B Integral bounds.

Thm Let $f : \mathbb{R}_+^0 \longrightarrow \mathbb{R}_+$ be continuous non-negative decreasing function.

Consider the positive series $\sum_{n \geq 1} f(n)$ then:

1. for every choice $N \geq M \geq 0$ $N, M \in \mathbb{N}$

$$\sum_{n=N+1}^{M+1} f(n) \leq \int_N^{M+1} f(x)\,dx \leq \sum_{n=N}^{M} f(n) \quad (IE)$$

2. $$\sum_{n=1}^{+\infty} f(n) < +\infty \text{ if and only if } \int_1^{+\infty} f(x)\,dx < +\infty$$

Proof: See the study of $\sum \frac{1}{n^p}$.

if $\sum_{n=1}^{+\infty} f(n) < +\infty$ sending $N \to +\infty$ in (IE)

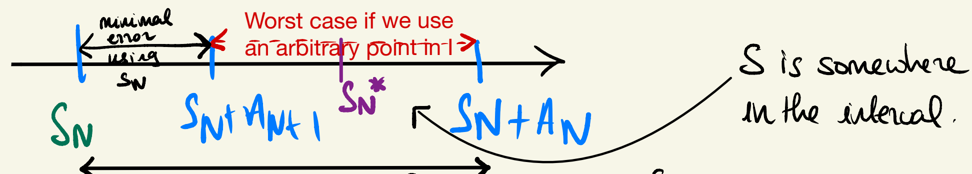
$$\underbrace{\int_{N+1}^{+\infty} f(x)\,dx}_{A_{N+1}} \leq \sum_{n=N+1}^{+\infty} f(n) \leq \underbrace{\int_N^{+\infty} f(x)\,dx}_{A_N}$$

This tells us that $S = \sum_{n=1}^{+\infty} f(n)$ is in the interval: $\left[S_N + A_{N+1} \, , \; S_N + A_N \right] = I$

Therefore <u>any number in this interval</u> approximates

$\boxed{S \text{ with error at most } A_N - A_{N+1}}$.

In particular, if we use $s_N^* = \frac{A_N + A_{N+1}}{2} + S_N$ (the midpoint of the interval) this gives a slightly better estimate than S_N.



Worst case error $= A_N$ if we use S_N to approximate S.

$$S - S_N^* = S - S_N - \frac{A_N + A_{N+1}}{2}$$

therefore:

$$-\frac{A_N - A_{N+1}}{2} \leqslant S - S_N^* \leqslant \frac{A_N - A_{N+1}}{2}$$

ie $\boxed{|S - S_N^*| \leqslant \frac{A_N - A_{N+1}}{2}.}$

whereas our best upper bound on $S - S_N$ is

$\boxed{0 \leqslant A_{N+1} \leqslant S - S_N \leqslant A_N}$

Example $\sum_{n\geq 1} \frac{1}{n^p}$ $\qquad f(x) = \frac{1}{x^p} \qquad p>1$

$$\int_{N+1}^{+\infty} \frac{1}{x^p}\,dx \leq \sum_{n=N+1}^{+\infty} \frac{1}{n^p} \leq \int_{N}^{+\infty} \frac{1}{x^p}\,dx$$

$$\frac{1}{p-1}\frac{1}{(N+1)^{p-1}} \leq \sum_{n=N+1}^{+\infty} \frac{1}{n^p} \leq \frac{1}{p-1}\frac{1}{N^{p-1}}$$

So $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ is in the interval $\left[\frac{1}{p-1}\frac{1}{(N+1)^{p-1}} + S_N, \frac{1}{p-1}\frac{1}{N^{p}} + S_N\right]$

S_N estimates $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ with error at most $\frac{1}{p-1}\frac{1}{N^{p-1}}$

Let's take $p=5$, $N=5$,

$$\boxed{\begin{array}{l} S_5 \text{ estimates } \sum_{n=1}^{+\infty} \frac{1}{n^p} \text{ with precision } \simeq 0{,}0004 \\ S_5^{+} \text{ estimates } \sum_{n=1}^{+\infty} \frac{1}{n^p} \text{ with precision } \simeq 0{,}0001 \end{array}}$$

it converges fast!