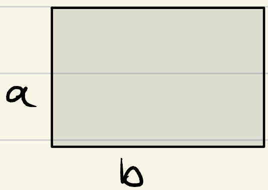
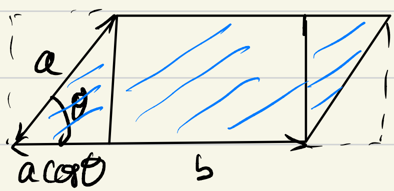
A introduction to integration in dimensions 2 and 3

Dimension 2

When we were kids we learnt how to compute the area of simple shapes:

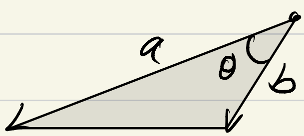




$A = a \times b$

$$A = a^2 \cos\theta \sin\theta + |(b - a\cos\theta)\, a \sin\theta|$$

$$A = ab|\sin\theta| = \underbrace{|\mathrm{Det}(\vec{a}, \vec{b})|}_{\text{geometric interpretation of the determinant}}$$

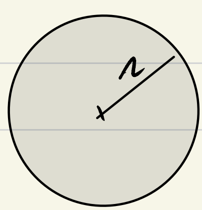


$$A(T) = \frac{a \times b\,|\sin\theta|}{2}$$

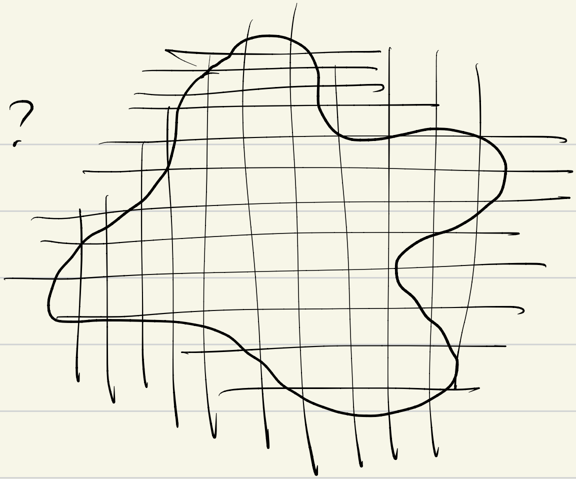
Then it got more complicated...

But our ancestors discovered..

$$\boxed{A(\mathcal{C}) = \pi r^2}$$



But what about ?



Can we define the area of any subset of $\mathbb{R}^2$? (1)

It may eventually be $+\infty$ as $A(\mathbb{R}^2) = +\infty$.

Is there a function $A : \underbrace{\mathcal{P}(\mathbb{R}^2)}_{\text{set of all subsets of } \mathbb{R}^2} \longrightarrow [0, +\infty]$ (2)

we have to allow $+\infty$ as certainly $A(\mathbb{R}^2) = +\infty$

that "measures" subsets of $\mathbb{R}^2$?

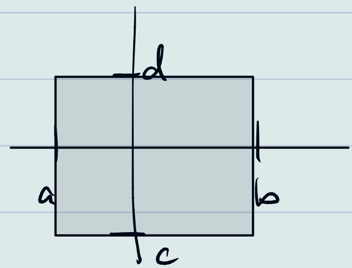
H. Lebesgue, E. Borel, amongst others developped measure theory which is also the language of modern probability theory.

The answer to one is $\boxed{\text{NO}}$, but the answer is positive if we restrict to a subset $\mathcal{B}(\mathbb{R}^2) \subset \mathcal{P}(\mathbb{R}^2)$

known as the "Borel σ-algebra". Sets of $\mathcal{B}(\mathbb{R}^2)$ are known as the Borel measurable set.

Theorem There is a unique measure λ: known as the Borel-Lebesgue measure such that:

$$\boxed{\lambda([a,b]\times[c,d]) = (b-a)\times(d-c)}$$



Remark One of the more subtle parts of measure theory is that we can enlarger $\mathcal{B}(\mathbb{R}^2)$ and extend λ to a slightly larger set $\mathcal{M}(\mathbb{R}^2)$ known as the set of Lebesgue measurable subsets. we will ignore this subtlety.

It is relatively difficult to find subsets that are not in $\mathcal{B}(\mathbb{R}^2)$, so we will not worry about defining it further. It contains all our usual sets, balls, rectangles, triangles, unions of these.

Integration with respect to a measure.

An alternative way of thinking about it:

Ever wondered what "dx" is in $\int f\,dx$ is?

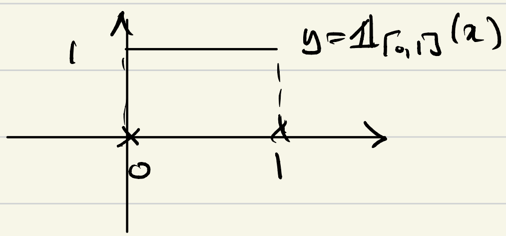
A possible answer is that it is the Borel-Lebesgue measure on $\mathbb{R}$.

Let A be a (Borel) measurable subset of $\mathbb{R}^2$

we call $\mathbb{1}_A$ or χ_A the characteristic function of A, the function:

$$\boxed{\mathbb{1}_A(x,y) = \begin{cases} 1 & \text{if } (x,y) \in A \\ 0 & \text{otherwise.} \end{cases}}$$

Example in 1-D,



A simple function is a function f given by:

$$f(x,y) = \sum_{i=1}^{n} c_i \, \mathbb{1}_{A_i}(x,y)$$

, for some measurable subsets $A_1, \dots, A_n$.

f is simple

The integral of f is a kind of weighted sum

$$\int_{\mathbb{R}^2} f(x,y)\,dx\,dy = \sum_{i=1}^{n} c_i \underbrace{\lambda(A_i)}_{\substack{\text{the measure of}\\ \text{the set when } f=c_i}}$$

or size

If f is a positive continuous function:

$$\boxed{\int_{\mathbb{R}^2} f(x,y)\,dx\,dy = \sup\left\{ \int_{\mathbb{R}^2} g(x,y)\,dx\,dy, \begin{array}{l} g \text{ simple} \\ g \le f \end{array} \right\}.}$$

It can either be a positive real number or $+\infty$.

If f is an arbitrary continuous function we say that

f is integrable if $\int_{\mathbb{R}^2} |f|\,dx\,dy < +\infty$.

In this case, $f = f_+ - f_-$, where f_+ and f_- are positive functions and we define

$$\boxed{\int_{\mathbb{R}^2} f\,dx\,dy = \int_{\mathbb{R}^2} f_+\,dx\,dy - \int_{\mathbb{R}^2} f_-\,dx\,dy}$$

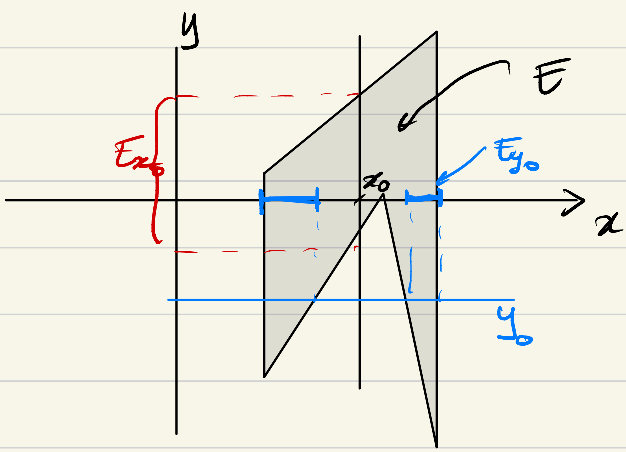
If $A \subset \mathbb{R}^2$, we define $$\boxed{\int_A f\,dx\,dy = \int_{\mathbb{R}^2} f\,\mathbb{1}_A \underbrace{dx\,dy}_{dA}}$$

Calculating integrals

This point of view is very nice and intuitive but now we need a theorem, how do we calculate it?

