors of vector bundles, to keep this discussion brief we refer the reader to [Lan95, Chapter III, §4], we shall simply quote a few important examples.

Example II.7. Consider first the contravariant functor that sends a Banach space $\mathbf{E}$ to its dual[7] $\mathbf{E}^\vee$; recall that the map between morphisms is given by: $u \mapsto {}^t u$. There is a corresponding functor of vector bundles of base M, $\pi : E \to M$, that yields a new vector bundle $E^\vee$ whose fibres are the dual spaces of those of E. Applied to the tangent bundle, this gives us the *cotangent bundle*.

Example II.8. If E_1 and E_2 are two vector bundles with the same base M one can form their direct sum[8], $E_1 \oplus E_2$ which is a vector bundle with base M and whose fibre at a point $x \in M$ is $E_{1x} \oplus E_{2x}$.

Example II.9. In finite dimensions, one can also form the tensor product of vector bundles.

II.E. Vector fields

We can now define:

Definition II.10: Vector fields

A vector field is a section of the tangent bundle $\pi : TM \to M$, i.e. a map $X : M \to TM$ such that $\pi \circ X = Id_M$.

In a local chart (U, ϕ) this is a map from U to $U \times \mathbf{E}$ of the form: $p \mapsto (p, X_U(p))$. The map X_U is called the local representative of the vector field, X is smooth if and only for every $p \in M$, one can find a chart (U, ϕ), such that that X_U is smooth. In finite dimensions, this translates to the fact that: $X = X^i \frac{\partial}{\partial x_i}$ is smooth if and only if the functions (X^i) are smooth. Note that under a change of chart $(U, \phi) \to (V, \psi)$, one has: $X_V = (\psi \circ \phi^{-1})' \circ \phi \cdot X_U$. Conversely, if for an atlas on M one is given a family of smooth maps, indexed by the charts, that verify this compatibility condition then one can construct a unique vector field locally described by the family.

If X is a vector field to avoid too many parentheses it is customary to write $X(p) = X_p$. We denote by $\Gamma(TM)$ the set of smooth vector fields on M; it is a $C^\infty(M)$-module; there is an injective map into the set of derivations of $C^\infty(M)$. A remarkable fact is that it is also a *Lie algebra*.

[7]Banach space of continuous linear forms on $\mathbf{E}$, equipped with the norm $||l|| = \sup_{x \neq 0} \frac{|l(x)|}{||x||}$

[8]also known as the Whitney sum

Theorem II.1

Let X, Y be two vector fields on M, ∂_X, ∂_Y the corresponding derivations, then there is a unique vector field $[X,Y]$ such that: $\partial_X \partial_Y - \partial_Y \partial_X = \partial_{[X,Y]}$. In a local chart (U,ϕ), the local representative is:

$$\begin{aligned}[X,Y]_U &= Y_{U*}(\phi_*^{-1} \cdot X_U) - X_{U*}(\phi_*^{-1} \cdot Y_U), \\ &= (Y_U \circ \phi^{-1})' \circ \phi \cdot X_U - (X_U \circ \phi^{-1})' \circ \phi \cdot Y_U.\end{aligned}$$

Proof. Uniqueness follows immediately from injectivity of the map: $X \mapsto \partial_X$. Let (U,ϕ) be a chart and let X_U, Y_U be the local representatives of X, Y in the chart. Let f be a smooth function, then, for $p \in U$: $(\partial_X f)(p) = (f \circ \phi^{-1})'(\phi(p)) \cdot X_U$, hence on U:

$$\begin{aligned}(\partial_X \partial_Y - \partial_Y \partial_X)(f) =& ((f \circ \phi^{-1})' \cdot Y_U \circ \phi^{-1})' \circ \phi \cdot X_U - ((f \circ \phi^{-1})' \cdot X_U \circ \phi^{-1})' \circ \phi \cdot Y_U \\ =& (f \circ \phi)'' \circ \phi \cdot (X_U, Y_U) + (f \circ \phi^{-1})' \circ \phi \cdot \big((Y_U \circ \phi^{-1})' \circ \phi \cdot X_U\big) \\ & - (f \circ \phi)'' \circ \phi \cdot (Y_U, X_U) - (f \circ \phi^{-1})' \circ \phi \cdot \big((X_U \circ \phi^{-1})' \circ \phi \cdot Y_U\big) \\ =& (f \circ \phi^{-1}) \circ \phi \cdot \big(Y_{U*}(\phi_*^{-1} \cdot X_U) - X_{U*}(\phi_*^{-1} \cdot Y_U).\big)\end{aligned}$$

Where in the final step we use the symmetry of the second derivative. We have therefore shown that, on U, $\partial_X \partial_Y - \partial_Y \partial_X$ is a derivation associated with a vector field over U locally represented by:

$$[X,Y]_U = Y_{U*}(\phi_*^{-1} \cdot X_U) - X_{U*}(\phi_*^{-1} \cdot Y_U).$$

It remains to ensure ourselves of the fact that this can be glued together to from a vector field on M. For this, is sufficient to check that it transforms correctly under a change of chart. Recall that if (V,ψ) is another chart then on $U \cap V$, $X_V = (\psi \circ \phi^{-1})' \circ \phi \cdot X_U$, to simplify notation let us write: $h = (\psi \circ \phi^{-1})' \circ \phi$ now:

$$[X,Y]_V = (h \cdot Y_U)_*(\psi_*^{-1} h \cdot X_U) - (h \cdot X_U)_*(\psi_*^{-1} h \cdot Y_U).$$

Now:

$$\psi_*^{-1} h \cdot X_U = \psi_*^{-1}(\psi_* \circ \phi_*^{-1}) \cdot X_U = \phi_*^{-1} X_U,$$

and:

$$(h \cdot X_U)_*(\phi_*^{-1} Y_U) = (\psi \circ \phi^{-1})'(X_U, Y_U) + h \cdot (X_U)_*(\phi_*^{-1} Y_U).$$

Again, using the symmetry of the second derivative we arrive at:

$$[X,Y]_V = h \cdot [X,Y]_U.$$

Therefore, it is clear that this patches together to define a smooth vector field on M, that we denote by $[X,Y]_U$. □

Remark II.6. If M is of finite dimension n and (U, x) is a local chart, the local formula becomes on U:

$$[X, Y] = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial Y^i}{\partial x_j} X^j - \frac{\partial X^i}{\partial x_j} \cdot Y^j \right) \frac{\partial}{\partial x_i},$$

where $X = X^i \frac{\partial}{\partial x_i}$, $Y = Y^i \frac{\partial}{\partial x_i}$ and $X^i, Y^j : U \to \mathbb{R}$.

Example II.10. In $\mathbb{R}^2$, $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, $Y = \frac{\partial}{\partial y}$:

$$[X, Y] = \frac{\partial}{\partial y}.$$

Proposition II.4

The bracket $[\cdot, \cdot]$ of vector fields is a Lie bracket, i.e.

- It is bilinear, antisymmetric.
- For any vector fields, X, Y, Z:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Proof. It is sufficient to prove this on an open subset U of $\mathbf{E}$. Identifying the vector field with its local representative on U, i.e. a map $X : U \to \mathbf{E}$ we have: $[X, Y] = X' \cdot Y - Y' \cdot X$. It is now a simple computation:

$$\begin{aligned}
[X, [Y, Z]] &= X' \cdot (Y' \cdot Z - Z' \cdot Y) - Y'' \cdot (Z, X) - Y' \cdot (Z' \cdot X) + Z'' \cdot (Y, X) + Z' \cdot (Y' \cdot X) \\
[Y, [Z, X]] &= Y' \cdot (Z' \cdot X - X' \cdot Z) - Z'' \cdot (X, Y) - Z' \cdot (X' \cdot Y) + X'' \cdot (Z, Y) + X' \cdot (Z' \cdot Y) \\
[Z, [X, Y]] &= Z' \cdot (X' \cdot Y - Y' \cdot X) - X'' \cdot (Y, Z) - X' \cdot (Y' \cdot Z) + Y'' \cdot (X, Z) + Y' \cdot (X' \cdot Z)
\end{aligned}$$

Summation of the three expressions shows the result, using once more symmetry of the second derivative. □

It also worth noting, for some calculations, that we have the following Leibniz rule for $[X, Y]$:

$$[X, fY] = (\partial_X f)Y + f[X, Y].$$

Proof. Exercise. □

II.F. Flow and bracket

Let $\gamma : I \to M$ be a curve on M defined on an open interval I. Note that $TI = I \times \mathbb{R}$ and we have a canonical section $j : t \mapsto (t, 1)$. The *canonical lift* of γ is the map $\dot{\gamma} = \gamma_* \circ j$, it is a curve on TM that satisfies $\pi \circ \dot{\gamma} = \gamma$, where $\pi : TM \to M$ is the projection of TM onto M.

Remark II.7. For each $t \in I$, $\dot{\gamma}(t)$ has the usual interpretation as the tangent (or velocity) vector to the curve at the point $\gamma(t)$. It is sometimes written:

$$\dot{\gamma}(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\bigg|_{t=t_0}.$$

We can reinterpret tangent vectors at p as velocity vectors to curves passing through p. Indeed if γ is a curve such that $\gamma(0) = p$ then $\dot{\gamma}(0) \in T_pM$. Conversely, if $v_p \in T_pM$ then there is a curve such that $\gamma(0) = p$, $\dot{\gamma}(0) = v_p$. To construct it let (U, ϕ) be a local chart such that $p \in U$, for small enough $\varepsilon > 0$, $\phi(p) + t\phi_{*p}(v_p) \in \phi(U)$ for all $t \in (-\varepsilon, \varepsilon)$, consider the curve: $\gamma : (-\varepsilon, \epsilon) \to M$ defined by:

$$\gamma(t) = \phi^{-1}(\phi(p) + t\phi_{*p}v_p).$$

Clearly, $\gamma(0) = 0$ and:

$$\dot{\gamma}(0) = \phi^{-1}_{*\phi(p)}(\phi_{*p}v_p) = v_p.$$

If you are not entirely convinced define: $\alpha : (-\varepsilon, \varepsilon) \to \phi(U) \subset \mathbf{E}$ by:

$$\alpha(t) = \phi(p) + t\phi_{*p}v_p.$$

Now the tangent map is given explicitly:

$$\alpha_* : T(-\varepsilon, \varepsilon) \cong (-\varepsilon, \varepsilon) \times \mathbb{R} \to T\phi(U) \cong \phi(U) \times \mathbf{E},$$

with: $\alpha_*(t, \lambda) = (\alpha(t), \alpha'(t) \cdot \lambda)$, here: $\alpha'(t) \in L(\mathbb{R}, \mathbf{E})$ is the usual derivative, indeed, it is naturally identified with $\alpha_{*t} : T_t(-\varepsilon, \varepsilon) \cong \mathbb{R} \to T_{\alpha(t)}\mathbf{E} \cong \mathbf{E}$. Now recall that we have the canonical section $j : (-\varepsilon, \varepsilon) \to T(-\varepsilon, \varepsilon) = (-\varepsilon, \varepsilon) \times \mathbb{R}$ given by: $j(t) = (t, 1)$. Hence: $\dot{\alpha}(t) = (\alpha_* \circ j)(t) = (\alpha(t), \alpha'(t) \cdot 1)$. By the chain rule:

$$\dot{\gamma}(0) = \phi^{-1}_{*\alpha(0)}\dot{\alpha}(0) = \phi^{-1}_{*\phi(p)}(\phi_{*p} \cdot v_p).$$

To understand the last inequality it is important to note that although in all rigour: $\phi^{-1}_{*\phi(p)} : T_{\phi(p)}\phi(U) \cong \{\phi(p)\} \times \mathbf{E} \to T_pM$ we *systematically* identify $\{\phi(p)\} \times \mathbf{E}$ with $\mathbf{E}$ and write:

$$\phi^{-1}_{*\phi(p)}(\phi(p), v) \equiv \phi^{-1}_{*\phi(p)}v.$$

It can sometimes be useful to note that if $f : M \to N$ and $v_p \in T_pM$ then for any curve $\gamma : I \to M$ such that $\gamma(0) = p, \dot{\gamma}(0) = v_p$ we have:

$$f_{*p}v_p = \dot{\widehat{(f \circ \gamma)}}(0), \tag{II.1}$$

this is just another application of the chain rule.

We now have enough tools to consider first order ordinary differential equations on manifolds. In particular, if X is a smooth vector field on TM, we can consider the Cauchy problem:

$$\begin{cases} \dot{\gamma} = X \circ \gamma, \\ \gamma(0) = p \in M. \end{cases}$$

In a local chart (U, ϕ), recall that $TU = U \times \mathbf{E}$, writing: $\phi \circ \gamma = \gamma_U : J \to U$, where $J \subset I$ such that $\gamma(J) \subset U$ and γ some curve defined an open interval containing 0 then: $\phi_* \dot{\gamma} = (\gamma_U, \gamma_U')$. Hence if $\phi_* X = (\mathrm{id}_U, X_U)$, the equation is equivalent in a local chart to:

$$\gamma_U'(t) = X_U(\gamma_U(t)), \qquad t \in J.$$

Apply the classical theory of ordinary differential equations in local charts, we have existence and uniqueness of maximal solutions to such problems; we will write the unique maximal solution as $t \mapsto \phi_t^X(p)$; solutions are referred to as *integral curves* of X.

