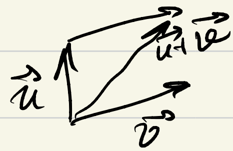
Vector space basics

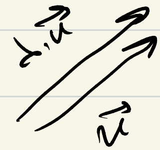
Def A (real) vector space is a set E

endowed with two operations :

→ $\boxed{\text{internal}}$ addition $+$



→ multiplication by a scalar $\lambda \in \mathbb{R}$



subject to the following long list of conditions
(that summarise completely all that you are used to.)

Properties of $+$ $\qquad \vec{x}+\vec{y} \in E$ (the sum of vectors is a vector)

- There is a $\vec{0}$ element such that $\vec{x}+\vec{0} = \vec{x}$ for any $\vec{x}$.

- $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$ for all $\vec{x},\vec{y}$

- For every $\vec{x} \in E$ there is an element $(-\vec{x})$ such that $\vec{x}+(-\vec{x}) = \vec{0}$.

- $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (commutative)

(Mathematicians summarise this by: $(E, +)$ is an abelian group).

Properties of $\cdot$

- $\cdot : \lambda \times E \longrightarrow E$
  $(\lambda, \vec{x}) \longmapsto \lambda\vec{x}$ } scaling of a vector.

- $\underbrace{\lambda}_{\in \mathbb{R}} \cdot \underbrace{(\mu \cdot \vec{x})}_{\in E} = \underbrace{\underbrace{(\lambda\mu)}_{\in \mathbb{R}} \cdot \underbrace{\vec{x}}_{\in E}}_{\in E}$ — multiplication in $\mathbb{R}$

- $1 \cdot \vec{x} = \vec{x}$

- $\lambda \cdot (\vec{x} + \vec{y}) = \lambda \cdot \vec{x} + \lambda\vec{y}$ (Distributivity)

- $(\lambda + \mu) \cdot \vec{x} = \lambda\vec{x} + \mu \cdot \vec{x}$

Example $\mathbb{R}^n$ with the following operations

- $(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$
- $\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$.

Let $(\vec{e_i})_{i\in I}$ an arbitrary family of vectors.

Let $(\lambda_i)_{i\in I}$ be a family of reals such that

$\lambda_i = 0$ for all $i \in I$ except a finite number.

We call $\underbrace{\sum_{i\in I}}_{\text{this is a finite sum}} \lambda_i \vec{e_i}$ a linear combination

of the vectors $\vec{e_i}$.

We define $\mathbb{R}^{(I)}$ to be the set composed of

all families $(\lambda_i)_{i\in I}$ such that $\lambda_i = 0$

for all but a finite number of indices.

It is a vector space, with the operations.

$$\begin{cases} (\lambda_i)_{i\in I} + (\mu_i)_{i\in I} = (\lambda_i + \mu_i)_{i\in I} \\ \alpha \cdot (\lambda_i)_{i\in I} = (\alpha \lambda_i)_{i\in I} \end{cases}$$

The zero vector is the family such that $\alpha_i = 0$ for all $i \in I$.

Linear maps / transformations.

Let E, F be two vector spaces, a map $u : E \longrightarrow F$ is said to be linear if it preserves the vector space structure:

i.e.: for every $\vec{x}, \vec{y} \in E$, $\alpha \in \mathbb{R}$

$$u(\alpha \vec{x} + \vec{y}) = \underbrace{\alpha u(\vec{x}) + u(\vec{y})}_{\in F}.$$

Fundamental example Let $(e_i)_{i\in I}$ be an arbitrary family of vectors (elements of E)

then the map:

$$\boxed{\begin{aligned}\phi : \mathbb{R}^{(I)} &\longrightarrow E \\ (\alpha_i)_{i\in I} &\longmapsto \sum_{i\in I} \alpha_i e_i\end{aligned}}$$

is linear.

Def ① If Φ is onto /surjective then we say that $(e_i)_{i\in I}$ generates E

② If Φ is one-to-one (injective) then $(e_i)_{i\in I}$ is said to be free.

③ If Φ is a one-to-one correspondence (bijective) then $(e_i)_{i\in I}$ is said to be a BASIS.

Theorem ① The cardinal of any basis of E is the same and is called the dimension of E.

② Every vector space has a basis.

The dimension represents the number of independant directions in E

Example: $\mathbb{R}^n$ is of dimension n.

It has the canonical basis

$$e_1 = (1,0,\dots,0) \ , \ e_2 = (0,1,0,\dots,0) \ \dots$$

$$\dots \ e_i = (0,\dots,0,\underset{\substack{\uparrow \\ i^{th}\text{ position}}}{1},0,\dots,0) \ , \ e_n = (0,\dots,0,1)$$

Matrices and linear maps

Let E and F be two finite dimensional vector spaces of dimensions $\begin{cases} \dim E = n \\ \dim F = m \end{cases}$

Let $(\vec{e_1}, \dots, \vec{e_n})$ be a basis of E

$(\vec{f_1}, \dots, \vec{f_m})$ be a basis of F.

By definition of a basis every vector $\vec{x} \in E$ can be written uniquely as a linear combination of the basis elements.

$$\vec{x} = \sum_{i=1}^{n} \underbrace{x_i}_{\in \mathbb{R}} \vec{e_i}$$

The real numbers (x_i) are the coordinates of $\vec{x}$ in the basis $(\vec{e_1}, \dots, \vec{e_n})$

Therefore : $$\underbrace{u(\vec{x})}_{\in E} = \sum_{i=1}^{n} x_i \underbrace{u(\vec{e_i})}_{\in F}$$ by

linearity. So the map u is completely determined by the vectors $u(\vec{e_i})$.