$\Rightarrow$ Relate it to integrals we are used to calculating.

Lemma Let E be (Borel) measurable subset of $\mathbb{R}^2$



Define $E_{x_0} = \{ y \in \mathbb{R},\quad (x,y) \in E \}$

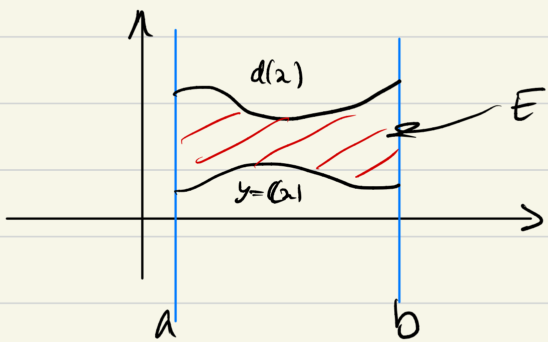
$E_{y_0} = \{ x \in \mathbb{R},\quad (x,y) \in E \}$

Then: $$\lambda(E) = \int_{\mathbb{R}} \underbrace{\lambda_{\mathbb{R}}(E_y)}_{\text{size of the set } E_y \text{ in } \mathbb{R}} dy = \int_{\mathbb{R}} \underbrace{\lambda_{\mathbb{R}}(E_x)}_{\text{size of the set } E_x \text{ in } \mathbb{R}.} dy$$

if it is empty then $\lambda_{\mathbb{R}}(\phi) = 0$

Important example

We say that E is <u>y-simple</u> if it is bounded by two vertical lines and the graphs of two functions $y = c(x)$, $y = d(x)$



in other words, $E_x = \{ c(x) \leq y \leq d(x) \}$ if $a \leq x \leq b$

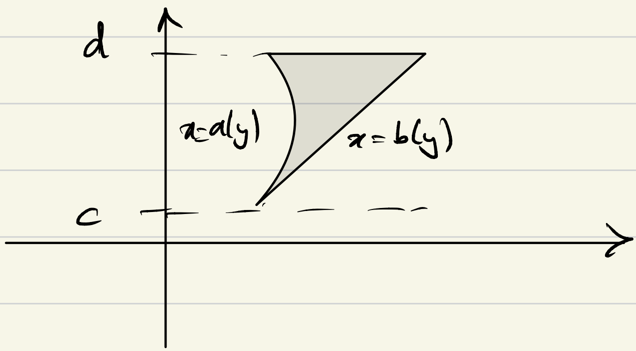
$E_x = \phi$ if $\begin{cases} x > a \\ x < b. \end{cases}$

In this case the theorem says that; the area of E

is $\boxed{\mathcal{A}(E) = \int_a^b \left[\left(\int_{c(x)}^{d(x)} 1 \right) dy \right] dx = \int_a^b \left(d(x) - c(x) \right) dx}$

Similarly E is <u>x-simple</u> if,

$$\begin{cases} E_y = \{ a(y) \leq x \leq b(y) \} & \text{if} \quad y \in [c, d] \\ E_y = \phi & \text{if} \quad y \notin [c, d] \end{cases}$$



A domain D is called regular if it is a union of sets E that are both x-simple and y-simple

Fubini - Tonelli theorem

If f is a positive continuous function, then

$$\boxed{\int_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x,y)\,dx\right)dy = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x,y)\,dy\right)dx}$$

Important example:

$$\int_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx.$$

Fubini's theorem

If a continuous function f is integrable

i.e. $\int_{\mathbb{R}^2} |f(x,y)|\, dx\,dy < +\infty$

$$\boxed{\int_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,dy \right) dx}$$

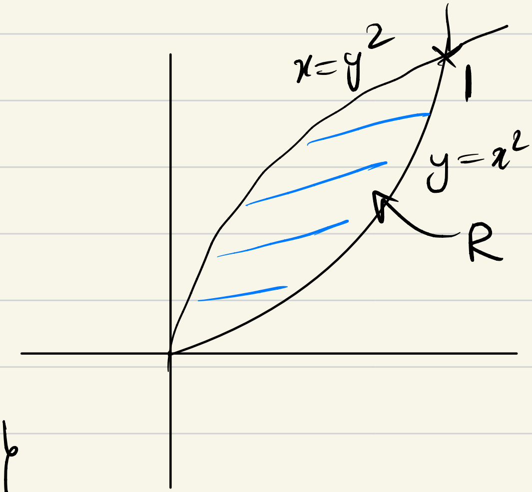
Note: if f is continuous and E is bounded

then $f * \mathbb{1}_E$ is integrable.

So Fubini's theorem will always apply when we look at bounded regular domains

Example

$$\int_R xy^2\, dA$$



R is regular.

$$R = \{x^2 \le y \le \sqrt{x},\ 0 \le x \le 1\}$$

$$\int_R xy^2\, dA = \int_0^1 x\left(\int_{x^2}^{\sqrt{x}} y^2\, dy\right) dx$$

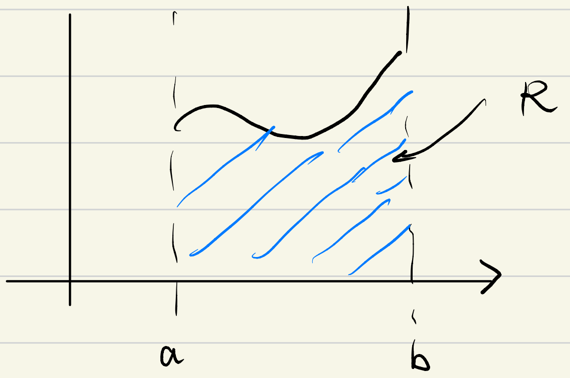
$$= \int_0^1 \frac{x}{3}\left(x^{3/2} - x^6\right) dx$$

$$= \frac{1}{3}\int_0^1 x^{5/2} - \frac{1}{3}\int_0^1 x^7\, dx$$

$$= \frac{2}{21} - \frac{1}{24} = \frac{3}{56}$$

Example: $f : [a,b] \longrightarrow \mathbb{R}$ continuous function

Consider $R = \{ a \leq x \leq b , \; 0 \leq y \leq f(x) \}$



$$A(R) = \int_R dA = \int_R dx\,dy = \int_a^b \left(\int_0^{f(x)} dy \right) dx$$

$$\boxed{A(R) = \int_a^b f(x)\,dx.}$$

The 1-D integral is the area under the curve.