Definition II.11: Flows

- A subset $\mathscr{D} \subset \mathbb{R} \times M$ such that for any $p \in M$, $\mathscr{D}^{(p)} = \{t \in \mathbb{R}, (t,p) \in \mathscr{D}\}$ is an open interval containing 0 is known as a *flow domain*.
- A (local) flow is a map $\theta : \mathscr{D} \to M$ where $\mathscr{D}$ is a flow domain such that for any $t \in \mathscr{D}^{(p)}$, $s \in \mathscr{D}^{(\theta(t,p))}$:

$$\theta(s, \theta(t,p)) = \theta(s + t, p).$$

With a bit of elbow grease one can show the following fundamental theorem of flows:

Theorem II.2: Fundamental theorem of flows

Let X be a smooth vector field on a smooth manifold M, then there is a unique maximal flow domain $\mathscr{D}$ such that:

1. $(t,p) \in \mathscr{D} \mapsto \phi_t^X(p)$ is a local flow,
2. For each $p \in M$, $\mathscr{D}^{(p)}$ is the interval of definition of the maximal solution $t \mapsto \phi_t^X(p)$,
3. If $s \in \mathscr{D}^{(p)}$ then $\mathscr{D}^{(\phi_s^X(p))} = \mathscr{D}^{(p)} - s$,
4. For each $t \in \mathbb{R}$ the set $M_t = \{p \in M : (t,p) \in \mathscr{D}\}$ is open in M and $p \mapsto \phi_t^X(p)$ is a diffeomorphism of M_t onto M_{-t} with inverse $\phi_{-t}^X(p)$.

We refer to [Lee03] for the proof. This gives another formula for the Lie bracket that is sometimes useful:

Proposition II.5: Lie bracket and flow

Let X, Y be two smooth vector fields on a smooth manifold then:

$$[X,Y](p) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\phi^X_{-t*\phi_t^X(p)} \cdot Y(\phi_t^X(p))\right)\bigg|_{t=0}.$$

This shows that $[X,Y]$ measures the infinitesimal change in Y under the effect of the flow of X.

Proof. Exercise... □

II.G. Appendix: Product manifolds

Let $(M,\hat{\mathcal{A}})$ be a smooth $\mathbf{E}$-manifold, $(N,\hat{\mathcal{B}})$ a smooth $\mathbf{F}$-manifold. $M \times N$ can be given naturally the structure of a smooth $(\mathbf{E}\times\mathbf{F})$-manifold. Equip $M \times N$ with its product topology and denote by π_M (resp. π_N) the projection onto the first (resp. second) factor. Define an atlas on M by:

$$\mathcal{C} = \{(U\times V, \phi\times\psi), (U,\phi)\in\hat{\mathcal{A}}, (V,\psi)\in\hat{\mathcal{B}}\}.$$

where $\phi\times\psi : (p,q) \to (\phi(p),\psi(q))$. It is straightforward to check that $\phi\times\psi$ is a homeomorphism of $U\times V$ onto $\phi(U)\times\psi(V)$. Note now that $(\phi\times\psi)^{-1} = \phi^{-1}\times\psi^{-1}$, so:

$$(\phi\times\psi)\circ(\phi'\times\psi')^{-1} = (\phi\circ\phi'^{-1})\times(\psi\circ\psi'^{-1}).$$

It is a smooth map; this follows from the fact that if $\mathbf{E}\times\mathbf{F}$ is equipped with its norm $||(x,y)||_{\mathbf{E}\times\mathbf{F}} = ||x|| + ||y||$, then $f : U \subset \mathbf{G} \to \mathbf{E}\times\mathbf{F}$ is differentiable if and only if $\pi_1 \circ f$ and $\pi_2 \circ f$ are differentiable. Equipping $M \times N$ with the maximal atlas $\hat{\mathcal{C}}$, this property generalises to manifolds.

Proposition II.6

Let $(M \times N)$ be equipped with the maximal atlas $\hat{\mathcal{C}}$, then:

1. the projections π_1, π_2 are smooth maps,
2. a map $f : L \to M \times N$ is smooth if and only if $\pi_1 \circ f$ and $\pi_2 \circ f$ are smooth.

Let $(p,q) \in M \times N$; then, as one might expect, we have the toplinear isomorphism:

$$T_{(p,q)}(M \times N) \cong T_pM \times T_qN.$$

This follows directly from Definition II.2.

II.H. Appendix: Vector bundle morphisms

Definition II.12: Vector bundle morphism

Let (E,π,M), (E',π',M) two vector bundles with same base M, a vector bundle morphism is a smooth map: $f : E \to E'$ such that

- $\pi' \circ f = \pi$,
- for each $p \in M$, the induced map between fibres: $f_p : \pi^{-1}(p) \to \pi'^{-1}(p)$ is continuous linear
- there are bundle charts $\tau : \pi^{-1}(U) \to U \times \mathbf{E}$, $\tau' : \pi'^{-1}(U) \to U \times \mathbf{E}'$ such that the map from $U \to L(\mathbf{E},\mathbf{E}')$ determined by $p \mapsto \pi_{\mathbf{E}'} \circ \tau' \circ f \circ \tau^{-1} \circ j_p$ is smooth; here, $j_p : v \in \mathbf{E} \mapsto (p,v)$.

Remark II.8.
- The final condition is superfluous for finite dimensional manifolds.
- The map $T : M \mapsto TM$ is in fact a functor between smooth manifolds and smooth vector bundles; the map between morphisms is given by the tangent map.

III. Geometry on manifolds, connections

From now on, for simplicity, all manifolds will be assumed finite dimensional.

Roughly, modern differential geometry can be thought of as the study of *connections* on manifolds. The point of view I adopt in these notes is due to E. Cartan and it generalises Klein's Erlengan program to a curved setting. Klein introduced the idea that geometry should be thought of as the study of the action of a group G on a space X. Restricting our attention to any of the orbits of X, we can assume that the action is transitive and hence X can be thought of as the quotient G/H, where H is the isotropy subgroup of any point of X. Affine, Euclidean, projective and conformal geometry fit nicely into this framework. Our starting point will be to revisit affine geometry from this point of view and identify the differential structures that one would like to reproduce on a manifold.

III.A. Lie groups and their Lie algebra

Definition III.1: Lie groups

- A Lie group $(G, \cdot)$ is a group equipped with the structure of a smooth finite-dimensional manifold that is compatible with its group operations, i.e. $\mu : (g_1, g_2) \mapsto g_1 \cdot g_2$ and $\iota : g \mapsto g^{-1}$ are smooth.
- A Lie subgroup is a subgroup that is also a smooth submanifold.

Example III.1. $(\mathbb{R}^*, \cdot)$, $GL_n(\mathbb{R})$, $O_n(\mathbb{R})$, $SL_n(\mathbb{R})$

A number of important examples follow from:

Definition III.2: The closed subgroup theorem (Cartan-Von Neumann)

Let G be a Lie group and H a closed subgroup, then H is a Lie subgroup of G.

A vector field X is said to be ***left-invariant***, if for any $p, g \in G$

$$X_{pg} = L_{g*p} X_p.$$

Left-invariant vector fields are completely determined by their value at the identity element; indeed for any $g \in G$: $X_g = L_{g*e} X_e$. They form a $\dim G$-dimensional subspace of $\Gamma(TG)$, and is in fact a ***Lie subalgebra***. To prove this it is custom to use the notion of ϕ-related vector fields. Let $\phi : M \to N$ be a smooth map, X a vector field on M, and Y a vector field on N. Y is said to ϕ-related to X, if for any $p \in M$, $Y_{\phi(p)} = \phi_{*p} X_p$.

Proposition III.1

Let X be ϕ-related to $\tilde{X}$, Y, ϕ-related to $\tilde{Y}$, then $[X, Y]$ is ϕ-related to $[\tilde{X}, \tilde{Y}]$. This is sometimes written:

$$\phi_*[X, Y] = [\phi_* X, \phi_* Y].$$

Proof. This is not the most elegant proof, but it is valid in infinite dimensions. Let us work in a local charts, suppose that (U, x) is a chart at p, (V, y) a chart at $\phi(p)$ such that $\phi(U) \subset V$. Let X_U and $\tilde{X}_V$ be the local representations of the fields near in the chart then they are ϕ-related means that for any $p \in U$:

$$\tilde{X}_V(\phi(p)) = (y \circ \phi \circ x^{-1})'(x(p)) \cdot X_U(p).$$

Now:

$$[\tilde{X}, \tilde{Y}]_V(\phi(p)) = (\tilde{Y}_V \circ y^{-1})'(y(\phi(p)) \cdot \tilde{X}_U(\phi(p)) - (\tilde{X}_V \circ y^{-1})'(y(\phi(p)) \cdot \tilde{Y}_U(\phi(p))$$

By the chain rule:

$$\begin{aligned}(\tilde{Y}_V \circ y^{-1})'(y(\phi(p)) \cdot \tilde{X}_U(\phi(p)) &= (\tilde{Y}_V \circ y^{-1})'(y(\phi(p)) \cdot (y \circ \phi \circ x^{-1})'(x(p)) \cdot X_U(p), \\ &= (\tilde{Y}_V \circ \phi \circ x^{-1})'(x(p)) \cdot X_U(p).\end{aligned}$$

Furthermore:

$$\begin{aligned}((y \circ \phi \circ x^{-1})'(x(p)) \cdot Y_U(p))' \cdot X_U(p) =& (y \circ \phi \circ x^{-1})''(x(p)) \cdot (X_U(p), Y_U(p)) \\ &+ (y \circ \phi \circ x^{-1})(x(p)) \cdot Y_U'(p) \cdot X_U(p).\end{aligned}$$

Treating the second term in exactly the same fashion and using the symmetry of the second derivative, we find that:

$$[\tilde{X}, \tilde{Y}]_V(\phi(p)) = (y \circ \phi \circ x^{-1})'(x(p)) \cdot [X, Y]_U(p).$$

Which proves that $[X, Y]$ and $[\tilde{X}, \tilde{Y}]$ are ϕ-related. $\square$

By definition a vector field is left-invariant if and only if it L_g-related to itself for all g; the above result implies directly that the Lie bracket of two left-invariant vector fields is left-invariant so that $\mathfrak{g}$ is a Lie subalgebra.

Definition III.3: Lie algebra of a group G

The Lie subalgebra $\mathfrak{g}$ of $\Gamma(TG)$ composed of left-invariant vector fields is known as the Lie algebra of the group G.

Since left-invariant vector fields are uniquely determined by their value at the identity, we can identify $\mathfrak{g}$ and T_eG as vector spaces, and transfer the Lie algebra structure to T_eG, i.e. by setting:

$$[X_e, Y_e] = [X, Y]_e,$$

where X (resp. Y) is the unique left-invariant vector field such that $X(e) = X_e$ (resp. $Y(e) = Y_e$).

III.B. The exponential map of a Lie group

Let $X \in \mathfrak{g}$ be a left-invariant vector field on a Lie group G, let us consider arbitrary integral curves of X passing through the identity, i.e. solutions γ to the initial value problem:

$$\begin{cases} \dot{\gamma} = X \circ \gamma, \\ \gamma(0) = e. \end{cases}$$

Call $u = X_e$, we shall show that $\gamma(t+s) = \gamma(t)\gamma(s)$. Fix $s \in \mathbb{R}$, it suffices to show that the curves $t \mapsto \gamma(s+t)$ and $t \mapsto \gamma(s)\gamma(t) = \sigma(t)$ satisfy the same Cauchy problem. First, they satisfy the same initial condition and:

$$\dot{\sigma}(t) = L_{\gamma(s)_{*\gamma(t)}} \cdot \dot{\gamma}(t) = L_{\gamma(s)_{*\gamma(t)}} \circ L_{\gamma(t)_{*e}} \cdot u = L_{\sigma(t)_{*e}} \cdot u = X_{\sigma(t)}.$$

By uniqueness of the solution we conclude that for every $t, s \in \mathbb{R}$, $\gamma(t+s) = \gamma(t)\gamma(s)$. It follows that:

Proposition III.2

- The integral curves of a left-invariant vector field are complete, i.e. defined on all of $\mathbb{R}$.
- $\gamma(t)^{-1} = \gamma(-t)$ for all $t \in \mathbb{R}$.

Corollary III.1. $\iota_{*e} : u \in T_eG \mapsto -u \in T_eG$.

Definition III.4: The exponential map

Let $X \in \mathfrak{g}$, we define the exponential map: $\exp : \mathfrak{g} \to G$, by

$$\exp(X) = \gamma(1).$$

Note that $\gamma(t) = \exp(tX)$ for every $t \in \mathbb{R}$.

Example III.2. Let $G = GL_n(\mathbb{R})$, the tangent space at I_n is identified with $M_n(\mathbb{R})$. The exponential map of $X \in \mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$ is given by the usual power series: $\sum_{k\in\mathbb{N}} \frac{1}{k!}X^k$. This enables us to determine the Lie bracket on $M_n(\mathbb{R})$. Using Proposition II.5 and Equation (II.1), we have:

$$\begin{aligned} [X,Y]_e &= \frac{d}{dt}\left(\frac{d}{ds}\exp(tX)\exp(sY)\exp(-tX)\right)\bigg|_{s,t=0}, \\ &= \frac{d}{dt}(\exp(tA)B\exp(-tA))\bigg|_{t=0}, \\ &= AB - BA. \end{aligned}$$

This englobes many examples as the Lie algebra of a Lie subgroup H can be identified with a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. One can prove this using the inclusion map $i : H \hookrightarrow G$ and noting that left-invariant vector fields on H are i-related to left-invariant vector fields on G.