Since they are elements of F they can b written uniquely :

$$\boxed{u(\vec{e_i}) = \sum_{j=1}^{m} M_{ji} \vec{f_j}}$$

but then :

$$u(\vec{a}) = \sum_{i=1}^{n} x_i \cdot \left(\sum_{j=1}^{m} M_{ji} \vec{f_j} \right)$$

$$\boxed{u(\vec{a}) = \sum_{j=1}^{m} \vec{f_j} \left(\sum_{i=1}^{n} M_{ji} x_i \right)}$$

So given the two bases we know everything about the transformation u if we know the numbers $(M_{ji})_{(j,i)\in\{1,\dots,n\}^2}$

We call $M=(M_{ji})_{(j,i)}$ the matrix of u

in the bases $(\vec{e_1},\dots,\vec{e_n})$ of E and $(\vec{f_1},\dots,\vec{f_q})$ of F.

We write:

$$\underbrace{\begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \\ M_{m1} & \cdots & M_{mn} \end{pmatrix}}_{n \text{ columns}} \left.\vphantom{\begin{pmatrix} M \\ \vdots \\ M \end{pmatrix}}\right\} m \text{ rows}$$

If $\vec{x} = \sum_{i=1}^{n} x_i e_i$ we call

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

the coordinate column vector of $\vec{x}$ in the basis $(\vec{e_1},\dots,\vec{e_n})$.

From the above, we see that the coordinate column vector of $u(\vec{x})$ in the basis $(\vec{f_1}, \dots, \vec{f_m})$ is given by

$$\underbrace{\begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \\ M_{m1} & \cdots & M_{mn} \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{X}$$

the matrix M depends on the choice of bases in E and F.

Example Consider $\mathbb{R}^3 = E$ with basis
$\vec{i} = (1,0,0)$ $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$

$F = \mathbb{R}^2$ with basis $\vec{e_1} = (1,0)$
$\vec{e_2} = (0,1)$

Define a <u>linear</u> map by :

$$u(\vec{i}) = 2\vec{e_1} + 3\vec{e_2}$$

$$u(\vec{j}) = \frac{1}{2}\vec{e_2}$$

$$u(\vec{k}) = \vec{e_1} - \vec{e_2}$$

The matrix M of u in the bases $(\vec{i}, \vec{j}, \vec{k})$ of $\mathbb{R}^3$ and $(\vec{e_1}, \vec{e_2})$ of $\mathbb{R}^2$ is :

$$M = \begin{pmatrix} 2 & 0 & 1 \\ 3 & \frac{1}{2} & -1 \end{pmatrix}$$

coordinates of $u(\vec{i})$

coordinates of $u(\vec{j})$

coordinates of $u(\vec{k})$

If $\vec{x} = \vec{i} + 2\vec{j} - \vec{k}$

then the coordinate vector of $\vec{x}$ in the basis $\vec{\imath}, \vec{\jmath}, \vec{k}$ is

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

the coordinate vector of $u(\vec{x})$ in the basis $(\vec{e_1}, \vec{e_2})$ of $\mathbb{R}^2$ is.

$$\begin{pmatrix} 2 & 0 & 1 \\ 3 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

so $\vec{u}(\vec{a}) = \vec{e_1} + 5\vec{e_2}$

To check we can calculate directly:

$\vec{u}(\vec{x}) = \vec{u}(\vec{\imath} + 2\vec{\jmath} - \vec{k})$

$$= \vec{u}(\vec{i}) + 2 \cdot \vec{u}(\vec{j}) - \vec{u}(\vec{k})$$

$$= (2\vec{e_1} + 3\vec{e_2}) + 2 \cdot \left(\frac{1}{2}\vec{e_2}\right) - (\vec{e_1} - \vec{e_2})$$

$$= \vec{e_1} + 5\vec{e_2}.$$

<u>Remark</u> Matrix multiplication is defined so that this works!

<u>Prop</u> if $u : E \to F$, $v : F \to G$

are two linear maps. and:

- $(\vec{e_1}, \dots, \vec{e_n})$ a basis of E $\equiv \mathcal{B}_E$
- $(\vec{f_1}, \dots, \vec{f_m})$ a basis of F $\equiv \mathcal{B}_F$
- $(\vec{g_1}, \dots, \vec{g_d})$ a basis of G $\equiv \mathcal{B}_G$

If M is the matrix of u in the basis $(\vec{e_1}, \dots, \vec{e_n})$ of E and $(\vec{f_1}, \dots, \vec{f_m})$ of F

if N is the matrix of v in the basis $(\vec{f_1}, \dots, \vec{f_m})$ of F and $(\vec{g_1}, \dots, \vec{g_d})$ of G

then the matrix of $v \circ u$ (composition) in the basis $(\vec{e_1}, \dots, \vec{e_n})$ of E and $(\vec{g_1}, \dots, \vec{g_d})$ of G is the matrix $N \cdot M$.

We sometimes write:

$$\boxed{\operatorname{Mat}_{B_E, B_G}(v \circ u) = \underbrace{\operatorname{Mat}_{B_F, B_G}(v)}_{\text{matrix of } v \text{ in the basis } B_F \text{ of } F \text{ and the basis } B_G \text{ of } G} \times \underbrace{\operatorname{Mat}_{B_E, B_F}(u)}_{\text{matrix of } u \text{ in the basis } B_E \text{ of } E \text{ and } B_G \text{ of } G.}}$$