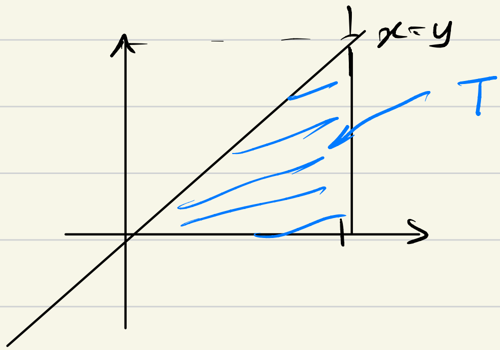
Everything is consistent.

Examples on finite domains

Ex 1: Evaluate
$$\int_0^1 \left(\int_y^1 e^{-x^2}\, dx \right) dy$$

Note here that the way it is written we cannot calculate it!

This is the same as the integral over the domain $T = \{ (x,y) \in \mathbb{R}^2,\ 0 \leq y \leq x \leq 1 \}$



As written above, we cannot evaluate the integral but using Fubini's theorem:

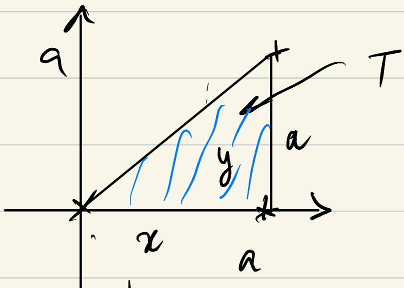
$$\int_0^1 \int_y^1 e^{-x^2} dx\, dy = \int_T e^{-x^2} dx dy = \left(\int_0^1 e^{-x^2} \int_0^x dy \right) dx$$

$$= \int_0^1 e^{-x^2} x\, dx = \left[\frac{1}{2} e^{-x^2} \right]_0^1$$

$$= \frac{1}{2}(1 - e^{-1}).$$

Ex 2

T = triangle with vertices $(0,0)$, $(a,0)$, (a,a)



$a > 0$

using Fubini's theorem

$$\int_T \sqrt{a^2-y^2}\,dx\,dy = \int_0^a \left(\int_0^x \sqrt{a^2-y^2}\,dy \right) dx$$

which one?

$$= \int_0^a \left(\int_y^a \sqrt{a^2-y^2}\,dx \right) dy$$

$$= \int_0^a (a-y)\sqrt{a^2-y^2}\,dy$$

$$= a^3 \int_0^1 (1-u)\sqrt{1-u^2}\,du$$

$u = \sin\theta$ $\quad = a^3 \int_0^{\frac{\pi}{2}} (1-\sin\theta)\cos^2\theta\,d\theta$

IBP and $\cos^2\theta + \sin^2\theta = 1$

$$= a^3 \left(\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta - \frac{1}{3} \right)$$

$$\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = \frac{\pi}{4}$$

$$= a^3 \left(\frac{\pi}{4} - \frac{1}{3} \right).$$

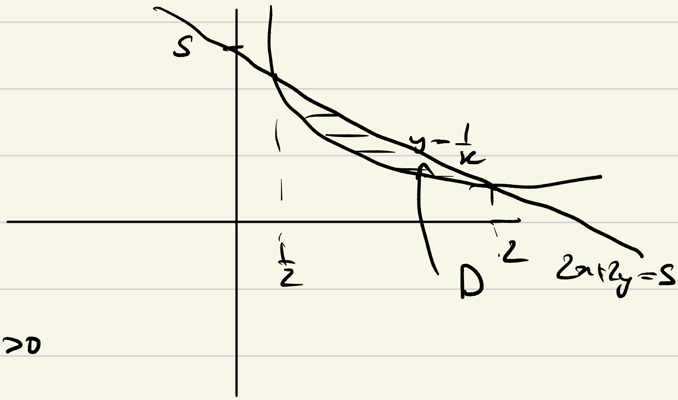
Alternatively

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = \frac{\pi}{4}$$

Ex 3

$$\int_D \ln x \, dx dy$$



$$\begin{cases} 2x+2y=5 \\ xy=1 \end{cases} \qquad x>0, y>0$$

Find the points of intersection

$$2x^2+2=5x \qquad x \in \left\{\frac{1}{2}, 2\right\}$$

$$\int_{\frac{1}{2}}^{2} \int_{\frac{1}{x}}^{\frac{5}{2}-x} (\ln x)\, dy\, dx \qquad y = \frac{5}{2} - x$$

$$\int_{\frac{1}{2}}^{2} (\ln x)\left(\frac{5}{2} - x - \frac{1}{x}\right) dx$$

$$= \frac{5}{2}\Big[x\ln x - x\Big]_{\frac{1}{2}}^{2} - \overset{0}{\cancel{\int_{\frac{1}{2}}^{2} \frac{\ln x}{x}}} - \int_{\frac{1}{2}}^{2} x \ln x \, dx$$

$$= \frac{5}{2}\left(2\ln 2 - 2 + \frac{1}{2}\ln 2 + \frac{1}{2}\right) - \left[\frac{x^2}{2}\ln x\right]_{\frac{1}{2}}^{2} + \frac{1}{2}\int_{\frac{1}{2}}^{2} x \, dx$$

$$= \frac{25}{4}\ln 2 - \frac{15}{4} - 2\ln 2 - \frac{1}{8}\ln 2 + \frac{1}{4}\left(4 - \frac{1}{4}\right)$$

$$= 4\ln 2 + \frac{1}{8}\ln 2 - 3 + \frac{3}{16} = \ln(16\sqrt[8]{2}) - \frac{45}{16}$$

On infinite domains

The definition of the integral allows for infinite domains too (or unbounded functions on finite domains)

→ if the function is positive: Fubini-Tonelli tells us that we can calculate it by writing it as an iterated integral how we want. (The result may be infinite but the result will be the same no matter what).

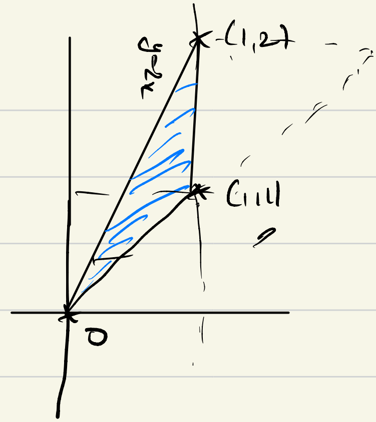
→ for functions that change sign: Fubini's theorem only applies if $\int |f(x,y)|\,dx\,dy < +\infty$

Examples

$$\int_{\mathbb{R}^2} e^{-(|x|+|y|)}\,dx\,dy = 4\int_0^{+\infty} e^{-y}\int_0^{+\infty} e^{-x}\,dx\,dy$$

$$= 4\left(\int_0^{+\infty} e^{-x}\,dx\right)^2 = 4$$

$$\int_{\mathbb{R}^2} e^{-|x+y|}\, dx\,dy = +\infty$$



$\rightarrow \int_T \frac{1}{x\sqrt{y}}\, dx\,dy$ where T

$\Rightarrow$ This function is not defined at $(0,0)$.

we can however still consider the integral :