III.C. Affine geometry

Recall that the affine group is the semi-direct product:

$$G = GA(n) = \mathbb{R}^n \rtimes GL_n(\mathbb{R}).$$

It is useful to see it as a Lie subgroup of $GL_{n+1}(\mathbb{R})$. For this, identify the affine plane with the hyperplane $A^n = \{x_{n+1} = 1\}$ in $\mathbb{R}^{n+1}$ and consider the subgroup G of $GL_{n+1}(\mathbb{R})$ that preserves the form $(0,\dots,0,1)$. It is composed of matrices of the form:

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}, A \in GL_n(\mathbb{R}), b \in \mathbb{R}^n.$$

G acts in the plane $x_{n+1} = 1$ and if $\begin{pmatrix} x \\ 1 \end{pmatrix} \in A^n$, $g = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$ then $g\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax+b \\ 1 \end{pmatrix}$. Hence G can be identified with the affine group; it is a closed subgroup of $GL_{n+1}(\mathbb{R})$ and therefore a Lie subgroup. We can also determine its Lie algebra (as a Lie subalgebra of $\mathfrak{gl}_{n+1}(\mathbb{R}) = M_{n+1}(\mathbb{R})$), we have:

$$\mathfrak{g} = \left\{\begin{pmatrix} M & b \\ 0 & 0 \end{pmatrix}, M \in M_n(\mathbb{R}), b \in \mathbb{R}^n\right\}.$$

G acts transitively on X and the isotropy subgroup of, say, $O = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is of course:

$$H = \left\{\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, A \in GL_n(\mathbb{R})\right\} \cong GL_n(\mathbb{R}).$$

It is standard that we have a bijection $G/H \cong A^n$; in fact G/H equipped with the quotient topology is a topological manifold and has a unique smooth structure such that the canonical projection $\pi : G \to G/H$ is a smooth submersion; it is then the case that G/H is ***diffeomorphic*** to A^n. We omit this development and instead make some further observations:

1. π admits a smooth global section $\begin{pmatrix} x \\ 1 \end{pmatrix} \in A^n \mapsto \begin{pmatrix} I_n & x \\ 0 & 1 \end{pmatrix}$. The interpretation of σ is as follows: at each point $x \in \mathbb{R}^n$ we choose a frame anchored at x and composed of the n vectors $(e_1, \dots, e_n)$ of the canonical basis of $\mathbb{R}^n$.

2. The section σ is equivalent[9] to a smooth diffeomorphism: $\phi : G \to A^n \times H$ defined by $\phi(\sigma(x)h) = (x, h)$.

3. H acts on the right on G and we have $\phi(gh) = \phi(g)h$, where on the rhs[10] this is the canonical right action on the product $A^n \times H$ i.e. $(x, h_1)h_2 = (x, h_1 h_2)$ This is our first example of a principle H-bundle that we will define below. With respect to this structure, G is interpreted as the "frame bundle" of A^n: at each point we glue the set of all possible linear frames anchored at x; the right-action can be thought of as a (global) change of frame.

4. Our final observation is that left-multiplication L_g by an element of g is a smooth diffeomorphism of G. The tangent map $L_{g^{-1}}{}_{*g}$ is a linear isomorphism from T_gG onto $T_eG \cong \mathfrak{g}$. This gives a smooth map $\omega : TG \to \mathfrak{g}$:

$$\omega_g(X_g) = L_{g^{-1}}{}_{*g}(X_g), g \in G, X_g \in T_gG \hookrightarrow TG.$$

   ω is usually interpreted as a $\mathfrak{g}$-valued differential form known as the *Maurer-Cartan* form, i.e. a section of the bundle $T^\vee G \otimes (G \times \mathfrak{g})$.

To describe the properties of the Maurer-Cartan form we need to recall some algebra.

[9] $\sigma(x) = \phi^{-1}(x, e)$
[10] right-hand side

III.D. Differential forms and the exterior algebra

Definition III.5: k-forms

Let V be a real vector space a k-form on V is multilinear map $\alpha : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$ that is *alternating*, i.e. for any $\sigma \in \mathfrak{S}_k$, $v_1, \ldots, v_k \in V$:

$$\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \varepsilon(\sigma)\alpha(v_1, \ldots, v_k),$$

where $\varepsilon(\sigma) \in \{-1, 1\}$ is the signature of σ. The set of k-forms on V is a linear space denoted by $\bigwedge^k V^\vee$.

Example III.3.

- 1-forms are elements of the dual.
- 0-forms are constants

Let us define a product on forms:

Definition III.6: Exterior product

Let α and β be respectively a k and l form on some vector space V, we define their exterior product to be the $k + l$ form given by:

$$(\alpha \wedge \beta)(v_1, \ldots, v_k) = \frac{1}{k!l!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \varepsilon(\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})$$

Example III.4. Let u and v be one forms, then: $(u \wedge v)(x, y) = u(x)v(y) - v(x)u(y)$.
Let V have finite dimension $n \in \mathbb{N}^*$, and $\mathcal{B} = (e_1, \ldots, e_n)$ a basis of V. Write the dual basis $(e_1^*, \ldots, e_n^*)$, then by induction one can show that:

$$(e_1^* \wedge \cdots \wedge e_n^*)(v_1, \ldots, v_n) = \det_{\mathcal{B}}(v_1, \ldots, v_n).$$

Proposition III.3

The multiplication $\wedge$ is bilinear and associative, making $\mathbb{R} \oplus \bigoplus_{k=1}^{\infty} \bigwedge^k V^\vee$ an associative algebra with unit called the exterior algebra. If α and β are respectively a k-form and l-form then:

$$(\alpha \wedge \beta) = (-1)^{kl} \beta \wedge \alpha.$$

If $\dim V = n$ then $\bigwedge^m V^\vee = 0$, if $m > n$ and if $(e_1, \ldots, e_n)$ is a basis for V, then:

$$\{e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*, 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

is a basis of $\bigwedge^k V^\vee$

Proof. Any textbook on linear algebra. □

Let U be an open subset of $\mathbb{R}^n$ a ***differential** k**-form*** on U is a smooth map $\alpha : U \to \bigwedge^k V^\vee$, we often write $\alpha(x) = \alpha_x$. Let $\Omega^k(U)$ denote the $C^\infty(U)$-module of differential forms on U.

Example III.5.

- A 0-form is just a smooth function on U.
- If f is smooth f' can be viewed as a differential 1-form.

We will encounter the following operations on differential forms in the sequel:

Definition III.7: Pullback of differential forms

Let $f : U \to V$, a smooth map between open subsets of $\mathbb{R}^k$, α a k-form on V, we call the pullback of α by f, the differential form on U defined by:

$$(f^*\alpha)_x(v_1, \ldots, v_n) = \alpha_{f(x)}(f'(x)v_1, \ldots, f'(x)v_n).$$

Pullback distributes over the wedge product (defined in a pointwise manner), i.e. $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$.

Proposition III.4: Exterior derivative

There is a ***unique*** linear map d defined on differential forms such that:

1. If f is a smooth function $df = f'$.
2. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$, if α is a k-form and β an l-form;
3. $d^2 = 0$.

Example III.6. Let U be an open set of $\mathbb{R}^n$, $(e_1, \dots, e_n)$ the canonical basis. Define, $x^i : U \to \mathbb{R}$ which maps p to its ith coordinate of p in the canonical basis. Then for any $p \in U$, $dx^i_p = e^*_i$. If f is a smooth function then we can write $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$.

Example III.7. Consider on $\mathbb{R}^2$, the one-form $\alpha = ydx - xdy$, then $d\alpha = 2dy \wedge dx$

Example III.8. Let f and g be two zero forms, then the wedge product between f and g is simply the product fg. Point 2 in Proposition III.4 is simply the product rule: $d(fg) = dfg + fdg$.

Example III.9. Let us consider polar coordinates on $\mathbb{R}^2$ defined by $x = r\cos\theta$, $y = r\sin\theta$. This are valid, for example, on the open set $\mathbb{R}^2 \setminus \{y = 0, x < 0\}$. At each point $p = (r\cos\theta, r\sin\theta)$ we have the corresponding basis $(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})$ of T_pM, related to the canonical basis by:

$$\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial\theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$

The dual basis is

$$dr = \frac{xdx + ydy}{\sqrt{x^2 + y^2}}, d\theta = \frac{xdy - ydx}{x^2 + y^2}$$

To extend this to manifolds, one can construct $\bigwedge^k T_p^\vee M$ at each point p, in the same way as for a vector space. Then, one can equip $\bigwedge^k T^\vee M = \coprod_{p\in M} \bigwedge^k T_p^\vee M$ with the structure of a vector bundle where the fibres are given by $\bigwedge^k T_p^\vee M$. This is an example of the more general procedure touched upon in Section II.D.

The gory details are as follows. Define $\pi : \bigwedge^k T^\vee M \to M$ by $\alpha_p \in T_p^\vee M \mapsto p$. Choose a chart, (x, U) and define a map $\hat{x} : \pi^{-1}(U) \to U \times \bigwedge^k (\mathbb{R}^n)^\vee$ by:

$$\hat{x}(\alpha_p) = (p, \alpha_p(x^{-1}_{*x(p)}, x^{-1}_{*x(p)}, \cdots, x^{-1}_{*x(p)})), \alpha_p \in \bigwedge^k T_p^\vee M.$$

These are our natural candidates for our vector bundle charts. Composing with the chart we get a bijective map:

$$\tilde{x} : \pi^{-1}(U) \to x(U) \times \bigwedge^k (\mathbb{R}^n)^\vee$$

that is a natural candidate for a coordinate chart. As with the tangent bundle we equip $\bigwedge^k T^\vee M$ with the coarsest topology making π and every $\tilde{x}$ (where x runs through the charts on M) continuous. The $\tilde{x}$ are easily seen to be homeomorphisms with this choice of topology. If (y, V) is another chart then:

$$\tilde{y} \circ \tilde{x}^{-1} : (u, \alpha) \mapsto ((y \circ x^{-1})(u), \alpha((x \circ y^{-1})'(y(p)), \dots, (x \circ y^{-1})'(y(p))).$$

Since M is a smooth manifold these transition charts are also smooth and so define a smooth structure on $\bigwedge^k T^\vee M$. This is precisely the smooth structure that makes the bundle charts $\hat{x}$ smooth diffeomorphisms. It is straightforward to check that $\bigwedge^k T^\vee M$ is a vector bundle.

Smooth sections of this bundle are called differential k-forms on M, they form a $C^\infty(M)$ module denoted by $\Omega^k(M)$.

Example III.10. Let $x^i = e_i^* \circ x$ where (x, U) is a local chart. Define a differential form on U, dx^i, by:

$$dx^i_p = x^i_{*p} = e_i^* \circ x_{*p}.$$

Then dx^i is a smooth section of $\bigwedge^1 T^\vee U \cong T^\vee U$, indeed, in the chart (x, U) this is locally represented by e_i^*. For each $p \in U$, $(dx^1_p, \dots, dx^n_p)$ is ***by definition*** the dual basis to the coordinate basis $(\left(\frac{\partial}{\partial x_1}\right)_p, \cdots, \left(\frac{\partial}{\partial x_n}\right)_p)$.

We can inductively define differential k-forms on U, $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, $1 \leq i_1 < \cdots < i_k \leq n$ that at each point p give a basis of $\bigwedge^k T_p^\vee M$. An arbitrary differential k-form can be represented locally:

$$\alpha = \sum_{0 \leq i_1 < \cdots < i_k \leq n} \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where $\alpha_{i_1 \dots i_k}$ are functions on U, the differential form is smooth if and only if they are smooth for every chart (x, U).

This is in a nutshell how we work with forms on manifolds in a very basic fashion and in a way that is designed to ressemble the case of an open set of $\mathbb{R}^n$.

Whilst this is satisfying for computations, to develop the theory it is sometimes nice to avoid using charts by adopting another point of view. Let $\alpha \in \Omega^k(M)$, if $X_1, \dots, X_k$ are vector fields on M then one can define a $C^\infty(M)$-linear k-alternating form $A : \Gamma(TM) \times \cdots \times \Gamma(TM) \to C^\infty(M)$ by:

$$A(X_1, \dots, X_k)(p) = \alpha_p((X_1)(p), \dots, (X_k)(p)).$$

On finite dimensional manifolds, it turns out that the two points of view are equivalent. The key to this correspondence is the following lemma:

Lemme III.1

Let $T : \Gamma(TM) \times \cdots \times \Gamma(TM) \to C^\infty(M)$ be a k-multilinear map on the $C^\infty(M)$ module $\Gamma(TM)$, then for each $p \in M$, and any $X_1, \ldots, X_k \in \Gamma(TM)$, the value of $T(X_1, \ldots, X_k)(p)$ only depends on the values of the vector fields $X_1, \ldots, X_k$ at the point p.
Consequently, for each p, T induces a k-multilinear map: $T_p : T_pM \times \cdots T_pM \to \mathbb{R}$. Furthermore, if T is alternating $p \mapsto T_p$ is a smooth section of $\bigwedge^k T^\vee M$.

Proof. It is sufficient to treat the case $k = 1$ and then use induction. Fix $p \in M$ and choose a chart (x, U) near p, let $b : M \to \mathbb{R}$ be a smooth bump function with support in U. Let $X \in \Gamma(TM)$ and decompose X_p on the coordinate basis as $X_p = v^i \left(\frac{\partial}{\partial x_i}\right)_p$. Set $Y = v^i b \frac{\partial}{\partial x_i}$. $C^\infty(M)$-linearity gives:

$$\alpha(bX - Y) = (X^i - v^i)\alpha(b\frac{\partial}{\partial x_i})$$

hence: $\alpha(bX - Y)(p) = 0$, but:

$$\alpha(X)(p) = b(p)\alpha(X)(p) = \alpha(bX)(p) = \alpha(Y)(p) = v^i\alpha(b(\frac{\partial}{\partial x_i}))(p).$$

Thus showing that $\alpha(X)(p)$ only depends on X_p. □

This correspondence is useful for defining tensor objects on manifolds without referring to a specific choice of chart or working in local coordinates. (Although to actually calculate things we do have to choose coordinates at some point). For instance, one can extend the exterior derivative to manifolds and we have the invariant formula:

Proposition III.5: Invariant formula for the exterior derivative on manifolds

Let α be a k-form on M, then its exterior derivative can be defined by the same conditions as in Proposition III.4. We have the following invariant formula:

$$\begin{aligned} d\alpha(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k}(-1)^{i+1}X_i(\alpha(X_1, \ldots, \tilde{X}_i, \ldots, X_{k+1})) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, \ldots, \tilde{X}_i, \ldots, \tilde{X}_j, \ldots, X_{k+1}) \end{aligned} \tag{III.1}$$

where $\tilde{}$ denotes omission. Recall that vector fields are to be thought of as derivations.

