"Equivalents"

⚠ THIS WILL <u>NOT</u> BE COVERED IN THE LECTURES
IT IS A TOOL YOU MAY USE TO HELP YOU
FIND A SERIES TO COMPARE TO.

IF YOU USE THIS I RECOMMEND YOU APPLY THE
LIMIT TEST AFTERWARDS TO CHECK YOU
HAVEN'T MADE A MISTAKE.

I – Introduction

Consider a series $\left(\sum_{n\geq 0} a_n\right)$, since, in general,
we cannot compute the partial sums $S_N = \sum_{n=0}^{N} a_n$,
we are reduced to trying to deduce
convergence of $\sum_{n\geq 0} a_n$ from properties

of the general term $(a_n)_{n\in\mathbb{N}}$.

We have seen, that if $\sum a_n$ converges then $\lim_{n\to+\infty} a_n$, but this is not enough in general. (Recall $\sum_{n\geq 1}\frac{1}{n}$ diverges!) What seems to matter is how fast (a_n) is converging to 0. $\left(\frac{1}{n}\right)$ is too slow but $\left(\frac{1}{n^\alpha}\right)_{\alpha>1}$ is fast enough.

We have also learnt that our main tool is to compare series.

Recall the main (and only underlying idea) is:

if $\quad 0\leq a_n \leq b_n$

and $\left|\begin{array}{l} \sum b_n \text{ converges then } \sum a_n \text{ converges} \\ \\ \sum a_n \text{ diverges then } \sum b_n \text{ diverges.} \end{array}\right.$

So to compare series we want to compare the behaviour of the general term.

"Equivalents" give a rigorous meaning to the idea "a_n" and "b_n" have the same behavior when $n \to +\infty$.

II The Definition.

This is the mathematical part, bear with me, its important because it gives you the rules.

Def We say that two sequences (a_n) and (b_n) are equivalent, if for every $\varepsilon \in \mathbb{R}_+^*$, there is $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, we have the estimate:

$$|a_n - b_n| \leq \varepsilon |b_n|$$

Remark, if $b_n \neq 0$ for $n \geq n_0$.

This means $$\left|\frac{a_n}{b_n} - 1\right| \leq \varepsilon$$

ie $$\lim_{n \to +\infty} \frac{a_n}{b_n} = 1$$

Prop If $\lim_{n \to +\infty} \frac{a_n}{b_n} = 1$ then $a_n \underset{n \to +\infty}{\sim} b_n$.

<u>Example</u>: $n^3 + 2n^2 + n \underset{n\to+\infty}{\sim} n^3$

Indeed $n^3 + 2n^2 + n = n^3 \left(1 + \underbrace{\frac{2}{n} + \frac{1}{n^2}}_{\underset{n\to+\infty}{\longrightarrow} 0} \right)$

So $\lim_{n\to+\infty} \frac{n^3 + 2n^2 + n}{n^3} = 1$.

In general any polynomial expression in n

is equivalent to the term of highest degree.

ie. $a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0 \underset{n\to+\infty}{\sim} a_k n^k$

<u>Example 2</u>: If (b_n) converges to a real $L \neq 0$

then $b_n \underset{n\to+\infty}{\sim} L$.

⚠ If a sequence (a_n) converges to 0 then it is not true in general that $\cancel{a_n \underset{n\to+\infty}{\sim} 0}$.

In fact for a sequence to be equivalent to 0 it must be constant and equal to 0 after a finite number of terms.

$\Rightarrow$ Do not replace a sequence converging to 0 by 0 when computing equivalents. Equivalents are all about finding out how (a_n) converges to 0

SHORT VERSION

$\Rightarrow$ Never write $a_n \underset{n\to+\infty}{\sim} 0$.

III - Rules of computation

It turns out that this definition is "good" in the sense that there is an associated calculus.

Here are the rules:

① $a_n \underset{n\to+\infty}{\sim} b_n \iff b_n \underset{n\to+\infty}{\sim} a_n$

② $\begin{cases} a_n \underset{n\to+\infty}{\sim} b_n \\ b_n \underset{n\to+\infty}{\sim} c_n \end{cases} \implies a_n \underset{n\to+\infty}{\sim} c_n$

③ $a_n \underset{n\to+\infty}{\sim} a_n$

This means that "$\underset{n\to+\infty}{\sim}$" behaves like "$=$".

Let (a_n) be a sequence then :

④ $c_n \underset{n\to+\infty}{\sim} d_n$ then $a_n c_n \underset{n\to+\infty}{\sim} a_n d_n$

We can <u>multiply</u> equivalents. !!