$x \leq y \leq 2x$

$$\int_0^1 \frac{1}{x} \int_x^{2x} \frac{1}{\sqrt{y}}\, dy\, dx$$

$$2\int_0^1 \frac{1}{x}\left(\sqrt{2x} - \sqrt{x}\right) dx = 4(\sqrt{2}-1) < +\infty$$

$$T = \{ x < y \leq 2x \quad , 0 \leq x \leq 1 \}$$

$$T = \{ \frac{y}{2} \leq x \leq y \quad , \underline{0 \leq x \leq 1} \}$$

$$\int_0^2 \left(\int_{\frac{y}{2}}^{\underline{\min(y,1)}} \frac{1}{x\sqrt{y}}\, dx \right) dy$$

$$= \int_0^1 \int_{\frac{y}{2}}^{y} \frac{1}{x\sqrt{y}}\, dx\, dy + \int_1^2 \int_{\frac{y}{2}}^{1} \frac{1}{x\sqrt{y}}\, dx\, dy$$

$$= \int_0^1 \frac{\ln 2}{\sqrt{y}}\,dy + \int_1^2 \frac{-\ln\left(\frac{y}{2}\right)}{\sqrt{y}}\,dy$$

$$= 2\ln 2 + \int_1^2 \frac{-\ln\left(\frac{y}{2}\right)}{\sqrt{y}}\,dy$$

$y = 2u^2$

$dy = 4u\,du$

$$= 2\ln 2 - \frac{8}{\sqrt{2}}\int_{\frac{1}{\sqrt{2}}}^{1} \ln u\,du$$

$$= 2\ln 2 - \frac{8}{\sqrt{2}}\Big[u\ln u - u\Big]_{\frac{1}{\sqrt{2}}}^{1}$$

$$= 2\ln 2 + \frac{8}{\sqrt{2}} + \frac{8}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\ln\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]$$

$$= \cancel{2\ln 2} + \frac{8}{\sqrt{2}} - \cancel{2\ln 2} - \frac{8}{2}$$

$$= 4\sqrt{2} - 4 = 4(\sqrt{2}-1) \quad !$$

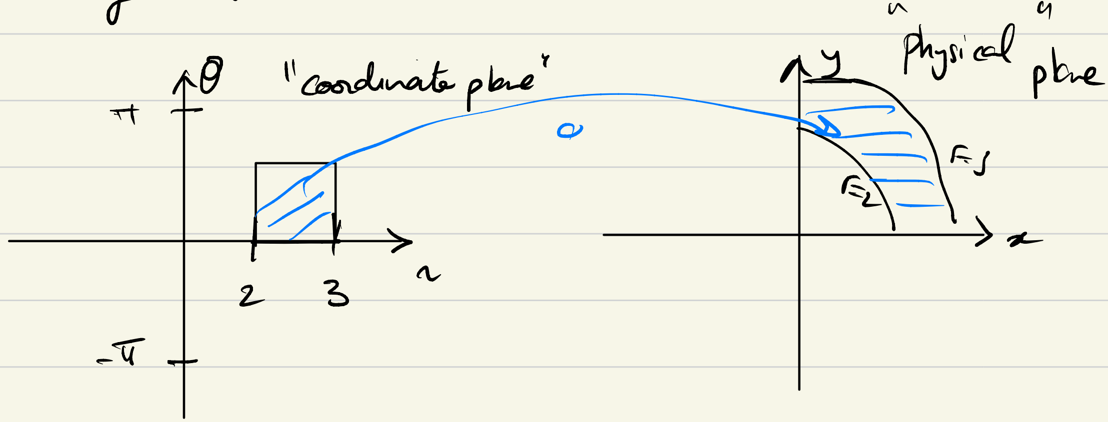
Identical as expected but it was simpler to integrate wrt y first!

Change of variable : polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$$\phi(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$



In the "physical" plane: $R = \{ 2 \leq \sqrt{x^2+y^2} \leq 3,\ x \geq 0, y \geq 0 \}$

It is nicer to describe in terms of (r,θ)

$R = \phi(\tilde{R})$ where $\tilde{R} = \{ 2 \leq r \leq 3,\ 0 \leq \theta \leq \frac{\pi}{2} \}$

But:
$$\int_{\tilde{R}} f(r,\theta)\,dr\,d\theta \neq \int_R f(x,y)\,dx\,dy.$$

For example take $f = 1$, $\int_{\tilde{R}} dr\,d\theta = \frac{\pi}{2}$, $\int dx\,dy = A(R) = \frac{\pi}{4}(9-4) = \frac{5\pi}{4}$

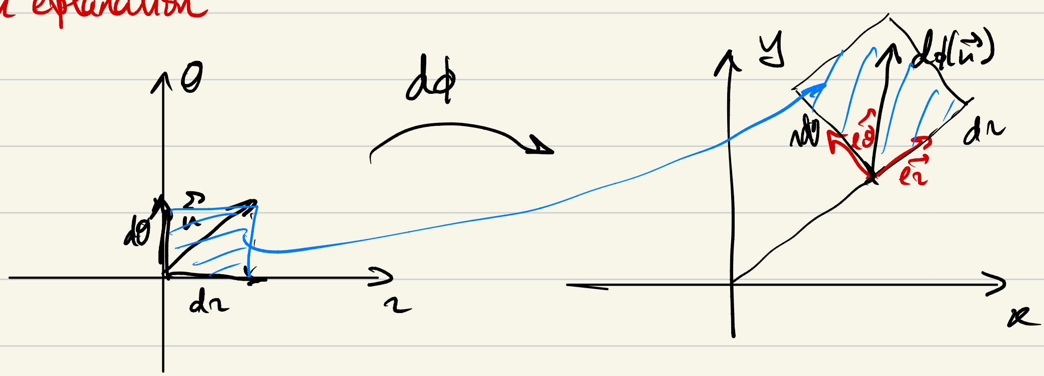
How do we modify the RHS so as to find the correct answer?

Prop

$$\int_{\phi^{-1}(\tilde{R})} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta = \int_{\tilde{R}} f(x,y)\, dx\, dy$$

where $\phi^{-1}(\tilde{R})$ is the part of the coordinate plane corresponding to $\tilde{R}$ in the "physical" plane.

Rough explanation



So the parallelogram supported by $dr\,\vec{i}$, $d\theta\,\vec{j}$ in the coordinate plane is mapped to the parallegram $dr\,\vec{e_r}$, $r\,d\theta\,\vec{e_\theta}$.

So a parallelogram with area $dr\,d\theta$ is mapped to a parallelgramme with area

$$\begin{vmatrix} dr & 0 \\ 0 & r\,d\theta \end{vmatrix} = r\,dr\,d\theta \text{ in the physical plane.}$$

remember the interpretation of the determinant we saw.

Example $f(R) = \int_R dx\,dy = \int_{R^2} r\,dr\,d\theta$

$$= \int_2^3 \int_0^{\frac{\pi}{2}} r\,dr\,d\theta$$

$$= \frac{5}{4}\pi$$

Example 2 $\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{+\infty}\int_0^{2\pi} r e^{-r^2}\,dr\,d\theta$

$$= \pi$$

But using Fubini's theorem:

$$\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy = \int_{\mathbb{R}} e^{-x^2}\,dx \int_{\mathbb{R}} e^{-y^2}\,dy$$

$$= \left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)^2$$

$$\Rightarrow \boxed{\int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi}}$$

The general change of coordinate formula

Let $\phi : U \longrightarrow V$, U, V open subsets of $\mathbb{R}^2$

$(u,v) \longmapsto \phi(u,v) = (x(u,v), y(u,v))$

a C^1 diffeomorphism then:

$$\boxed{\int_U f(x(u,v), y(u,v)) \, |\mathrm{Jac}\, \phi \, (u,v)| \, du \, dv = \int_V f(x,y) \, dx \, dy}$$

absolute value of determinant

sometimes written $\left| \frac{\partial(x,y)}{\partial(u,v)} \right|$

N.B - There are more refined versions.