Example III.11. The most important case is when α is a one form then the formula reads:
$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]).$$

III.E. Lie algebra valued differential forms and the Maurer-Cartan equation

Now we know a little bit about differential forms and what we can do with them, we shall make one further generalisation and consider $\mathfrak{g}$-valued differential forms. The difference is that we get at each point an alternating linear map, $\alpha_p : T_pM \times \cdots \times T_pM \to \mathfrak{g}$.

Whilst the notion of pullback by a diffeomorphism extends directly, Definition III.6 no longer makes sense. We can nevertheless recover a notion of exterior product by using the bracket of the Lie algebra, we define:

$$[\alpha,\beta](v_1,\ldots,v_{k+l}) = \frac{1}{k!l!}\sum_{\sigma\in\mathfrak{S}_{k+l}} \varepsilon(\sigma)[\alpha(v_{\sigma(1)},\ldots,v_{\sigma(k)}),\beta(v_{\sigma(k+1)},\ldots,v_{\sigma(k+l)})].$$

Example III.12. Let α be a $\mathfrak{g}$-valued one form, then $[\alpha,\alpha](v_1,v_2) = 2[\alpha(v_1),\alpha(v_2)]$

The exterior derivative can be extended to $\mathfrak{g}$-valued forms by choosing an arbitrary basis E^i of $\mathfrak{g}$ and declaring that d acts on components (which are usual k-forms) like the usual exterior derivative; one should check that this does not depend on the choice of basis. The invariant formula in Proposition III.5 then extends without any modification.

We can now summarise the main properties of the Maurer-Cartan form. Recall that for every $p \in G$, $X_p \in T_pG$,
$$\omega_p(X_p) = L_{p^{-1}}{}_{*p}X_p \in T_eG \cong \mathfrak{g}.$$

Now H is a closed subgroup of G and H acts on G from the right, let us denote by $R_h : G \to G$ the map $p \mapsto ph$. Calculating the pullback of ω under R_h, we have:

$$\begin{aligned}(R_h^*\omega)_g &= \omega_{gh} \circ R_{h*g} = L_{h^{-1}g^{-1}}{}_{*hg} \circ R_{h*g}\\ \text{(Proposition II.1)} &= (L_{h^{-1}g^{-1}} \circ R_h)_{*g}\\ &= (\mathrm{Ad}_{h^{-1}} \circ L_{g^{-1}})_{*g} = \mathrm{Ad}_{h*e}\omega_g\end{aligned} \tag{III.2}$$

where he have defined $\mathrm{Ad}_h : p \mapsto h^{-1}ph$. Its tangent map at the identity element, Ad_{h*e} – which we will abusively also denote Ad_h in the sequel – defines a representation of H with representing space $\mathfrak{g}$, called the *adjoint representation*. We also say that $\mathfrak{g}$ is a H-module.

The second property we have is that for any $X \in \mathfrak{h} \subset \mathfrak{g}$, one can consider a vector in T_pG for any other p defined by:

$$X_p^* = \frac{d}{dt} p \exp(tA)\bigg|_{t=0},$$

then:

$$\omega_p(X_p^*) = X. \tag{III.3}$$

Note that the vector X_p^* is in $\ker \pi_{*p}$ where $\pi : G \mapsto G/H$ is the canonical projection. Tangent vectors in $\ker \pi_{*p}$ are known as vertical vectors; intuitively they point in the direction of the fibres of G above G/H. In fact all vertical vectors can be obtained as fundamental vectors. The vector field $p \mapsto X_p^*$ is called the ***fundamental vector field*** associated to X. Summarising this:

Proposition III.6

Let G be a Lie group, H a closed subgroup and ω the Maurer-Cartan form of G, then:

1. For each $p \in G$, ω_p is a vector space isomorphism.
2. For each $h \in H$, ${R_h}^*\omega = \mathrm{Ad}_{h^{-1}}\omega$.
3. For each $X \in \mathfrak{h}$, $\omega(X^*) = X$.

The Maurer-Cartan form satisfies one further equation; later we will interpret this as the local expression of the flatness of affine space.

Proposition III.7: Maurer-Cartan structure equation

Let G be a Lie group, ω its Maurer-Cartan form, then:

$$d\omega + \frac{1}{2}[\omega, \omega] = 0$$

Proof. Let us work with the invariant formulae. First ω is a one form so for any vector fields X, Y:

$$d\omega(X, Y) = X(\omega(Y)) - X(\omega(Y)) - \omega([X, Y]),$$

and

$$\frac{1}{2}[\omega, \omega] = [\omega(X), \omega(Y)].$$

Now, recall from Lemma III.1 that the value at any given point $g \in G$ of the function $d\omega(X, Y) + \frac{1}{2}[\omega, \omega]$ only depends on the values of X and Y at g. Therefore it is sufficient to study the case where X and Y are ***left-invariant vector fields***, In this case:

$$d\omega(X, Y) = -\omega([X, Y]) = -[X_e, Y_e] = -[\omega(X), \omega(Y)] = -\frac{1}{2}[\omega, \omega].$$

□

III.F. Interpretation of the Maurer-Cartan form by an example

Let us consider 2-dimensional affine space, choose an arbitrary origin O and set up an affine frame (O, e_x, e_y). To see the structure at work, introduce polar coordinates (r, θ) on $U = \mathbb{R}^2 \backslash \{x < 0, y = 0\}$ defined by: $x = r\cos\theta, y = r\sin\theta$, and its "moving coordinate frame":

$$\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}.$$

Perhaps a more familiar moving frame is given by:

$$e_r = \frac{\partial}{\partial r}, e_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}.$$

At each point $p \in U$, choosing p as origin gives us an affine frame attached to each point. Expressed differently, we have a section σ of $\pi : G \to G/H \cong \mathbb{R}^2$ ($G = \mathbb{R}^2 \rtimes GL_2(\mathbb{R})$):

$$p \mapsto \begin{pmatrix} \cos\theta(p) & -\sin\theta(p) & r(p)\cos\theta(p) \\ \sin\theta(p) & \cos\theta(p) & r(p)\sin\theta(p) \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark III.1. Recall that a chart of a 2-dimensional manifold like affine space is defined as a map from an open set of a manifold onto an open subset of $\mathbb{R}^2$. Equivalently this can be described by two maps: $r : U \to \mathbb{R}$ and $\theta : U \to \mathbb{R}$.

Let us now consider the pullback of the Maurer-Cartan form of G onto U. First, let us determine what the Maurer-Cartan form is in matrix terms. By definition: $\omega_g = L_{g^{-1}}{}_{*g}$ i.e. it is the derivative at g of the restriction to G of the linear map: $q \mapsto g^{-1}q$. It can be written:

$$\omega_g = g^{-1}dg$$

where dg is to be understood as the derivative at p of the injection $\mathfrak{g} \hookrightarrow \mathfrak{gl}_{n+1}(\mathbb{R})$. The pullback along σ is given by:

$$\sigma^*\omega_p = \sigma^{-1}(p)\sigma_{*p} \qquad p \in U$$

Since:

$$\sigma^{-1}(p) = \begin{pmatrix} \cos\theta(p) & \sin\theta(p) & -r(p) \\ -\sin\theta(p) & \cos\theta(p) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\sigma_{*p} = \begin{pmatrix} -\sin\theta(p)d\theta & -\cos\theta(p)d\theta & \cos\theta(p)dr - r(p)\sin\theta(p)d\theta \\ \cos\theta(p)d\theta & -\sin\theta(p)d\theta & \sin\theta(p)dr + r(p)\cos\theta d\theta \\ 0 & 0 & 0 \end{pmatrix}$$

We have:

$$\sigma^*\omega_p = \begin{pmatrix} 0 & -d\theta & dr \\ d\theta & 0 & rd\theta \\ 0 & 0 & 0 \end{pmatrix}.$$

Analysing this result, notice that the last column is the dual basis of (e_r, e_θ); this can be interpreted as the infinitesimal displacement vector, expressed in the basis e_r, e_θ as we move from a point p to $p + \delta p$. On the other hand, the matrix:

$$\begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix},$$

is a matrix describing the infinitesimal change in the basis vectors. Indeed, differentiate in the classical sense the equations defining e_r and e_θ we find:

$$\begin{aligned} de_r &= -\sin\theta d\theta e_x + \cos\theta d\theta e_y = d\theta e_\theta, \\ de_\theta &= -\cos\theta d\theta e_x - \sin\theta d\theta e_y = -d\theta e_r. \end{aligned}$$

Hence, to first order, it is the change of basis matrix from (e_r, e_θ) to $(e_{r+\delta r}, e_{\theta+\delta\theta})$. In other words the Maurer-Cartan form measures how the moving basis (section) changes as we move from point to point.

III.G. The Frame Bundle

To extend this structure to manifolds, we need to find a way to replicate this. The first step is to notice that the structure of $\pi : G \mapsto G/H$ we highlighted in Section III.C is a (trivial) example of a principal H-bundle over a manifold:

Definition III.8: Principal bundles

Let $\pi : P \to M$ be a smooth map between smooth manifolds. Suppose that there is a smooth right-action of a Lie group H on P and that π satisfies the following properties:

1. $\forall r \in P, h \in H, \pi(r \cdot h) = \pi(r)$,

2. Each point $p \in M$ admits an open neighbourhood U, called a trivialising neighbourhood, and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times H$ such that: $\phi(rh) = \phi(r) \cdot h$, where $U \times H$ is equipped with the canonical right-action: $(p, h_1) \cdot h_2 = (p, h_1 h_2)$, and the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi"] \arrow[d, "\pi"] & U \times H \arrow[dl, "\pi_U"] \\ U & \end{tikzcd}$$

Then π is a principal H-bundle with total space P, base M and structure group H.

Remark III.2.
- The right-action is free, and the orbits are the fibres:

$$\pi^{-1}(\{\pi(p)\}) = \{p \cdot h, h \in H\}$$

- The fibres are diffeomorphic to H but do not have a natural group structure.

Every smooth n-dimensional manifold M has a natural smooth $GL_n(\mathbb{R})$-principal bundle, $L(TM)$ known as the ***frame bundle***. The fibre at $p \in M$ of this bundle will be the set of linear frames of T_pM, described as follows

$$L(TM)_p = GL(\mathbb{R}^n, T_pM).$$

There is a natural right-action of $GL(\mathbb{R}^n) \cong GL_n(\mathbb{R})$ on $L(TM) = \coprod_{p \in M} L(TM)_p$ given by $u \cdot g = u \circ g$, the natural projection defined by: $\pi : u_p \in L(\mathbb{R}, T_pM) \hookrightarrow L(TM) \mapsto p$, certainly satisfies $\pi(ug) = \pi(u)$; i.e. preserves the fibres.

Let (x, U) be some chart on M, then for each $p \in U$ we have a natural map: $L(\mathbb{R}^n, T_pM) \to \{p\} \times GL(\mathbb{R}^n) \cong GL_n(\mathbb{R})$ given by: $u_p \mapsto (p, x_{*p} \circ u_p)$, hence we get a bijective map $\phi : \pi^{-1}(U) \to U \times GL_n(\mathbb{R})$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi"] \arrow[d, "\pi"] & U \times GL(\mathbb{R}^n) \arrow[dl, "\pi_U"] \\ U & \end{tikzcd}$$

We just need to define a topology and a smooth structure to make all of this smooth; we proceed as for the tangent bundle. Compose ϕ with the chart of the manifold and define:

$$\hat{\phi} : \pi^{-1}(U) \to x(U) \times GL(\mathbb{R}^n),$$

these will be our local charts. Endow $L(TM)$ with the coarsest topology that makes π and all $\hat{\phi}$ for any chart (x, U) continuous; they are then homeomorphisms. If (x, U), (y, V) $U \cap V \neq \emptyset$ are two charts and $\hat{\phi}$, $\hat{\psi}$ are the corresponding charts on $L(TM)$, then: $\hat{\psi} \circ \hat{\phi}^{-1} : U \cap V \times GL(\mathbb{R}^n) \to U \cap V \times GL(\mathbb{R}^n)$ is given by:

$$\hat{\psi} \circ \hat{\phi}^{-1}(p, g) = (y \circ x^{-1}(p), (y \circ x^{-1})'(p) \circ g).$$

This is smooth (because M is smooth) and so the maximal atlas determined by the charts $\hat{\phi}$ is a smooth structure on $L(TM)$ with respect to which the maps ϕ are smooth diffeomorphisms and π is smooth !

Definition III.9: Local section

Let $\pi : P \to M$ be a smooth principal H bundle, a local section of P is a smooth map: $\sigma : U \to P$ where U is an open set of M and $\pi \circ \sigma = \mathrm{id}_U$.

Example III.13. Let M be a smooth manifold and $L(TM)$ its frame bundle, a local section: $\sigma : U \to L(TM)$ is a smooth choice of linear frame of T_pM for each $p \in U$. Indeed by definition for each $p \in U$, $\sigma(p) \in L_p(TM) = GL(\mathbb{R}^n, T_pM)$ hence setting for each $i \in \{1, \dots, n\}$, $E_i(p) = \sigma(p) \cdot e_i$, we get n-smooth vector fields that span T_pM for each $p \in U$.

In particular if (x, U) is a local chart: $p \in U \mapsto x^{-1}_{*x(p)}$ is a smooth section of $L(TM)$ on U and the corresponding frame is the coordinate frame $(\frac{\partial}{\partial x_i})$.

Remark III.3. Smooth sections on $L(TM)$ are equivalent to local bundle trivialisations, indeed, define:

$$\begin{array}{rccc} \phi : & U \times H & \longrightarrow & \pi^{-1}(U) \\ & (p, h) & \longmapsto & \sigma(p) \cdot h \end{array}$$

This is a smooth bijective map since the H-action is smooth, the derivative: $\phi_{*(p,h)} = R_{h*\sigma(p)}\sigma_{*p} + L_{\sigma(p)_{*h}}$ this is a bijective map at each p and so by the local inversion theorem ϕ is invertible and ϕ^{-1} is a bundle chart. Conversely if $\phi : \pi^{-1}(U) \to U \times H$, then $\sigma(p) = \phi^{-1}(p, e)$ is a local section on M. It follows in particular that a principal bundle is trivial (i.e. a product) if and only if there is a global section. Moreover, in general we should only expect there to be *local* sections.