Now if $a_n \underset{n\to+\infty}{\sim} b_n$ and $c_n \underset{n\to+\infty}{\sim} d_n$

$a_n c_n \underset{n\to+\infty}{\sim} a_n d_n$

$a_n d_n \underset{n\to+\infty}{\sim} b_n d_n$

so $a_n c_n \underset{n\to+\infty}{\sim} b_n d_n$

We can also deduce that if $a_n \neq 0$, $b_n \neq 0$ after a finite number of terms then :

$$a_n \underset{n\to+\infty}{\sim} b_n \Rightarrow \frac{1}{a_n} \underset{n\to+\infty}{\sim} \frac{1}{b_n}.$$

Exponential

(5) If $a_n \underset{n\to+\infty}{\sim} b_n$ $\boxed{\text{and } \lim_{n\to+\infty} a_n - b_n = 0}$

then $e^{a_n} \underset{n\to+\infty}{\sim} e^{b_n}$

Proof: $\dfrac{e^{a_n}}{e^{b_n}} = e^{a_n - b_n}$

Since $a_n - b_n \xrightarrow[n\to+\infty]{} 0$ $\dfrac{e^{a_n}}{e^{b_n}} \xrightarrow[n\to+\infty]{} 1$

So $e^{a_n} \underset{n\to+\infty}{\sim} e^{b_n}$

(6) Some other functions

- Suppose $\lim_{n\to+\infty} a_n = 0$ $\begin{cases} \ln(1+a_n) \underset{n\to+\infty}{\sim} a_n \\ \sin(a_n) \underset{n\to+\infty}{\sim} a_n \end{cases}$

- If $\boxed{\lim_{n\to+\infty} a_n = 0}$ and $a_n \underset{n\to+\infty}{\sim} b_n$

then $\ln(a_n) \underset{n\to+\infty}{\sim} \ln(b_n)$

⚠ We cannot in general "take functions" of equivalence.

⚠ Summation of equivalences does not work.

$$\begin{cases} a_n \underset{n\to+\infty}{\sim} b_n \\ c_n \underset{n\to+\infty}{\sim} d_n \end{cases}$$ then it is NOT in general true that $a_n + c_n \underset{n\to+\infty}{\sim} b_n + d_n$

e.g. $a_n = n$ $\quad a_n \underset{n\to+\infty}{\sim} n$ BUT $a_n + b_n \underset{n\to+\infty}{\sim} 1$

$b_n = -n + 1$ $\quad b_n \underset{n\to+\infty}{\sim} -n$

DON'T ADD EQUIVALENCES

IV - Applications

Thm If $a_n \underset{n\to+\infty}{\sim} b_n$ then $\sum a_n$ converges if and only if $\sum b_n$ converges.

N.B. This is just a restatement of the limit test.

Examples $$\sum_{n\geq 0} \frac{n^3+2n+1}{n^{10}+3n+1}$$

$$\frac{n^3+2n+1}{n^{10}+3n+1} \underset{n\to+\infty}{\sim} \frac{1}{n^7}$$ so $\sum \frac{n^3+2n+1}{n^{10}+3n+1}$ converges

because $\sum \frac{1}{n^7}$ converges.

$$\sum_{n \geq 1} \frac{\ln\left(1+\frac{1}{n^3}\right) \sin\left(\frac{1}{n}\right)}{n^5}$$

Since $\begin{cases} \frac{1}{n^3} \xrightarrow[n\to+\infty]{} 0 \\ \frac{1}{n} \xrightarrow[n\to+\infty]{} 0 \end{cases}$ $\quad \ln\left(1+\frac{1}{n^3}\right) \underset{n\to+\infty}{\sim} \frac{1}{n^3}$

$$\sin\left(\frac{1}{n}\right) \underset{n\to+\infty}{\sim} \frac{1}{n}$$

therefore : $\dfrac{\ln\left(1+\frac{1}{n^3}\right) \sin\left(\frac{1}{n}\right)}{n^5} \underset{n\to+\infty}{\sim} \underbrace{\frac{1}{n^9}}_{\text{general term of a convergent series}}$

Calculate $\lim\limits_{n\to+\infty} \left(1+\frac{x}{n}\right)^n$, $x \in \mathbb{R}$ <u>constant</u>

$$\left(1+\frac{x}{n}\right)^n = e^{n \ln\left(1+\frac{x}{n}\right)}$$

Since $\frac{x}{n} \xrightarrow[n\to+\infty]{} 0$ $\quad \ln\left(1+\frac{x}{n}\right) \underset{n\to+\infty}{\sim} \frac{x}{n}$

$$n\ln\left(1+\frac{x}{n}\right) \underset{n\to+\infty}{\sim} x$$

This means that $\lim\limits_{n\to+\infty} n\ln\left(1+\frac{x}{n}\right) = x$

in particular $\lim\limits_{n\to+\infty} \left(n\ln\left(1+\frac{x}{n}\right) - x\right) = 0$

therefore, $\left(1+\frac{x}{n}\right)^n = e^{n\ln\left(1+\frac{x}{n}\right)} \underset{n\to+\infty}{\sim} e^x$.