Example $\mathcal{E} = \left\{ (x,y) \in \mathbb{R}^2 , \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1 \right\}$

(an ellipse)

$$A(\mathcal{E}) = \int_{\mathcal{E}} dx\,dy \qquad = ab \int_{\mathcal{E}'} du\,dv$$

set $x = au$, $y = bv$

$\phi(u,v) = (au, bv)$

$|\mathrm{Jac}\, \phi(u,v)| = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$

But ϕ maps $\mathcal{E}$ to $\{u^2 + v^2 \leq 1\}$

a disk of radius 1.

$\Rightarrow \boxed{A(\mathcal{E}) = ab\,\pi}$

Note: The attentive student may have noticed that the theorem was stated for open subsets but that above I seemingly applied it on the closed subset $\mathcal{E}$. The "trick" is that if $\mathring{\mathcal{E}}$ denotes the interior then

$$\int_{\mathring{\mathcal{E}}} \cdots = \int_{\mathcal{E}} \cdots$$

Because the boundary $\partial\mathcal{E}$ (the ellipse) satisfies:

$$\boxed{A(\partial\mathcal{E}) = 0}$$

It has 0-area, or is of measure zero.

As one might expect:

$$\boxed{\text{If} \quad A(E) = 0 \quad \text{then} \quad \int_E f \, dA = 0.}$$

whilst one can find "area filling curves", for the things we will encounter will always be "nice enough" so that the boundary has this property.

Points also have zero measure and since

$$A\left(\bigcup_n A_n\right) = \sum_{n=0}^{+\infty} A(A_n) \quad \text{if } (A_n) \text{ are pairwise disjoint}$$

countable sets are of measure 0 too!

Advanced remark

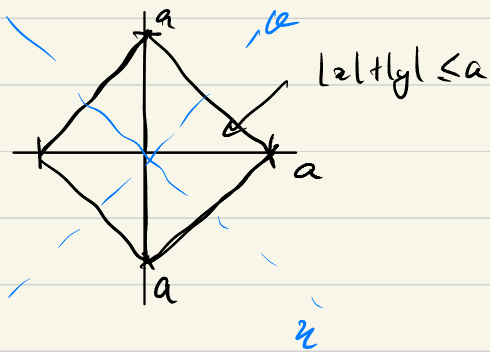
One of the subtleties of measure theory is that there are sets E such that $E \notin \mathcal{B}(\mathbb{R}^n)$ for which one can find $A \in \mathcal{B}(\mathbb{R}^n)$ with $\lambda(A) = 0$ and $E \subset A$!

It would seem natural that such an E should be measurable and $\lambda(E) = 0$ ... this leads to the notion of "completion" of a measure...

Example

$$\int_{|x|+|y| \leq a} e^{x+y} \, dx dy$$

To devise some coordinates let us look at the geometry of the situation.



Set $\begin{cases} u = x+y \\ v = y-x \end{cases}$ $\qquad y = \dfrac{u+v}{2} \qquad x = \dfrac{u-v}{2}$

$$\phi(u,v) = \left(\frac{u-v}{2}, \frac{u+v}{2}\right)$$

$$|\mathrm{Jac}\, \phi(u,v)| = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = \frac{1}{2}$$

$$\boxed{\frac{1}{2} \int_{[-a,a]\times[-a,a]} e^{u} \, du dv = \int_{|x|+|y| \leq a} e^{x+y} \, dx dy}$$

$$\boxed{= a\int_{-a}^{a} e^{u}\,du = a\left(e^{a}-e^{-a}\right) = 2a\,\mathrm{sh}\,a}$$

Mean-value of a function

Let D be a bounded domain we define the mean-value of f to be:

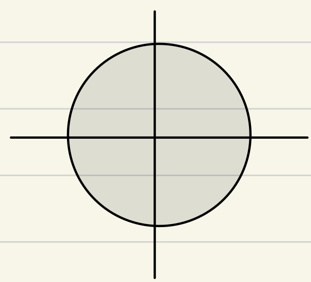
$$\boxed{\frac{1}{A(D)}\int_D f(x,y)\,dA}$$

<u>Example</u>: Let $f(x,y) = \sqrt{x^2+y^2}$ $\quad D = \{x^2+y^2 \leq 1\}$

$$\int_D f(x,y)\,dx\,dy = \int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta$$

$$= \frac{2\pi}{3}$$

$A(D) = \pi$



$$\boxed{\frac{1}{A(D)}\int_D f(x,y)\,dA = \frac{2}{3}}$$

Integration in $\mathbb{R}^3$

The theory can be developed as in the 2D case and relies on the existence of a "volume function".

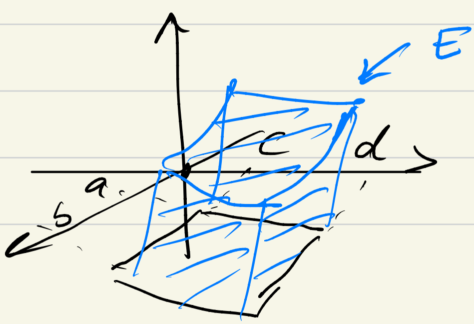
Theorem There is a unique measure $\vartheta : \mathcal{B}(\mathbb{R}^3) \longrightarrow [0,+\infty]$ such that:

$$\vartheta([a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]) = (b_1-a_1)\cdot(b_2-a_2)\cdot(b_3-a_3)$$

The Fubini (Tonelli) theorems are identical and the integral is defined in the same way as before

Example Let $f : [a,b]\times[c,d] \longrightarrow \mathbb{R}$ be a continuous function and consider

$$E = \{(x,y,z)\in\mathbb{R}^3,\ x\in[a,b],\ y\in[c,d],\ z\leq f(x,y)\}$$



"solid under the graph"

$$\mathcal{V}(E) = \int_{\mathbb{R}^3} \mathbb{1}_E \, d\mathcal{V} \overset{\text{definition}}{=} \int_E d\mathcal{V}$$

volume

$$= \int_a^b \int_c^d \int_0^{f(x,y)} dz\,dy\,dx$$

$$= \int_a^b \int_c^d f(x,y)\,dx\,dy$$

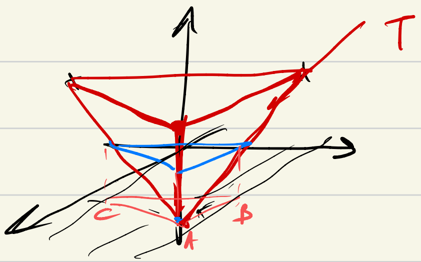
$$= \int_{[a,b]\times[c,d]} f(x,y)\,dA$$

The double integral of a function on a domain is the volume of the solid "under" the graph.

⇒ can be useful to calculate $\underbrace{\text{double integrals}}_{\text{i.e. integrals on } \mathbb{R}^2}$ "by inspection".