The frame bundle $L(TM)$ on M will play the same role as $G = \mathbb{R}^n \rtimes GL_n(\mathbb{R})$ in the case of affine space, we now want an analogue of the Maurer-Cartan form ω_G...

III.H. Affine connections on manifolds and curvature

Throughout this section let $G = \mathbb{R}^n \rtimes GL_n(\mathbb{R})$, $H = GL_n(\mathbb{R})$ and $\mathfrak{g}$, $\mathfrak{h}$ their Lie algebras.

We are one step away from our definition of affine connections. Since the right action of $GL_n(\mathbb{R})$ preserves the fibres we have a natural generalisation of a fundamental vector field.

Definition III.10

Let $X \in \mathfrak{h}$, then define a vector field X^* on $L(TM)$ by:

$$X^*_p = \frac{d}{dt}(p \exp(Xt))\bigg|_{t=0}$$

X^* is a smooth vector field on $L(TM)$ called the fundamental vector field associated with $X \in \mathfrak{h}$.

We can now state:

Definition III.11: (Cartan) Affine connections

Let R_h denote right-multiplication by an element of H in $L(TM)$. We will call an affine connection a $\mathfrak{g}$-valued one form on $L(TM)$ such that:

1. $\omega_p : T_pL(TM) \to \mathfrak{g}$ is a linear isomorphism.
2. $R_h^*\omega = \mathrm{Ad}_{h^{-1}}\omega$, $h \in H$
3. $\omega(X^*) = X$ for any $X \in \mathfrak{h}$.

Since we have no notion of left-invariant vector fields on $L(TM)$, we cannot expect ω to satisfy an analogue of Proposition III.7. Instead, this will be a measure of curvature:

Definition III.12: Curvature of a connection

Let ω be an affine connection on a smooth manifold, the curvature form Ω is the $\mathfrak{g}$-valued 2-form defined by:

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

We shall say that the connection is ***torsion-free*** if Ω is $\mathfrak{h}$-valued.

The intuitive meaning of this definition is that we are using affine geometry as a local model for a geometry on M. The Cartan connection tells us how to "slide" affine space along M and the curvature Ω is an infinitesimal measure of the difference between M and the model space. The curvature has the following remarkable property:

Proposition III.8

$$d\Omega + [\omega, \Omega] = 0.$$

Proof. This is a short computation that reduces to the Jacobi identity of the Lie algebra $\mathfrak{g}$:

$$\begin{aligned} d\Omega + [\omega, \Omega] &= [d\omega, \omega] + [\omega, \Omega] = [\Omega - \frac{1}{2}[\omega, \omega], \omega] + [\omega, \Omega] \\ &= -\frac{1}{2}[[\omega, \omega], \omega]. \end{aligned}$$

However, for any vector fields $X, Y, Z \in \Gamma(TM)$:

$$\begin{aligned} [[\omega, \omega], \omega](X, Y, Z) &= [[\omega(X), \omega(Y)], \omega(Z)] + [[\omega(Y), \omega(Z)], \omega(X)] \\ &\qquad + [[\omega(Z), \omega(X)], \omega(Y)] \\ &= 0 \end{aligned}$$

$\square$

Our definition of affine connection is slightly more general than the classical definition. Let us make an algebraic observation:

Proposition III.9

Let $\mathfrak{g}$ be the Lie algebra of $\mathbb{R}^n \rtimes GL_n(\mathbb{R})$, $\mathfrak{h}$ the Lie subalgebra of $GL_n(\mathbb{R})$, then the $GL_n(\mathbb{R})$-module $\mathfrak{g}$ (with the adjoint representation) admits a H-module decomposition:

$$\mathfrak{g} = \mathbb{R}^n \oplus \mathfrak{h}.$$

Proof. Recall that:

$$\mathfrak{g} = \left\{ \begin{pmatrix} M & b \\ 0 & 0 \end{pmatrix}, M \in M_n(\mathbb{R}), b \in \mathbb{R}^n \right\}.$$

which gives us immediately the vector space decomposition, we just need to check that each component is indeed $\mathrm{Ad}(H)$-invariant. Let $h = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, let us first consider the $\mathbb{R}^n$ component and an element: $x = \begin{pmatrix} I_n & b \\ 0 & 1 \end{pmatrix}$, then:

$$hxh^{-1} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_n & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_n & Ab \\ 0 & 1 \end{pmatrix}.$$

This proves that the $\mathbb{R}^n$ component is $\mathrm{Ad}(H)$-invariant, and the adjoint representation reduces to the standard representation of $GL_n(\mathbb{R})$. A similar calculation shows that if: $x = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$ then:

$$hxh^{-1} = \begin{pmatrix} ABA^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

We conclude that the $GL_n(\mathbb{R})$ component of the direct sum decompostion is also $\mathrm{Ad}(H)$ invariant and reduces to the standard adjoint representation of H on $\mathfrak{h} = M_n(\mathbb{R})$. $\square$

This means that any affine connection ω can be decomposed as: $\omega = \theta + \gamma$, where θ is a $\mathbb{R}^n$ valued one-form with $R_h^*\theta = h^{-1}\theta$ and γ a $\mathfrak{h}$-valued one form with $R_h^*\gamma = \mathrm{Ad}_{h^{-1}}\gamma$.

Remark III.4. For some calculations it is nice to notice how to calculate the bracket in terms of the semi-direct product of Lie algebras, $\mathfrak{g} = \mathbb{R}^n \oplus \mathfrak{h}$, we have:

$$[b + A, b' + A'] = Ab' - A'b + [A, A']. \tag{III.4}$$

It turns that there is a canonical choice for the θ component on $L(TM)$.

Definition III.13: Solder form

Let $\pi : L(TM) \to M$ be the frame bundle then for every $u_p \in L(\mathbb{R}^n, T_pM) \hookrightarrow L(TM)$, define:

$$\theta_{u_p} = u_p^{-1} \circ \pi_{*u_p}.$$

This defines a smooth $\mathbb{R}^n$-valued one-form on $L(TM)$ known as the canonical one-form or solder form.

Now:
$$(R_h^*\theta)_{u_p} = (u_p \circ h)^{-1} \circ (\pi \circ R_h)_{*u_p} = h^{-1} \circ \theta_{u_p}.$$
(Where we have used that $\pi \circ R_h = \pi$), it therefore satisfies the correct transform rule (we say that it is $GL_n(\mathbb{R})$-equivariant). Furthermore: $\theta(X^*) = 0$ for any $X \in \mathfrak{h}$.

Proposition III.10

Let γ be a $\mathfrak{h}$-valued one form on $L(TM)$ such that:

1. $R_h^*\gamma = \mathrm{Ad}_{h^{-1}}\alpha$,
2. $\gamma(X^*) = X$ for any $X \in \mathfrak{h}$,

then $\omega = \theta + \gamma$, where θ is the canonical one-form, is an affine Cartan connection on M.

Remark III.5. γ is known as a *principal connection* on $L(TM)$.

To specify an affine connection, it is therefore sufficient to give a principal connection on $L(TM)$, we will sometimes refer to γ as a linear connection.

We will now assume that all the affine connections we consider arise in this manner from a linear connection.

The curvature form Ω can also be split up into two parts:

Proposition III.11

Let $\omega = \theta + \gamma$ be an affine connection; with respect to the H-module decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^n$ the curvature splits as follows:

$$\Omega = \underbrace{d\theta + [\gamma, \theta]}_{\mathbb{R}^n\text{-valued 2-form}} + \underbrace{d\gamma + \frac{1}{2}[\gamma, \gamma]}_{\mathfrak{gl}_n(\mathbb{R})\text{ valued 2-form}}.$$

where $[\gamma, \theta](X, Y) = \gamma(X) \cdot \theta(Y) - \gamma(Y) \cdot \theta(X)$, $X, Y \in \Gamma(L(TM))$.

- $T = d\theta + [\gamma, \theta]$ is called the *torsion* of the connection ω.
- The form $R = d\gamma + \frac{1}{2}[\gamma, \gamma]$ is the curvature of the linear connection γ.

Proposition III.8 has the following expression in terms of T and R:

Proposition III.12: Bianchi identities

1. $dT + [\gamma, T] = [R, \theta]$
2. $dR + [\gamma, R] = 0$

where:

$$[R, \theta](X_1, X_2, X_3) = \frac{1}{2} \sum_{\sigma \in \mathfrak{S}_3} \varepsilon(\sigma) R(X_{\sigma(1)}, X_{\sigma(2)}) \cdot \theta(X_{\sigma(3)}).$$

To understand this formula recall that R is $\mathfrak{gl}_n(\mathbb{R})$-valued and therefore acts naturally on the $\mathbb{R}^n$-valued form θ.

III.I. Local gauge definition

Theoretically, the definition above is very satisfying; it encodes a lot of information into a global object. However, it is not very practical for local considerations: to calculate things we need to work in charts. Besides, we are more familiar with working with forms on TM rather than on $L(TM)$. We therefore encode the data of ω in another way.

Let $\sigma : U \to L(TU)$ be a local section of $\pi : L(TM) \to M$. Interpreted as a choice of frame in T_pM at each $p \in U$. The pullback of ω by σ, $\sigma^*\omega = \omega_U$ is $\mathfrak{g}$-valued one-form on M known as a *local gauge* or *local connection form*. This is the differential information telling us how the frame moves. We shall now seek the consequences of the properties of ω on ω_U.

First: $\sigma^*\omega_p : T_pM \to \mathfrak{g}/\mathfrak{h} \cong \mathbb{R}^n$ is an isomorphism, this follows directly from the fact that $\omega_p : T_pM \to \mathfrak{g}$ is an isomorphism. When $\omega = \theta + \gamma$, where θ is the solder form, this amounts to projecting onto this component and considering: $\sigma^*\theta$, which we can evaluate:

$$(\sigma^*\theta) = \theta_{\sigma(p)} = \sigma(p)^{-1} \circ \underbrace{(\pi \circ \sigma)_{*p}}_{\mathrm{id}_{T_pM}}.$$

Recall that: $\sigma(p) : \mathbb{R}^n \to T_pM$. Define, $E_i(p) = \sigma(p) \cdot e_i$ where (e_i) is the canonical basis of $\mathbb{R}^n$, and define $(\omega^i(p))$ the dual basis in $T_p^\vee M$, one can therefore, express, $\sigma^*\theta$ as the $\mathbb{R}^n$ valued one form:

$$\sigma^*\theta = \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix}. \tag{III.5}$$

The pullback of the canonical one form is hence simply the map that to a vector associates its coordinates in the basis determined by the frame $\sigma(p)$ and is clearly an isomorphism in the fibres.

The property $R_h^*\omega = \mathrm{Ad}_{h^{-1}}\omega$, tells us how to change local gauge. Suppose that $\tilde{\sigma} : V \to L(TM)$ is another local gauge defined on V such that $U \cap V \neq \emptyset$. Let us set $\omega_V = \tilde{\sigma}^*\omega$ and compare ω_V and ω_U on $U \cap V$. First note that there is in fact a smooth function $h : U \cap V \mapsto GL_n(\mathbb{R})$ such that $\tilde{\sigma} = \sigma h = R_h \circ \sigma$. For every $p \in U \cap V$, $h(p)$ is simply the map that changes basis of the frame at p. We claim that:

$$(\tilde{\sigma}^*\omega) = \mathrm{Ad}_{h^{-1}}\omega_U + h^*\omega_H,$$

where ω_H is the Maurer-Cartan form of H. To prove this, we need to evaluate:

$$\omega_{\sigma(p)h(p)} \circ (R_h \circ \sigma)_{*p},$$

The key is to understand how to calculate the derivative of $L(TM) \times H \to L(TM)$, $(r,h) \mapsto rh$, in terms of those of the maps: R_h and $L_r : H \to L(TM)$, $h \mapsto rh$, using the isomorphism: $T_{(r,h)}(L(TM) \times H) = T_r L(TM) \times T_h H$ it can be described as the map:

$$(X, Y) \mapsto R_{h*r}X + L_{r*h}(Y).$$

We therefore have:

$$(\tilde{\sigma}\omega)_p = \underbrace{\omega_{\sigma(p)h(p)} \cdot R_{h(p)_{*r}}\sigma_{*p}}_{\sigma^*(R^*_{h(p)}\omega)=\sigma^*(\mathrm{Ad}_{h(p)^{-1}}\omega)=\mathrm{Ad}_{h(p)^{-1}}\omega_U} + \omega_{\sigma(p)h(p)} \cdot L_{\sigma(p)_{*h(p)}} \circ h_{*p}.$$

The last term looks rather mysterious, however for any $X_p \in T_pM$.

$$L_{\sigma(p)_{*h(p)}} \circ h_{*p} = \frac{d}{dt}\sigma(p)\gamma(t)\bigg|_{t=0}.$$

Where $\gamma : I \to H$ is any curve such that $\gamma(0) = h(p)$ and $\dot{\gamma}(0) = h_{*p}X_p$, but we can take: $\gamma(t) = h(p)\exp(t\omega_{H\,h(p)}h_{*p}X_p)$ (recall that $\omega_{H\,h(p)} = L_{h(p)^{-1}{}_{*h(p)}}$). It then follows that:

$$L_{\sigma(p)_{*h(p)}} \circ h_{*p}X_p = (\omega_{H\,h(p)}h_{*p}X_p)^*_{\sigma(p)h(p)}.$$

But, ω sends fundamental vector fields to their generator, so that:

$$\omega_{\sigma(p)h(p)} \cdot L_{\sigma(p)_{*h(p)}} \circ h_{*p} = h^*\omega_H.$$

Proposition III.13

Let $(U_i)_{i\in I}$ be a covering of M by open sets, $\sigma_i : U_i \to L(TM)$ a family of local sections and (ω_{U_i}) a family of $\mathfrak{g}$-valued one forms such that the previous properties are satisfied. Then there exists a unique $\mathfrak{g}$-valued one form ω on $L(TM)$ such that $\omega_{U_i} = \sigma_i^*\omega$ for each $i \in I$.