In otherwords $\lim\limits_{n\to+\infty} \left(1+\frac{x}{n}\right)^n \underset{n\to+\infty}{\sim} e^x$.

Notice that although $1+\frac{x}{n} \underset{n\to+\infty}{\sim} 1$

$$\left(1+\frac{x}{n}\right)^n \cancel{\underset{n\to+\infty}{\sim}} 1^n = 1$$

$\Rightarrow$ HANDLE WITH CARE.

Equivalences can be used to calculate limits:

Proposition

if $a_n \underset{n\to+\infty}{\sim} b_n$ and $\lim\limits_{n\to+\infty} a_n = L$

where $L \in \mathbb{R} \cup \{+\infty, -\infty\}$

then $\lim\limits_{n\to+\infty} b_n = L$.

Proof: Suppose L is finite; then (a_n) is bounded. Let $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, $|a_n - L| \leq \varepsilon/2$

There is also $n_1 \in \mathbb{N}$ such that whenever $n \geq n_1$, $|b_n - a_n| \leq \frac{\varepsilon |a_n|}{2M} \leq \frac{\varepsilon}{2}$

now $|b_n - L| \leq \underbrace{|b_n - a_n|}_{\varepsilon/2} + \underbrace{|a_n - L|}_{\leq \varepsilon/2} \leq \varepsilon$

whenever $n \geq \max(n_0, n_1)$, so $\lim\limits_{n\to+\infty} b_n = L$

If $\lim_{n\to+\infty} a_n = +\infty$

Let $A>0$, there is $n_0 \in \mathbb{N}$ such that

when $n \geq n_0$, $a_n \geq 2A > 0$ Now one can

also find $n_1 \in \mathbb{N}$, such that $|b_n - a_n| \leq \frac{|a_n|}{2}$

so when $n \geq \max(n_1, n_0)$

$$b_n - a_n \geq -\frac{|a_n|}{2} = -\frac{a_n}{2}$$

so $b_n \geq \frac{a_n}{2} \geq A.$

The proof is similar if $L = -\infty$.

Example ① $n \sin\left(\frac{1}{n}\right) \underset{n\to+\infty}{\sim} n \times \frac{1}{n} \underset{n\to+\infty}{\sim} 1$

so $\lim_{n\to+\infty} n\sin\left(\frac{1}{n}\right) = 1$

② $\frac{n^3+2n+1}{3n^3+50n+225} \underset{n\to+\infty}{\sim} \frac{1}{3}\frac{n^3}{n^3} \underset{n\to+\infty}{\sim} \frac{1}{3}$

$\lim_{n\to+\infty} \frac{n^3+2n+1}{3n^3+50n+225} = \frac{1}{3}.$

V - Concluding remarks

- There is a very famous formula, known as Stirling's formula, that gives an equivalent for $n!$

Theorem

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

This gives a very good approximation of $n!$ for very large n.

- My advice: if you feel that this makes sense to you, use it in your draft book (so use the known equivalents above) but check your answer using the limit test.

If you think this could be useful to you and want to practice try to show:

Ex 1: $$\frac{n \ln\left(1+\sin^2\left(\frac{1}{n}\right)\right)}{\sin\left(\frac{1}{n}\right)} \underset{n\to+\infty}{\sim} 1$$

Ex 2: $$n^3 \ln\left(1+\frac{2}{n}\right)\ln\left(1+\frac{3}{n}\right) \underset{n\to+\infty}{\sim} 6n$$

Ex 3 $$\sqrt{n^3+2n^2+1} \underset{n\to+\infty}{\sim} n^{3/2}$$ (calculate $\lim_{n\to+\infty} \frac{\sqrt{n^3+2n^2+1}}{n^{3/2}}$)

Ex 4 $$n(\ln(n+1)-\ln n) \underset{n\to+\infty}{\sim} 1$$

Ex 5 $$e^{-n^2+\frac{1}{n}} - e^{-n^2} \underset{n\to+\infty}{\sim} \frac{e^{-n^2}}{n}$$

Hint:

(use that: if $a_n \to 0$ then $e^{a_n}-1 \underset{n\to+\infty}{\sim} a_n$)

can you see why this one works? cf l'Hospitals rule.

Ex 6 $$\frac{(e^{\frac{1}{n}}-1)\sin\frac{1}{n}}{n^2+n^3} \underset{n\to+\infty}{\sim} \frac{1}{n^5}$$

Ex 7 $$\ln\left(\ln\left(e+\frac{1}{n}\right)\right) \underset{n\to+\infty}{\sim} \frac{1}{en}$$

Ex 8 Using l'Hospital's rule : show that

$$\lim_{x\to 0} \frac{\tan x}{x} = 1 \quad , \quad \text{show that}$$

$$\tan\left(\sin\left(\frac{1}{\sqrt{n^2+3}}\right)\right) \underset{n\to+\infty}{\sim} \frac{1}{n}$$