Example : $\int_T x$ where T is the tetrahedron bounded by the planes $x=1, y=1, z=1, x+y+z=2$

Try to draw:



T is z-simple $0 \leq z \leq 1$

A z-slice

$$T_z = \{(x,y) \in \mathbb{R}^2, (x,y,z) \in T\}$$

is a triangle

The 3-dimensional version of the Fubini-Tonelli theorem will enable us first to write:

$$\int_T x \, dV = \int_0^1 \underbrace{\left(\int_{T_z} x \, dA \right)}_{\text{now we want to apply}} dz .$$

now T_z is in the (x,y) plane and has vertices $A(1,1)$, $B(1-z, 1)$, $C(1, 1-z)$

$$\int_0^1 \left(\int_{1-z}^1 \left(\int_{2-y-z}^1 x \, dx \right) dy \right) dz$$

$$= \frac{1}{2}\int_0^1 \left(\int_{1-z}^1 1-(2-y-z)^2 \, dy\right) dz$$

$$= \frac{1}{2}\int_0^1 \left(z + \frac{1}{3}(1-z)^3 - \frac{1}{3}\right) dz$$

$$= \frac{1}{2}\left(\frac{1}{2} \times \frac{1}{3} + \frac{1}{12}\right) = \frac{1}{12} + \frac{1}{24} = \frac{3}{24} = \boxed{\frac{1}{8}}$$

Remark: This is an application of Fubini's theorem

$$\int_{\mathbb{R}^3} x \, \mathbb{1}_T \, dV = \int x \, \mathbb{1}_T(x,y,z) \, dx \, dy \, dz$$

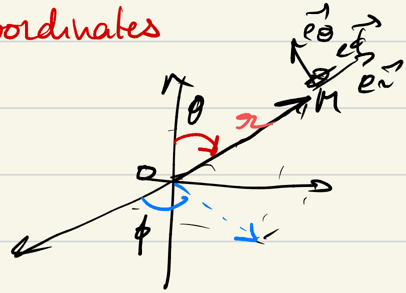
$$= \int_{\mathbb{R}} \int_{\mathbb{R}^2} x \, \underbrace{\mathbb{1}_T(x,y,z)} \, dA \, dz$$

if z is fixed

$$\mathbb{1}_T(x,y,z) = \mathbb{1}_{T_z}(x,y) \, \mathbb{1}_{[0,1]}(z)$$

$$= \int_{\mathbb{R}} \mathbb{1}_{[0,1]}(z) \left(\int_{T_z} x \, dA\right) dz = \int_0^1 \left(\int_{T_z} x \, dA\right) dz$$

Change of variable : spherical coordinates



$$\underbrace{\left\{\begin{array}{ll} x = r\sin\theta\cos\phi & r \in [0,+\infty) \\ y = r\sin\theta\sin\phi & \theta \in [0,\pi] \\ z = r\cos\theta & \phi \in [0,2\pi]. \end{array}\right.}_{\psi(r,\theta,\phi)}$$

"physicist/engineering" convention

Theorem

$$\int_{[0,+\infty)\times[0,\pi]\times[0,2\pi]} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_{\mathbb{R}^3} f(x,y,z)\, dx\, dy\, dz$$

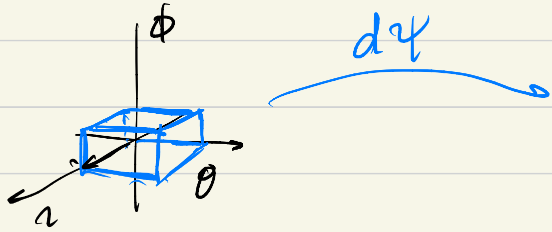
Justification

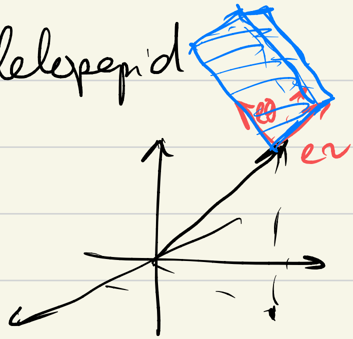
$$d\psi(dr\,\vec{i}) = \sin\theta\cos\phi\,\vec{i} + \sin\theta\sin\phi\,\vec{j} + \cos\theta\,\vec{k}$$

$$d\psi(d\theta\,\vec{j}) = + r\cos\theta(\cos\phi\,\vec{i} + \sin\phi\,\vec{j}) - r\sin\theta\,\vec{k}$$

$$d\psi(d\phi\,\vec{k}) = -r\sin\theta\sin\phi\,\vec{i} + r\sin\theta\cos\phi\,\vec{j}$$

So infinitesimally the parallelepiped





coordinate space

physical space

$$\begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\cos^2\theta\sin\theta \quad + r^2\sin^2\theta\sin\theta$$

$$= r^2\sin\theta$$

so $\boxed{r^2\sin\theta\, dr\, d\theta\, d\phi = dx\, dy\, dz}$

Example The volume of a sphere

$$\int dx\,dy\,dz = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{r} r^2\, dr$$

$$= \frac{2\pi r^3}{3} \times 2 = \frac{4\pi r^3}{3}$$

Mean-value of $G(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$

when $\{ 4 \le x^2+y^2+z^2 \le 9 \} = R$

$$V(R) = 4\pi \times \int_2^3 r^2\, dr = \frac{20\pi}{3}$$

$$\int_R G\, dV = 4\pi \int_2^3 dr = 4\pi$$

$$\boxed{\frac{1}{V(R)} \int_R G\, dV = \frac{3}{5}}$$

General change of coordinates

Everything is identical, shift $2 \to 3$

$\phi : U \longrightarrow V$ a C^1 diffeomorphism between open subsets of $\mathbb{R}^3$.

$$\phi(u,v,w) = \begin{pmatrix} x(u,v,w) \\ y(u,v,w) \\ z(u,v,w) \end{pmatrix}$$

then:

$$\int_U f(x(u,v,w), y(u,v,w), z(u,v,w)) \, |\mathrm{Jac}\, \phi(u,v,w)| \, du\, dv\, dw$$

$$= \int_V f(x,y,z) \, dx\, dy\, dz$$

Sometimes $|\mathrm{Jac}\, \phi(u,v,w)| = \left| \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \right|$

Example $\mathcal{E} = \left\{ 0 \leq \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$ $a, b, c > 0$

set $x = au$, $y = bv$, $z = cw$ ie.

$$\phi(u, v, w) = \begin{pmatrix} au \\ bv \\ cw \end{pmatrix}$$

then $|\operatorname{Jac} \phi(u, v, w)| = abc$

and
$$\int_{\mathcal{E}} dx\,dy\,dz = abc \int_{\mathcal{E}'} du\,dv\,dw$$

where $\mathcal{E}' = \{ 0 \leq u^2 + v^2 + w^2 \leq 1 \}$

$\to$ spherical coordinates $u = r \ldots$ $v = r \ldots$ $w = \ldots$

$$\boxed{\begin{aligned} \int_{\mathcal{E}} dx\,dy\,dz &= abc \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^1 r^2\, dr \\ &= \frac{4\pi}{3} abc. \end{aligned}}$$

Additional material: optimisation under a "nice" constraint, - Lagrange multipliers

For definiteness let us consider in $\mathbb{R}^2$,

$$C = \{ (x,y) \in \mathbb{R}^2, \ \phi(x,y) = 0 \}$$

where ϕ is a smooth function. Since C is closed (it is a level set) our differential calculus doesn't apply directly...

The aim of this note is to see how we can extend these techniques to this situation.