Remark III.6. Of course the θ component is known so one only needs the data that determines γ. If $\tilde{\sigma} = \sigma h$ for some map $h : U \cap V \to GL_n(\mathbb{R})$ then

$$\forall p \in U \cap V,\ (\tilde{\sigma}^*\theta)_p = h(p)^{-1}(\sigma^*\theta)_p,$$

simply because $\theta(X^*) = 0$, so it follows that:

$$\tilde{\sigma}^*\gamma = h^*\omega_H + \mathrm{Ad}_{h^{-1}}\sigma^*\gamma.$$

Hence, we only need a family of $\mathfrak{h}$-valued one forms, γ_{U_i} such that the above is satisfied between two local sections.

IV. Covariant derivatives and geodesics

IV.A. Covariant derivatives of vector fields

In affine space, since we have a canonical identification of tangent spaces at each point, it is clear how to define the derivative of vector fields. This amounts to choosing a constant basis and differentiating the components. A connection will enable us to extend this notion to manifolds. First a formal definition:

Definition IV.1: Covariant derivative

A covariant derivative on the tangent bundle TM is an $\mathbb{R}$-bilinear operator:

$$\begin{array}{rcl}\nabla : \Gamma(TM) \times \Gamma(TM) & \longrightarrow & \Gamma(TM) \\ (X, Y) & \longmapsto & \nabla_X Y,\end{array}$$

such that:

1. for any $X \in \Gamma(TM)$, $Y \mapsto \nabla_Y X$ is $C^\infty(M)$-linear,
2. for any smooth function $f \in C^\infty(M)$, $X, Y \in \Gamma(TM)$

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y.$$

An affine connection $\theta + \gamma$ determines a covariant derivative on TM in the following manner.

Let $\sigma : U \to L(TM)$ be a local section, i.e. a smooth choice of linear frame of T_pM at each $p \in U$. Recall that:

$$\sigma_U^*\omega = \sigma_U^*\theta + \sigma_U^*\gamma,$$

the first part tells us how the point moves (the origin of the affine frame) and the second how the basis vectors are changing. Let us define n vector fields on U by

$E_i(p) = \sigma_U(p) \cdot e_i$. Any vector field X can be written on U, $X = X^i E_i$ where X^i are smooth functions on U. On U define:

$$\nabla_Y X = Y(X^i)E_i + E_i[(\sigma_U^*\gamma)^i_j \cdot Y]X^j. \tag{IV.1}$$

Recall that $(\sigma_U^*\gamma) \cdot Y \in \mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$ is matrix, equivalently one can view $\sigma_U^*\gamma$ as a matrix whose components are one-forms, which justifies the notation: $(\sigma_U^*\gamma)^i_j \cdot Y$.

We must check that the above formula defines a vector field on all of M. For this, we must investigate how this transforms under a change of local section. Suppose $\sigma_V : V \to L(TM)$ is another local section defining vector fields $\tilde{E}_i = \sigma_V \cdot e_i$. Let $h : U \cap V \to GL_n(\mathbb{R})$ be the smooth map such that, $\sigma_V = \sigma_U h$. Write $X = \tilde{X}^i \tilde{E}_i$ then:

$$\begin{pmatrix} X^1 \\ \vdots \\ X^n \end{pmatrix} = h \begin{pmatrix} \tilde{X}^1 \\ \vdots \\ \tilde{X}^n \end{pmatrix}.$$

So

$$\begin{pmatrix} Y(X^1) \\ \vdots \\ Y(X^n) \end{pmatrix} = h_*(Y) \begin{pmatrix} \tilde{X}^1 \\ \vdots \\ \tilde{X}^n \end{pmatrix} + h \begin{pmatrix} Y(\tilde{X}^1) \\ \vdots \\ Y(\tilde{X}^n) \end{pmatrix}$$

It follows that:

$$Y(X^i)E_i = Y(\tilde{X}^i)\tilde{E}_i + \begin{pmatrix} \tilde{E}_1 & \cdots & \tilde{E}_n \end{pmatrix} h^{-1} h_*(Y) \begin{pmatrix} \tilde{X}^1 \\ \vdots \\ \tilde{X}^n \end{pmatrix}$$

Now $\sigma_V^*\gamma = \mathrm{Ad}_{h^{-1}} \sigma_U^*\gamma + h^*\omega_{\mathfrak{gl}_n(\mathbb{R})}$, or: $\sigma_U^*\gamma = \mathrm{Ad}_h \sigma_V^*\gamma - \mathrm{Ad}_h h^*\omega_{\mathfrak{gl}_n(\mathbb{R})}$ so:

$$\begin{aligned} X^j(\sigma_U^*\gamma)^i_j \cdot Y E_i &= E_i(h[(\sigma_V^*\gamma) \cdot Y]h^{-1})^i_j X^j - E_i(h_*(Y)h)^i_j X^j \\ &= \tilde{E}_i(\sigma_V^*\gamma)^i_j \cdot Y \tilde{X}_j - \tilde{E}_i(h^{-1}h_*(Y))^i_j \tilde{X}^j. \end{aligned}$$

Hence, the two expressions coincide on $U \cap V$ and can be glued together to define a vector field on M.

Proposition IV.1

Conversely, a covariant derivative ∇ on TM determines an affine connection on $L(TM)$. If $(E_1, \ldots, E_n)$ is a local frame and $(\omega^1, \ldots, \omega^n)$ the dual frame, then the corresponding $\mathfrak{gl}_n(\mathbb{R})$-valued one form on U is given by:

$$\gamma^i{}_j = \omega^i(\nabla e_j).$$

Proof. See Proposition III.13. $\square$

IV.B. Torsion and curvature in terms of the covariant derivative ∇

Recall that ω is said to be *torsion free* if the torsion form T vanishes. We will now translate this into a property of the covariant derivative:

Proposition IV.2

Let $\omega = \theta + \gamma$ be an affine connection and ∇ the corresponding covariant derivative. Then ω is torsion-free if and only if:

$$\nabla_X Y - \nabla_Y X = [X, Y], \qquad X, Y \in \Gamma(TM)$$

Proof. $\omega = \theta + \gamma$ is torsion-free means that that $T = d\theta + [\gamma, \theta] = 0$. Let $\sigma : U \to L(TM)$ be a local section, introduce the frame $E_i = \sigma \cdot e_i$ and ω^i the dual frame. Pulling back the equation $T = 0$ using σ, gives:

$$d(\sigma^*\theta) + [\sigma^*\gamma, \sigma^*\theta] = 0.$$

Using Equation (III.5), and writing $(\Gamma^i_j) = \sigma^*\gamma$ the local connection form we get the following system of equations:

$$d\omega^i + \sum_j \Gamma^i_j \wedge \omega^j = 0.$$

Let us calculate:

$$d\omega^i(X, Y) = X(\omega^i(Y)) - Y(\omega^i(X)) - \omega^i([X, Y])$$

Now by definition:

$$\omega^i(\nabla_X Y) = X(\omega^i(Y)) + \Gamma^i_j(X)\omega^j(Y).$$

Hence:

$$d\omega^i(X, Y) = \omega^i(\nabla_X Y - \nabla_Y X - [X, Y]) - \underbrace{(\Gamma^i_j(X)\omega^j(Y) - \Gamma^i_j(Y)\omega^j(X))}_{(\Gamma^i_j \wedge \omega^j)(X,Y)}.$$

Therefore:

$$d\omega^i(X, Y) + (\Gamma^i_j \wedge \omega^j)(X, Y) = \omega^i(\nabla_X Y - \nabla_Y X - [X, Y]).$$

It follows that:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0 \Leftrightarrow d\theta + [\gamma, \theta] = 0.$$

$\square$

We will now investigate how the other part of the curvature of the connection – the curvature form of γ – translates in terms of the covariant derivative. As is apparent from the above, curvature measure commutators of derivatives, let us define the Riemann tensor of the connection:

Proposition IV.3: Riemann tensor

Let $X, Y, Z \in \Gamma(TM)$, define a vector field:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

The expression is $C^\infty(\mathbb{R})$-linear in all variables, and antisymmetric in the first two. It is therefore a $(3,1)$ tensor field on M known as the Riemann tensor.

Proof. Exercice. □

We shall now derive the link between $R(X,Y)Z$ and $R = d\gamma + \frac{1}{2}[\gamma, \gamma]$. Choose a local section $\sigma : U \to L(TM)$ and pull back R to a $\mathfrak{h}$-valued one form on U that we shall denote by (Ω^i_j). Again write $\sigma^*\gamma = (\Gamma^i_j)$. Now for two matrix-valued one-forms α, β, define:

$$(\alpha \wedge \beta)^i_j = \alpha^i_k \wedge \beta^k_j.$$

We can then write:

$$[\sigma^*\gamma, \sigma^*\gamma] = 2(\Gamma \wedge \Gamma).$$

It follows that:

$$\Omega^i_j = d\Gamma^i_j + \Gamma^i_k \wedge \Gamma^k_j.$$

Proposition IV.4

Let σ be a local section of $L(TM)$, $(E_1, \ldots, E_n)$ and $(\omega^1, \ldots, \omega^n)$ the corresponding frame and dual frame, then:

$$\omega^l(R(E_i, E_j)E_k) = \Omega^l_k(E_i, E_j).$$

Proof. Exercice. □

IV.C. Acceleration of a curve

Let $\gamma : I \to M$ be a curve, with the extra structure induced by the connection we can now define a more familiar notion of acceleration.

A smooth vector field along a curve $\gamma : I \to M$ is a smooth map: $V : I \mapsto TM$ such that $\pi \circ V = \gamma$, the canonical lift (see II.F), $\dot{\gamma}$, is a vector field along γ. They form a $C^\infty(I)$ module that we denote $C^\infty_\gamma(M)$.

Remark IV.1. An alternative way of thinking about a vector field along γ is as a section of the pullback of the tangent bundle along γ. This is a vector bundle on I defined by:

$$\gamma^*TM = \{(t,v) \in I \times M, \pi(v) = \gamma(t).\}$$

We now state:

Proposition IV.5

Let $\gamma : I \to M$ be a smooth curve, a covariant derivative ∇ on M induces a unique $\mathbb{R}$-linear operator $\frac{D}{dt}$ on vector fields along curves such that:

1. $\forall t \in I, \frac{D}{dt}(fY)(t) = \dot{f}(t)Y(t) + f(t)\frac{D}{dt}Y(t)$, $f : I \to \mathbb{R}$ smooth,
2. Let $t_0 \in I$, if Z is a vector field on M such that $Z_{\gamma(t_0)} = \dot{\gamma}(t_0)$, X a vector field on M such that on some open neighbourhood J of $t_0 \in I$, $X_{\gamma(t)} = Y(t)$, then:
$$\frac{D}{dt}Y(t) = (\nabla_Z X)_{\gamma(t)}.$$

Sometime we write $\frac{D}{dt}Y = \nabla_{\dot{\gamma}}Y$. If $\sigma : U \longrightarrow L(TM)$ is a local section of the tangent bundle, let $(E_1, \ldots, E_n)$ denote the local frame it determines and consider the local gauge $\sigma^*\gamma = (\Gamma^i_j)$, then writing $Y = Y^i(t)E_i(\gamma(t))$

$$(\nabla_{\dot{\gamma}}Y)(t) = \dot{Y}^i(t)E_i(\gamma(t)) + Y^j(t)(\Gamma^i_j \cdot \dot{\gamma}(t))E_i(\gamma(t)).$$

Remark IV.2. In essence, this is just the pullback of the connection to the pullback bundle.

Definition IV.2

Let $\gamma : I \to M$ be a smooth curve, we define its acceleration to be: $\nabla_{\dot{\gamma}}\dot{\gamma}$. γ is said to be a geodesic if:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0.$$

A vector field Y along the curve γ is said to be parallel along the curve if:

$$\nabla_{\dot{\gamma}}Y = 0.$$

V. Riemannian geometry

Up to now we have developed a curved version of affine geometry built around the homogeneous model G/H where G is the affine group and H the linear group. Euclidean geometry can also be thought of as the study of a homogeneous space G/H, where $G = \mathbb{R}^n \rtimes O(n)$ and $H = O(n)$. There is again a Maurer-Cartan form and one can again interpret G as a frame bundle of Euclidean space. However, the relevant frames for Euclidean geometry are ***orthonormal frames***.

To get a curved version of this we can still base our construction on the frame bundle $L(TM)$, but this is a $GL_n(\mathbb{R})$-bundle and not a $O_n(\mathbb{R})$-bundle. We therefore need to reduce $L(TM)$ to an $O_n(\mathbb{R})$-bundle, which is essentially specifying which frames are orthonormal.

On the model this is achieved by the scalar product; so we can expect that to generalise this to manifolds we should choose a scalar product on each T_pM, this leads to the following definition:

Definition V.1: Riemannian metric

A Riemannian metric on M is a $C^\infty(M)$-bilinear symmetric map:

$$g : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M),$$

such that at each point the induced map $g_p : T_pM \times T_pM \to \mathbb{R}$ is positive-definite.

We call (M, g) a ***Riemannian manifold***. Given this extra structure there is a canonical choice of connection on $L(TM)$

Proposition V.1: Levi-Civita connection

There is a unique ***torsion-free*** covariant derivative ∇ on TM that is compatible with the metric g in the sense that for any vector fields $X, Y, Z \in \Gamma(TM)$:

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Proof. Let us derive a formula for $g(\nabla_X Y, Z)$.

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{V.1}$$

$$Y(g(X, Z)) = g(\nabla_Y X, Z) + g(X, \nabla_Y Z) \tag{V.2}$$

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \tag{V.3}$$

Now add the first two equations and substract the last one, and use that the connection is torsion free to find:

$$\begin{aligned} X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) = 2g(\nabla_X Y, Z) - g([X, Y], Z) \\ + g([X, Z], Y) + g(X, [Y, Z]). \end{aligned}$$

Hence:

$$2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ + g([X,Y],Z) + g([Z,X],Y) + g([Z,Y],X).$$

We leave it as an exercice to show that this formula determines a covariant derivative (and hence a connection) on $L(TM)$. □

Remark V.1. The covariant derivative can be extended in a unique way to tensor fields by imposing the Leibniz rule (with to the tensor product), i.e. define: ∇g by

$$X(g(Y,Z)) = (\nabla_X g)(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

i.e.

$$(\nabla_X g)(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z).$$

The compatibility condition between ∇ and g can then simply be stated as the vanishing of the covariant derivative of g.

Let V be a vector space $S^2V^\vee$ the space of symmetric bilinear forms on V. If $\alpha, \beta \in V^\vee$, define:

$$(\alpha \odot \beta)(v_1, v_2) = \frac{1}{2}\left(\alpha(v_1)\beta(v_2) + \alpha(v_2)\beta(v_1)\right).$$

If $(e_1, \dots, e_n)$ is a basis of V, then: $(e_i \odot e_j)_{i \leq j}$ is a basis of $S^2V^\vee$.

On manifolds, define $S^2(T^\vee M) = \coprod_{p \in M} S^2 T_p^\vee M$ and equip it with a topology and smooth structure as we have done for the other bundles. Any local section $\sigma : U \to L(TM)$, corresponding to vector fields $(E_1, \dots, E_n)$ that at each point $p \in U$ spans T_pM, gives rise to smooth fields: $(\omega^i \odot \omega^j)_{i \leq j}$ that at each $p \in U$ span $S^2 T_p^\vee M$ where (ω^i) is the dual frame. We will often use this to expression the metric locally.

Example V.1. The standard Riemannian metric on $\mathbb{R}^3$ can be written, in the standard global frame:

$$g = dx \odot dx + dy \odot dy + dz \odot dz.$$

It is custom to write: $dx \odot dx = dx^2$. Let us now consider the unit sphere S^2 in $\mathbb{R}^3$, and introduce spherical coordinates according to $x = \cos\phi \sin\theta$, $y = \sin\phi \sin\theta$, $z = \cos\theta$. Then on S^2 we have the relations:

$$dx = -\sin\phi \sin\theta d\phi + \cos\phi \cos\theta d\theta, \\ dy = \cos\phi \sin\theta d\phi + \sin\phi \cos\theta d\theta, dz = -\sin\theta d\theta$$

Restricting g to S^2 we get the standard round metric on S^2:

$$g = d\theta \odot d\theta + \sin^2\theta d\phi \odot d\phi = d\theta^2 + \sin^2\theta d\phi^2.$$

Let us determine the Levi-Civita connection in the local orthonormal frame: $E_1 = \frac{\partial}{\partial\theta}$, $E_2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}$. Let $\omega^1 = d\theta$, $\omega^2 = \sin\theta d\phi$. By definition it is the unique one-form valued matrix (Γ^i_j) such that:

$$d\omega^i + \Gamma^i_j \wedge \omega^j = 0, \qquad \text{(torsion-free)}.$$

The compatibility condition written for $Y = E_i, Z = E_j$ becomes, since $g(E_i, E_j) = \delta_{ij}$.

$$\forall X \in \Gamma(TM), 0 = \Gamma^j_i(X) + \Gamma^i_j(X)$$

i.e. $(\Gamma^i_j(X)) \in \mathfrak{so}_2(\mathbb{R})$. By uniqueness if we can find such a matrix (Γ^i_j) then it is the Levi-Civita connection form. we have: $d\omega^1 = d^2\theta = 0$ and $d\omega^2 = \cot\theta\omega^1 \wedge \omega^2$.

We must have:

$$0 = \Gamma^1_2 \wedge \omega^2,$$
$$\cot\theta\omega^1 \wedge \omega^2 = -\Gamma^2_1 \wedge \omega^1,$$
$$\Gamma^2_1 = \cot\theta\omega^2,$$

determines a $\mathfrak{so}_2(\mathbb{R})$-valued form that solves the torsion free equation, by uniqueness this is the Levi-Civita connection.

VI. Associated vector bundles

To conclude these notes, it seems appropriate to add some remarks that allow us to make the bridge between the Cartan connection point of view we developed and the more classical approach.

The most effective way to jump between invariant objects on $L(TM)$ and M are the notions of equivariant horizontal vector valued forms on $L(TM)$ and associated vector bundles. We start with the latter, let $\pi : P \to M$ be a (smooth) principal H-bundle for some Lie group H and $\rho : H \to GL(V)$ a linear representation of H on a vector space V. One can then construct a vector bundle with fibre V over M, the associated vector bundle, written $P \times_H V$. This is defined as the quotient set of $P \times V$ under the equivalence relation:

$$(p, v) \sim (p', v') \Leftrightarrow \exists h \in H, p' = ph, v' = h^{-1}v.$$

Let us denote an equivalence class by $[p, v]$. The idea behind this is that the couple (p, v) consisting of a frame p and the expression of the vector in the frame p. The vector is then the equivalence class of all couples under a change of frame. The projection, $\tilde{\pi}$ is given by factorisation of the map $(p, v) \to \pi(p)$; this is possible because: $\pi(ph) = \pi(p)$ and so $\pi(p)$ only depends on the equivalence class of $[p, v]$.

If $\phi : \pi^{-1}(U) \to U \times H$ is a local trivialisation, we construct a vector bundle chart by factorisation of:

$$\begin{array}{rcl} \hat{\pi}^{-1}(U) = \pi^{-1}(U) \times V & \longrightarrow & U \times V \\ (p, v) & \longmapsto & (\pi(p), \pi_H(\phi(p))v) \end{array}$$

Where π_H denotes the projection onto the H component in $U \times H$. If we replace (p, v) by $(ph, h^{-1}v)$ then:

$$\pi_H(\phi(ph))h^{-1}v = \pi_H(\phi(p))hh^{-1}v = \pi_H(\phi(p))v.$$

Since $\phi(ph) = \phi(p) \cdot h$. These will be our local vector bundle charts, it is then an exercise to show that equipping $P \times_H V$ with the ***quotient topology***, constructing charts from these in the usual way defines a smooth structure on $P \times_H V$ that makes these smooth vector bundle charts.

Example VI.1. Let $P = L(TM)$ be the frame bundle of a smooth manifold M and ρ the standard representation of $GL_n(\mathbb{R})$ on $V = \mathbb{R}^n$. Then $P \times_H L(TM) \cong TM$. The isomorphism is given by $[u_p, v] \mapsto u_p \cdot v \in T_pM \hookrightarrow TM$, this is a vector space isomorphism between the fibres and therefore a vector bundle isomorphism. All tensor bundles arise in this way as associated bundles to the frame bundle.

We will now use associated vector bundles to relate objects on $L(TM)$ to objects on M. First let us consider the $\mathfrak{h}$-valued one form γ, $\gamma_{u_p} : T_{u_p}P \to \quad \mathfrak{h}$ is no longer an isomorphism and has a kernel. Define $V_{u_p} = \ker \pi_{*u_p}$, where $\pi : P \to M$ is the projection . We shall refer to these vectors as vertical vectors; intuitively these are vectors that point in the fibres of $L(TM)$. Since $\mathrm{im}\gamma_{u_p} = V_p$ by the property $\gamma(X^*) = X$, its kernel is a complementary subspace to V_p that we shall call the horizontal subspace, H_p. Intuitively, this is a choice of subspace that is isomorphic to T_pM in $L_{u_p}(TM)$. Any vector X in $T_{u_p}L(TM)$ therefore has a decomposition $X = X^H + X^V$.

Now consider a representation ρ of H on some vector space V, and say that a V-valued k-form α on $L(TM)$ ***equivariant*** if:

$$R_h^*\alpha = \rho(h)^{-1} \cdot \alpha.$$

We shall say that it is ***horizontal***, if, $\alpha(X_1, \ldots, X_k) = 0$ whenever at least one X_i is a vertical vector field. We shall say that α is tensorial if α is both horizontal and equivariant. It is then possible to show that:

Proposition VI.1

Let ρ be a given representation of H on V, then tensorial k-forms on $L(TM)$ are equivalent to sections of $L(TM) \times_H V$.

Example VI.2.

- The curvature $d\alpha + \frac{1}{2}[\alpha, \alpha]$ is equivariant (with respect to the adjoint representation) and horizontal; the corresponding field on M is the Riemann tensor.
- The connection α is equivariant (with respect to the adjoint representation), but not horizontal.
- The solder form is a horizontal and equivariant one form (with respect to the standard representation).

A. Compactness in second-countable spaces

It is well known that for metric spaces compactness is equivalent to sequential compactness, i.e. the following two notions of compactness coincide:

Definition: Compactness

Let X be a Hausdorff topological space. We say that:

- X is *compact* if satisfies the Heine-Borel property: every cover of X with open sets has a finite subcover.
- X is *sequentially compact* if every sequence has a convergent subsequence.

For second-countable spaces, these notions also coincide. Let $\mathscr{B} = (V_n)_{n\in\mathbb{N}}$ be a countable basis for the topology:

- Assume first compactness, then X is sequentially compact. Recall first that:
  - A family of closed sets $(F_i)_{i\in I}$ in X is said to satisfy the ***finite intersection property*** if for any finite subset $J \subset I$, $\cap_{j\in J} F_j \neq \emptyset$.
  - A Hausdorff topological space X is compact if and if for any family of closed subsets $(F_i)_{i\in I}$ that satisfy the finite intersection property one has $\cap_{i\in I} F_i \neq \emptyset$.
    Indeed, suppose first that there is a family $(F_i)_{i\in I}$ of closed subsets that satisfy the finite intersection property but such that $\cap_{i\in I} F_i = \emptyset$. Then $(X \setminus F_i)_{i\in I}$ is a family of open subsets that has no finite subcover, hence X is not compact. Conversely, suppose that X is not compact, then there is an open cover $(U_i)_{i\in I}$ of X that has no finite subcover, but then $(F_i = X \setminus U_i)_{i\in I}$ is a family of closed sets with the finite intersection property such that $\cap_{i\in I} F_i = X \setminus \cup_{i\in I} U_i = \emptyset$.

  The decreasing family of closed subsets $(F_n)_{n\in\mathbb{N}}$, defined by $F_n = \overline{\{x_m, m \geq n\}}$ for some sequence $(x_n)_{n\in\mathbb{N}}$ clearly satisfies the finite intersection property, hence, X being compact $\cap_{n\in\mathbb{N}} F_n \neq \emptyset$. Choose $x \in \cap_{n\in\mathbb{N}} F_n$ then one can construct a subsequence converging to x in the following manner. Observe that one can construct from $\mathscr{B}$ a non-increasing basis of open neighbourhoods of x, $(V_m^x)_{m\in\mathbb{N}}$; this will play a role similar to open balls in the metric case. We construct an increasing extraction map $\phi : \mathbb{N} \to \mathbb{N}$ by induction. First, for $n = 0$, since V_0^x is a neighbourhood of x there is an infinite number of $n \in \mathbb{N}$ such that $x_n \in V_0^x$, let $\phi(0) = \min\{n \in \mathbb{N}, x_n \in V_0^x\}$. Suppose that we have defined $\phi(k)$ for $k \in [\![0, n]\!]$ such that $\phi(0) < \phi(1) < \cdots < \phi(n)$ and for all $k \in \mathbb{N}$, $x_\phi(k) \in V_k^x$. To define $\phi(n+1)$, using the fact that $\{n \in \mathbb{N}, x_n \in V_{n+1}^x\}$ is infinite one can again choose $\phi(n+1) > \phi(n)$ such that $x_{\phi(n+1)} \in V_{n+1}^x$. $(x_{\phi(n)})_{n\in\mathbb{N}}$

converges to $\mathbb{N}$, because if V is any neighbourhood of x, then there is $n_0 \in \mathbb{N}$ such that $V_{n_0}^x \subset V$, by definition of ϕ for any $n \geq n_0$, $x_{\phi(n)} \in V_n^x \subset V_{n_0}^x \subset V$.

- Assume now sequential compactness and let $(U_i)_{i\in\mathbb{N}}$ be an arbitrary cover of X by open sets.
    1. $(U_i)_{i\in\mathbb{N}}$ has a countable subcover, indeed for every $x \in X, \exists i_x \in I, x \in U_{i_x}$, but one can find n_x such that $V_{n_x} \subset U_{i_x}$. However: $\{n_x, x \in X\} = C$ is countable and $\cup_{c\in C} V_c$ covers X. Moreover by construction, for each $c \in C$, one can find at least one $i(c)$ such that $V_c \subset U_{i(c)}$. $(U_{i(c)})_{c\in C}$ is then a countable subcover.
    2. Countable covers have finite subcovers. Let $(U_n)_{n\in\mathbb{N}}$ be a countable cover of X by open sets. Suppose that it has no finite subcover then for every $n \in \mathbb{N}$ there is $x_n \in X \setminus \cup_{i=1}^n U_i$. By sequential compactness, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence. Let us call the limit $x \in X$; by construction: $x \in X \setminus \cup_{i=1}^n U_i$ for all $n \in \mathbb{N}$. Which contradicts the fact that $(U_n)_{n\in\mathbb{N}}$ covers X. Therefore $(U_n)_{n\in\mathbb{N}}$ has a finite subcover.

B. Tensor algebra, exterior algebra, differential forms (to be completed)

The following is a short review on the notion of tensor algebra, it gives a more abstract foundation to our notion of exterior algebra. We begin with the basic proposition that we shall understand as our definition for tensor products:

Proposition B.1: Tensor product

Let V_1, V_2 be two vector spaces over the same field $\mathbb{K}$, then up to isomorphism there is a unique $\mathbb{K}$-vector space written $V_1 \otimes V_2$ equipped with a bilinear map: $j : V_1 \times V_2 \to V_1 \otimes V_2$ satisfying the following universal property: for any *bilinear* map $b : V_1 \times V_2 \to V_3$, there is a unique *linear* map: $l : V_1 \otimes V_2 \to V_3$ such that the following diagram commutes:

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "b"] \arrow[d, "j"] & V_3 \\ V_1 \otimes V_2 \arrow[ur, "l"] & \end{tikzcd}$$

Proof. See your favourite reference in general algebra; for instance: [Hun74] or [Lan05]. □

Remark B.1.