(A) Critical points on C

Let $(x,y) \in C$, then we have seen that the only directions $\vec{h}$ for which we stay on the curve C are those such that:

$$\boxed{d\phi(\vec{h}) = 0}$$

By assumption

we defined $T_{(x,y)}C = \ker d\phi(x,y) \neq \mathbb{R}^2$

if $f: \mathbb{R}^2 \to \mathbb{R}$ is a function then as long as we work on $\mathcal{C}$

we should only consider these directions, hence

we infer that a point (x,y) should be critical

relative to $\mathcal{D}$ if

$$\boxed{df(x,y)(\vec{h}) = 0 \text{ for all } \vec{h} \in T_{(x,y)}C}$$

⚠ This is a weaker condition than being a critical point for f! Reasoning on an open subset we see that it is necessary if a local extremum is attained at (x,y)

In terms of the gradient, $\vec{\nabla} f(x,y)$, this means that:

$$\boxed{(\vec{\nabla} f(x,y) \mid \vec{h}) = 0 \text{ for every } \vec{h} \in T_{(x,y)}C}$$

In other words, $\vec{\nabla} f(x,y)$ is normal to $T_{(x,y)}C$!

But $\vec{\nabla}\phi(x,y)$ is a normal vector to $T_{(x,y)}C$ and all such vectors are colinear, hence there is $\lambda \in \mathbb{R}$ such that:

$$\boxed{\vec{\nabla} f(x,y) = \lambda \vec{\nabla}\phi(x,y)}$$

Lagrange multiplier.

Theorem A necessary condition for a local extremum of f relative to D be attained at $(x,y) \in D$ is that there is $\lambda \in \mathbb{R}$ such that:

$$\boxed{\vec{\nabla} f(x,y) = \lambda \vec{\nabla}\phi(x,y)}$$

$\phi(x,y) = 2x^2+y^2-1$

<u>Example</u>: $f(x,y) = xy$, $C = \{ 2x^2+y^2 = 1 \}$

$\vec{\nabla} f(x,y) = y\vec{i} + x\vec{j}$, $\vec{\nabla}\phi(x,y) = 4x\vec{i} + 2y\vec{j}$

Suppose there is $\lambda \in \mathbb{R}$ such that:

$$2x^2+y^2 = 1$$

$$\begin{cases} y = 4\lambda x & (1) \\ x = 2y\lambda & (2) \end{cases}$$

Note that $\lambda \neq 0$ because $(0,0) \notin C$.

plug (1) into (2)

$$y = 8\lambda^2 y \quad \Rightarrow \quad (8\lambda^2-1)y = 0$$

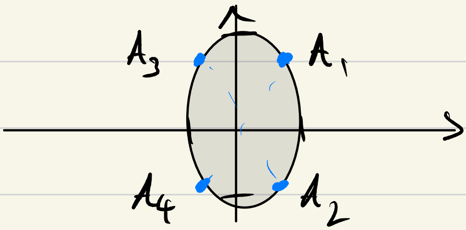
Since $y=0 \Rightarrow x=0$, this means that $\lambda = \pm \frac{1}{2\sqrt{2}}$

so $\begin{cases} y = \pm \frac{2}{\sqrt{2}} x \\ x = \pm \frac{1}{\sqrt{2}} y \end{cases} \iff y = \pm \sqrt{2} x$

Now $2x^2+y^2=1$, so $x = \pm \frac{1}{2}$

The critical points relative to C are:

$A_1\left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right), A_2\left(\frac{1}{2}, -\frac{\sqrt{2}}{2}\right), A_3\left(-\frac{1}{2}, \frac{\sqrt{2}}{2}\right), A_4\left(-\frac{1}{2}, -\frac{\sqrt{2}}{2}\right)$



$f(A_1) = \frac{\sqrt{2}}{4} = f(A_4)$, $f(A_2) = -\frac{\sqrt{2}}{4} = f(A_3)$

Since C is closed and bounded, there is a min and a max

$\Rightarrow$ the maximum value on C is $\frac{\sqrt{2}}{4}$, the minimal value is $-\frac{\sqrt{2}}{4}$

In 3-dimensions we have:

Theorem Let S be the surface defined by $\phi(x,y,z)=0$ where ϕ is a smooth function and $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ a smooth function. Then a necessary condition for f to have a local extrema relative to S at $(x,y,z) \in S$ is that there is $\lambda \in \mathbb{R}$, such that:

$$\boxed{\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} \phi(x,y,z)}$$

Example Set $P=(3,0,0)$ and S be the surface $z = x^2 - y^2$ let us find the distance $d(P,S)$ which is defined to be

$$\boxed{d(P,S) = \min_{M \in S} d(P,M)}$$

(P is not on S so $\neq 0$)

Here $f(x,y,z) = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$, and we wish to minimize relative to S.

We look for critical points relative to S:

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} \phi(x,y,z) \quad , \lambda \in \mathbb{R}$$

means:

$$(S)\begin{cases} \dfrac{(x-3)}{f(x,y,z)} = 2\lambda x \\ \\ \dfrac{y}{f(x,y,z)} = -2\lambda y \\ \\ \dfrac{z}{f(x,y,z)} = -\lambda \end{cases} \qquad (z = x^2 - y^2)$$

The middle equation is $y\left(\dfrac{1}{f(x,y,z)} + 2\lambda\right) = 0$

which means $y = 0$ or $\lambda = -\dfrac{1}{2f(x,y,z)}$

<u>Case 1</u> if $y = 0$ $z = x^2$

$$(S) \Leftrightarrow \begin{cases} x^2 = z \\ (x-3) = -2x^3 \\ \dfrac{x^2}{f(x,y,z)} = -\lambda \end{cases} \Leftrightarrow \begin{cases} x^2 = z \\ (x-1)(2x^2+2x+3) = 0 \\ \dfrac{x^2}{f(x,y,z)} = -\lambda \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x = 1 \\ \lambda = -1 \\ z = 1 \end{cases}$$

So this leads to the solution $(1, 0, 1)$

$f(1,0,1) = \sqrt{5}$

Case 2 $\lambda = -\frac{1}{2f(x,y,z)}$ then $\begin{cases} (x-3) = -x \\ x^2-y^2 = z \\ z = +\frac{1}{2} \end{cases}$, $x = \frac{3}{2}$

and $z = x^2-y^2 \Rightarrow \frac{1}{2} = \frac{9}{4} - y^2 \rightarrow y^2 = \frac{7}{4}$

$y = \pm \frac{\sqrt{7}}{2}$

so if $\lambda = -\frac{1}{2f(x,y,z)}(s) \Leftrightarrow \begin{cases} x = \frac{3}{2} \\ z = \frac{1}{2} \end{cases}$ $y = \pm\frac{\sqrt{7}}{2}$

In either case, $f(x,y,z) = \sqrt{\left(\frac{3}{2}-3\right)^2 + \frac{7}{4} + \frac{1}{4}}$

$= \sqrt{\frac{9}{4} + \frac{8}{4}} = \frac{1}{2}\sqrt{17} < \sqrt{5}$

so we conclude that: $\boxed{d(P,S) = \frac{\sqrt{17}}{2}}$

And is attained at two points. $\left(\frac{3}{2}, \pm\frac{\sqrt{7}}{2}, \frac{1}{2}\right)$.