- The same construction is valid for modules over rings.
- It is custom to write $j(v_1, v_2) = v_1 \otimes v_2$
- Note that elements of $V_1 \otimes V_2$ are *not* all of the form $v_1 \otimes v_2$, they are linear combinations of such products. Indeed, let $l : V_1 \otimes V_2 \to \mathbb{K}$ be a linear map that vanishes on any element of the form $v_i \otimes v_j$, then l fits into the commutative diagram:

  $$\begin{tikzcd} V_1 \times V_2 \arrow[r, "0"] \arrow[d, "j"] & \mathbb{R} \\ V_1 \otimes V_2 \arrow[ur, "l"] & \end{tikzcd}$$

  However, this diagram the null map fits into this same diagram, therefore by uniqueness: $l = 0$.
- If $\dim V_1 = n, \dim V_2 = m$ and $(e_i), (f_j)$ are bases of V_1 and V_2 respectively, then: $(e_i \otimes e_j)$ is a basis for $V_1 \otimes V_2$.

- $\mathbb{K} \otimes V \cong V$ (map $v \in V$ to $1 \otimes v$)

The tensor product of two spaces solves the problem of exchanging bilinear maps for linear maps. As an exercise let us prove that $\otimes$ is associative in the following sense: $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$. Using the universal property there is a unique linear map $L : (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3)$ such that $(v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3)$ for all $(v_1, v_2, v_3) \in V_1 \times V_2 \times V_3$. It is the required isomorphism, indeed construct in the same way the map $\tilde{L} : V_1 \otimes (V_2 \otimes V_3) \to (V_1 \otimes V_2) \otimes V_3$ such that $v_1 \otimes (v_2 \otimes v_3) \mapsto (v_1 \otimes v_2) \otimes v_3$ then $L \circ \tilde{L}$ fits into the diagram:

$$\begin{tikzcd}
V_1 \otimes (V_2 \otimes V_3) \arrow[r, "j"] \arrow[d, "j"] & V_1 \otimes (V_2 \otimes V_3) \\
V_1 \times (V_2 \otimes V_3) \arrow[ur, "L\circ\tilde{L}"] &
\end{tikzcd}$$

But this diagram is also satisfied by id, hence, by uniqueness $\tilde{L} \circ L = \mathrm{id}$. Inversing the roles of the spaces we also get $L \circ \tilde{L} = \mathrm{id}$.

We are particularly interested in the case where $V_1 = V_2$, in this case we will write $V \otimes V = \otimes^2 V$, then define by induction: $\otimes^n V = V \otimes (\otimes^{n-1} V)$ and set:

$$\mathcal{T}(V) = \bigoplus_{k=0}^{+\infty} \otimes^k V,$$

where $\otimes^0 V = \mathbb{K}$ and $\otimes^1 V = V$. This is an associative unit algebra, equipped with multiplication given by $\otimes$, the unit is $1 \in \mathbb{K}$ known as the *tensor algebra*. Note that it is *not* commutative !

The tensor algebra is in some sense the most general associative unit algebra one can construct from a vector space V; it is therefore a powerful tool in constructing other associative algebras. To illustrate this, we will now use it to construct the *exterior algebra*.

Consider $\mathcal{I}$ the bilateral (e.g. left and right) ideal generated by elements of the form $v \otimes v, v \in V$ we define:

$$\bigwedge V = \mathcal{T}(V)/\mathcal{I}.$$

We will note multiplication in $\bigwedge V$ by $\wedge$, it is generated by elements of v. By construction if $v \in V, v \wedge v = 0$ and $0 = (v_1 + v_2) \wedge (v_1 + v_2) = v_1 \wedge v_2 + v_2 \wedge v_1$. Note that the grading of the tensor algebra passes to $\wedge V$ and we have: $\bigwedge = \bigoplus_{k=1}^{n} \bigwedge^k V$ where $\bigwedge^k V$ is the vector space spanned by elements of the form $v_1 \wedge \cdots \wedge v_k$, $v_1, \ldots, v_k \in V$. Note that by construction: $v_1 \wedge \cdots \wedge v_k$ vanishes if $(v_1, \ldots, v_k)$ is a linearly dependent family, it follows immediately that if $\dim V = n$, then the exterior algebra is finite dimensional. $\bigwedge^k V$ satisfies the following universal property, for any $u : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to W$, k-multilinear, alternated, there is a unique linear map $\tilde{u} : \bigwedge^k V \to W$ such that if j is the map: $(v_1, \ldots, v_k) \mapsto v_1 \wedge \cdots \wedge v_k$ then $u = \tilde{u} \circ j$. (This can be shown using the universal property of $\otimes^k V$ and taking the quotient). With this in mind we can show that if $(e_1, \ldots, e_k)$ is a basis of V then,

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k}, 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

is a basis of $\bigwedge^k V$. Suppose now that $V = E^*$ for some vector space E. Recall that a k-form on E is $\mathbb{K}$-valued k-multilinear alternated map.

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Glossary

Basis (Topology) A collection $\mathscr{B}$ of open subsets of a topological space X such that any open subset is a union of elements of $\mathscr{B}$. A collection of open subsets $\mathscr{B}$ is a basis if and only if for any $x \in X$ and any open subset U such that $x \in U$, one can find $B \in \mathscr{B}$ such that $x \in \mathscr{B} \subset U$. Example: open balls in metric spaces.

Bump function A bump function is a smooth map $f : M \to \mathbb{R}$ with compact support. For any $n \in \mathbb{N}^*$, one can find bump functions $f : \mathbb{R}^n \to \mathbb{R}$. Indeed, first consider: $f : \mathbb{R} \to \mathbb{R}_+$ defined by:

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \leq 0 \end{cases}.$$

f is of class C^∞; now define, $g : \mathbb{R}^n \to \mathbb{R}$ by $g(x) = f(1 - ||x||^2)$. Where $|| \cdot ||$ is the Euclidean norm on $\mathbb{R}^n$; it follows that g is smooth with support $\overline{B(0,1)} = \{x \in \mathbb{R}^n, ||x|| \leq 1\}$. See also *Uryhson's lemma*.

Closed graph theorem Let E, F be Banach spaces and $u \in L(E, F)$ then u is continuous if and only if its graph:

$$\mathscr{G} = \{(x, f(x)), x \in E\}$$

is closed in $E \times F$.

Comparison of topologies Let $\mathscr{T}_1, \mathscr{T}_2$, be two topologies on the same set X, we say that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$, written: $\mathcal{T}_2 \subset \mathcal{T}_1$ if $U \in \mathcal{T}_1 \Rightarrow U \in \mathcal{T}_2$. In this case $\mathcal{T}_2$ is also said to be coarser than $\mathcal{T}_1$. The analogy is that of grains of sand, the finer the topology the more "grains of sand" i.e. open sets.

Functor This is a general notion from *Category theory*, where mathematicians do a lot of abstract nonsense. Jokes aside, on a basic very level, I find it very useful to organise general ideas about mathematics that are common to many different areas and extract the essential information in certain constructions, in the form of ***universal properties***. In brief, a category $\mathcal{C}$ is a collection of mathematical objects, a collection of disjoint sets $\mathrm{Mor}(O_1, O_2)$ of morphisms (or arrows) where O_1, O_2 are two objects, and, for any three objects O_1, O_2, O_3 a composition law:

$$\begin{array}{ccc} \mathrm{Mor}(O_2, O_3) \times \mathrm{Mor}(O_1, O_2) & \longrightarrow & \mathrm{Mor}(O_1, O_3) \\ (g, f) & \longmapsto & g \circ f, \end{array}$$

subject to the conditions:

1. $\circ$ is associative and for each object O_1 of $\mathcal{C}$,
2. there is a morphism: $\mathrm{id}_{O_1} : O_1 \to O_1$ which acts as a left and right identity under the composition law.

The class of all sets is a category where morphisms are maps between sets, topological vector spaces form a category where morphisms are continuous linear maps, smooth manifolds form a category where the morphisms are smooth maps, Banachisable spaces form a category where morphisms are continuous linear maps, etc. A functor between categories $\mathcal{C}_1$ and $\mathcal{C}_2$, is a mapping between categories that:

- maps any object O_1 of $\mathcal{C}_1$ to an object $F(O_1)$ of $\mathcal{C}_2$
- maps any morphism $f : O_1 \to O_2$ to a morphism $F(f) : F(O_1) \to F(O_2)$, such that $F(\mathrm{id}_{O_1}) = \mathrm{id}_{F(O_1)}$ and $F(f \circ g) = F(f) \circ F(g)$.

Maps between categories that satisfy all the above conditions except that they reverse the order of composition, i.e. $F(f \circ g) = F(g) \circ F(f)$ are called *contravariant*-functors. They are functors in the above sense on the *opposite category*. See [Hun74, Chapter I.7, Chapter X] for a more detailed account. The map that maps a Banach space to its dual is a contravariant functor.

Homeomorphism An isomorphism in the category of topological spaces, i.e. a continuous bijection with continuous inverse.

Inverse function theorem Let E and F be two Banach spaces and $f : U \to E \to F$ be a C^k map, $k \geq 1$, defined on an open subset U of E. Let $x_0 \in U$, suppose that $f'(x_0) \in GL(E, F)$ then are open neighbourhoods V and W of a and $f(a)$ respectively, such that the restriction of $f|_V : V \to W$ is a C^k diffeomorphism. It extends naturally to manifolds.

Locally compact topological space A Hausdorff topological space such that every point has a neighbourhood with compact closure (i.e. relatively compact).

Modules and bimodules A R-*module* M is to a ring R, what a k-vector space is to field k, i.e. $(M, +)$ is an abelian group endowed with external multiplication by elements of R, i.e. a map: $(r, m) \in R \times M \mapsto r \cdot m$ satisfying the same rules as external multiplication in a vector space. When multiplication is written on the left, M is said to be a left-module. One could also define external multiplication on the right, i.e. a map: $(r, m) \in R \times M \mapsto m \cdot r$. M is then said to be a right-module. A priori, the two structures are different since, for a right-module, we have:

$$m \cdot (r_1 r_2) = (m \cdot r_1) \cdot r_2.$$

If we defined: $r \cdot m = m \cdot r$ then this property would translate to:

$$(r_1 r_2) \cdot m = r_2 \cdot (r_1 \cdot m).$$

If R is commutative the difference is purely notational. A *bimodule* is a module equipped with both left and right multiplication..

Open map A continuous map f such that $f(U)$ is open whenever U is open.

Open mapping theorem Let E, F be Banach spaces and $u \in L(E, F)$ a continuous surjective map, then u is an open map. **Corollary**: If $u \in L(E, F)$ is a bijective continuous linear map between *Banach spaces* then u^{-1} is bounded. See [Lax02, Chapter 15, Theorem 11].

Paracompact topological space A Hausdorff space with the property that every open covering admits a *locally finite* refinement. Guarantees the existence of partitions of unity. All metric spaces are in fact paracompact [Rud69].

Regular space A topological Hausdorff space X in which one can separate points and closed sets with open sets. i.e. if $x \in X$, $F \subset X$ closed and $x \notin F$ then there are disjoint open sets U and V such $x \in U$, $F \subset V$. This is equivalent to the statement that for any $x \in X$ and any open subset U of X, one can find an open subset V such that $x \in V \subset \overline{V} \subset U$.

Separable space A topological space with a countable dense subset.

The Einstein summation convention When it is convenient and no ambiguity can arise, I will use the Einstein summation convention which states that one should sum over repeated indices, generally one "up", one "down". For instance: $\sum_i x^i e_i \equiv x^i e_i$.

The subspace topology Let $(X, \mathscr{T})$ be at topological space, $A \subset X$, then the topology of X induces a natural topology on A, defined by:

$$\mathcal{T}_A = \{U \cap A, U \in \mathcal{T}\}.$$

It is the initial topology of the inclusion map.

Topological group A group $(G, \cdot)$ equipped with a topology compatible with its group structure, i.e. the maps $\mu : G \times G \to G$, $\iota : G \to G$ defined by:

$$\mu(g_1, g_2) = g_1 \cdot g_2, \quad \iota(g) = g^{-1}, \qquad g_1, g_2, g \in G,$$

are continuous.

Topological space A topology on a set X is a collection $\mathscr{T}$ of subsets of X that satisfies the following stability conditions:

1. $\emptyset, X \in \mathscr{T}$,
2. Arbitrary unions of elements of $\mathscr{T}$ are elements of $\mathscr{T}$,
3. Finite intersections of elements of $\mathscr{T}$ are elements of $\mathscr{T}$.

Elements of $\mathscr{T}$ are called *open sets*. A topology is the minimum set of data required to talk about limits, neighbourhoods, continuous maps, etc.

French-English Dictionary

arbitrary quelconque.

bundle fibré.

chain-rule règle de composition des dérivées.

chart carte.

closure fermeture, adhérence.

coarsest moins fin.

countable dénombrable.

data données.

decreasing strictement décroissant.

factor (to) verbe: factoriser, nom: facteur (dans un produit).

Hausdorff space espace topologique séparé.

hence d'où.

increasing strictement croissant.

induction récurrence.

isotropy subgroup stabilisateur.

manifold variété.

map application.

neighbourhood voisinage.

non-decreasing croissant.

non-increasing décroissant.

one to one injectif.

one to one correspondence bijection.

onto surjectif.

quote (to) citer.

record (to) verbe: enregistrer.

regular regulier.

second-countable à base dénombrable d'ouverts. On parle aussi du deuxième axiome de dénombrabilité.

sequence suite.

smooth lisse.

span espace vectoriel engendré par, vect.

split scindée (en parlant d'une suite exacte courte).

submanifold sous-variété.

subsequence sous-suite.

subspace topology topologie induite.