The Lagrange multiplier method generalises to subsets defined by more equations, like a curve in $\mathbb{R}^3$

Example $\quad f(x,y,z) = x + y^2 z \qquad \begin{cases} y^2 + z^2 = 2 \\ z = x \end{cases} \quad (\mathcal{C})$

we can think of $(\mathcal{C})$ as a level set of the function

$$\begin{aligned} \phi : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x,y,z) &\longmapsto \begin{pmatrix} y^2 + z^2 - 2 \\ z - x \end{pmatrix} \end{aligned}$$

$\mathcal{C} = \phi^{-1}\{0\}$

Reasoning as we did for surfaces we can define (at points where $d\phi(x,y,z)$ is onto) the tangent vector space

$$\boxed{T_{(x,y,z)}\mathcal{C} = \ker d\phi(x,y,z)}$$

Here: $(h_1, h_2, h_3) \in T_{(x,y,z)}\mathcal{C}$

$$\Leftrightarrow \begin{cases} 2y h_2 + 2z h_3 = 0 \\ h_3 = h_1 \end{cases} \Leftrightarrow \begin{cases} (\vec{N_1}, \vec{h}) = 0 \\ (\vec{N_2}, \vec{h}) = 0 \end{cases}$$

The rank condition guarantees that these two equations are linearly independant where $\vec{N_1} = (0, 2y, 2z)$ $\vec{N_2} = (1, 0, -1)$

$d\phi(x,y,z)$ is surjective as long as $(y,z) \neq (0,0)$ which is the case on $\mathcal{C}$.

Now if $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a function and (x,y,z) is a critical point relative to $\mathcal{C}$ then infact we must have:

$$\boxed{\vec{\nabla} f(x,y,z) = \lambda_1 \vec{N_1} + \lambda_2 \vec{N_2}}$$

$$\begin{cases} 1 = \lambda_2 \\ 2yz = 2\lambda_1 y \\ y^2 = 2\lambda_1 z - \lambda_2 \end{cases} \iff \begin{cases} \lambda_2 = 1 \\ y(\lambda_1 - z) = 0 \\ y^2 = 2\lambda_1 z - 1 \end{cases}$$

If $y = 0$, $z = x = \pm\sqrt{2} = f(x,y,z)$

Therefore: $\lambda_1 = z$

$$\begin{cases} y^2 + z^2 = 2 \\ z = x \end{cases}$$

$$\begin{cases} y^2 - 2z^2 = -1 \\ y^2 + z^2 = 2 \\ z = x \end{cases} \iff \begin{cases} z^2 = 1 \\ y^2 = 1 \\ z = x \end{cases}$$

So the critical points are $(1,1,1)$ $(1,-1,1)$ $(-1,-1,-1)$ $(-1,1,-1)$ $(\sqrt{2},0,\sqrt{2})$, $(-\sqrt{2},0,-\sqrt{2})$

$f(1,1,1) = 2 \quad , \quad f(1,-1,1) = 2 \quad , \quad f(-1,-1,-1) = -2$

$f(-1,1,-1) = -2 \quad f(\sqrt{2},0,\sqrt{2}) = \sqrt{2} \quad f(-\sqrt{2},0,-\sqrt{2}) = -\sqrt{2}$

so $\min_{\mathcal{C}} f = -2 \quad , \max_{\mathcal{C}} f = 2.$

We can check this by parametrising the curve

$y = \sqrt{2}\cos\theta$

$z = \sqrt{2}\sin\theta \qquad \gamma(\theta) = \sqrt{2}\begin{pmatrix} \sin\theta \\ \cos\theta \\ \sin\theta \end{pmatrix}$

$$f(\gamma(\theta)) = \sqrt{2}\sin\theta + 2\sqrt{2}\cos^2\theta\sin\theta = g(\theta)$$

$$g(\theta) = \sqrt{2}\sin\theta\left(1 + 2\cos^2\theta\right)$$

$$g'(\theta) = +\sqrt{2}\cos\theta\left(1+2\cos^2\theta\right) - 4\sqrt{2}\sin^2\theta\cos\theta$$

$$= \sqrt{2}\cos\theta\left(1 + 2\cos^2\theta - 4\sin^2\theta\right)$$

$$= 3\sqrt{2}\cos\theta\cos(2\theta)$$ ← good old trig identities

θ	$-\pi$		$-\frac{3\pi}{4}$		$-\frac{\pi}{2}$		$-\frac{\pi}{4}$		$\frac{\pi}{4}$		$\frac{\pi}{2}$		$\frac{3\pi}{4}$		π
$\cos\theta$		$-$		$-$	0	$+$		$+$		$+$	0	$-$		$-$	
$\cos(2\theta)$		$+$	0	$-$		$-$	0	$+$	0	$-$		$-$	0	$+$	
		$-$	0	$+$	0	$-$	0	$+$	0	$-$	0	$+$	0	$-$	
	0	$\searrow$	-2	$\nearrow$	$-\sqrt{2}$	$\searrow$	-2	$\nearrow$	2	$\searrow$	$+\sqrt{2}$	$\nearrow$	2	$\searrow$	0

so we arrive at the same conclusions

Critical points: $\gamma(-\frac{\pi}{2}) = \sqrt{2}\begin{pmatrix}-1\\0\\-1\end{pmatrix}$, $\gamma(\frac{\pi}{2}) = \sqrt{2}\begin{pmatrix}1\\0\\1\end{pmatrix}$

$\gamma(-\frac{3\pi}{4}) = \begin{pmatrix}-1\\-1\\-1\end{pmatrix}$, $\gamma(-\frac{\pi}{4}) = \begin{pmatrix}-1\\1\\-1\end{pmatrix}$, $\gamma(\frac{\pi}{4}) = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$\gamma(\frac{3\pi}{4}) = \begin{pmatrix}1\\-1\\1\end{pmatrix}$

A general statement

Let $\phi : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map and $c \in \mathbb{R}^m$. Assume that: $d\phi(x,y,z)$ is of full rank for every $(x,y,z) \in \phi^{-1}(\{c\}) = M$

$(\phi^{-1}(\{c\}) = \{(x_1, \dots, x_n) \in \mathbb{R}^n,\ f(x_1, \dots, y_n) = c\})$

Let $f : \mathbb{R}^n \to \mathbb{R}$. Then a necessary condition for $(x,y,z) \in M$ to be a local extrema of f relative to M is that:

$$\boxed{\begin{gathered} df(x_1, \dots, x_n)(\vec{h}) = 0 \\ \text{for every } \vec{h} \in T_{(x,y,z)}M = \ker d\phi(x,y,z) \end{gathered}}$$

This means that; one can find $\lambda_1, \dots, \lambda_m$ such that if $\phi = (\phi^1, \dots, \phi^m)$

$$df(x_1, \dots, x_n) = \sum_{i=1}^{m} \lambda_i \, d\phi^i(x_1, \dots, x_n)$$

Dually, if $\vec{N}_i$ is a normal vector to the hyperplane $\ker d\phi^i(x_1, \dots, x_n)$ then:

$$\boxed{\vec{\nabla} f(x_1, \dots, x_n) = \sum_{i=1}^{m} \lambda_i \vec{N}_i}$$

Note: This is really a theorem about submanifolds of $\mathbb{R}^n$, which are the subsets of $\mathbb{R}^n$ where we can do differential calculus.