Math 262 31 août '23

0] Brief recap' on sequences (9-1)

① Real numbers $(\mathbb{R}, +, \cdot, \leq)$

$\mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N}$

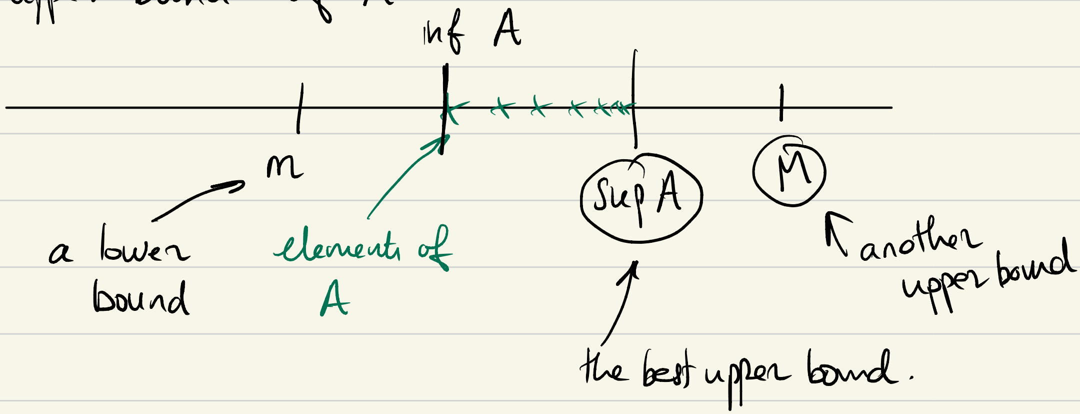
real rationals $\{\underline{0}, 1, 2, 3, \dots\}$

R satisfies the least upper bound axiom :

> Every non-empty subset A that has an upper bound has a least upper bound, written $\sup A$

Ex: $A = \{ q \in \mathbb{Q}, \ q^2 < 2 \}$ admits 2 as an upperbound, there is therefore $x = \sup A$ such that if $\varepsilon > 0$, $x - \varepsilon$ is not an upperbound.

In the example: $\boxed{\sup A = \sqrt{2}} \in \mathbb{R} \setminus \mathbb{Q}$ irrationnel sup A is the optimal solution to the problem of finding an upper bound of A



There are no upperbounds smaller than sup A

Similarly, inf A is the optimal solution to the problem of finding a lower bound of A.

> ⚠ inf A and sup A are not necessarily in the set A.

A sequence $(a_n)_{n\in\mathbb{N}}$ (with real values) is a map from $\mathbb{N}$ (or some subset $[n_0, +\infty) \subset \mathbb{N}$) to $\mathbb{R}$. For each $n \in \mathbb{N}$, $a_n \in \mathbb{R}$ is the "n^{th} term" of the sequence.

Examples : Fibonacci sequence

① $a_0 = 1$ $a_0 = 1$

For $n \geq 0$, $a_{n+2} = a_{n+1} + a_n$.

② $\forall n \in \mathbb{N}^*$, $a_n = \frac{1}{n}$. $\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, \dots\}$

Def Let $l \in \mathbb{R}$, we say that a sequence $(a_n)_{n\in\mathbb{N}}$ converges to l, we write : $\boxed{\lim_{n\to+\infty} a_n = l}$, if :

$$\boxed{\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}^*, \quad \forall n \geq N, \quad |a_n - l| \leq \varepsilon}$$

"For any error/tolerance ε, one can find some integer N such that for all integers greater than N, a_n approximates l to the given error ε."

As an example of how this definition works we will illustrate the proof of $\lim_{n\to+\infty} \frac{1}{n}$.

For this we will apply our axiom to discuss first:

Lemma: $\mathbb{R}$ is archimedean.

i.e Let $0 < a < b$ then there is $n \in \mathbb{N}$ such that $na > b$.

"There are no infinitely small real numbers"
i.e. if $0<a<b$, and I add a to itself n times, $\underbrace{a+\cdots+a}_{n}$ for large enough n the result will be greater than b."

Proof: Let $E = \{ n \in \mathbb{N},\ na < b \}$.
and consider $A = \{ na \ ; \ n \in E \} \subset \mathbb{R}$

$A = \emptyset$ because $a = 1 \cdot a < b$.

Since for every $n \in E$ $na < b$ it follows that
↑ is an upper bound
there is a lowest upper bound $\sup A$.

By definition $\sup A - a$ is not an upper bound

so there is $n_0 \in \mathbb{N}$ such that $\sup A - a < n_0 a$

but then $\sup A < (n_0+1)a$, so $(n_0+1)a \notin A$
i.e $(n_0+1)a \geq b$

$$\text{C19} \qquad \lim_{n \to +\infty} \frac{1}{n} = 0.$$

Proof Let $\varepsilon \in \mathbb{R}_+^*$, without loss of generality

we can choose $0 < \varepsilon < 1$. Now since $\mathbb{R}$ is archimedean, $\exists n_0 \in \mathbb{N}$, $n_0 \varepsilon \geq 1$

but if $n \geq n_0$ $n\varepsilon \geq n_0 \varepsilon \geq 1$

hence, for all $n \geq n_0$, $0 < \frac{1}{n} \leq \varepsilon$

$$\Updownarrow$$

$$0 < \frac{1}{n} - 0 \leq \varepsilon$$

$$\Updownarrow$$

$$\left|\frac{1}{n} - 0\right| \leq \varepsilon.$$

therefore $\boxed{\lim_{n \to +\infty} \frac{1}{n} = 0.}$

Another basic but important limit...

Let $q \neq 0$, what is: $\lim\limits_{n\to+\infty} q^n = ?$

if $|q|<1$ $\quad \lim\limits_{n\to+\infty} q^n = 0$

if $q=1$ $\quad \lim\limits_{n\to+\infty} q^n = 1$

if $q>1$ $\quad \lim\limits_{n\to+\infty} q^n = +\infty$

if $q \leq -1$ then the sequence has no limit

Partial proof: if $q>0$ $\quad q^n = e^{n\ln q}$.

If $0<q<1$, $\ln q<0$ and so $\lim\limits_{n\to+\infty} n\ln q = -\infty$

however $\lim\limits_{x\to-\infty} e^x = 0$, so $\lim\limits_{n\to+\infty} q^n = 0$; etc...

Monotone convergence theorem

Thm • Every increasing seq. (a_n) with upper bound converges. $\lim\limits_{n\to+\infty} a_n = \sup\{a_n, n\in\mathbb{N}\}$

• Every decreasing seq. (b_n) with a lower bound converges.

greatest lower bound

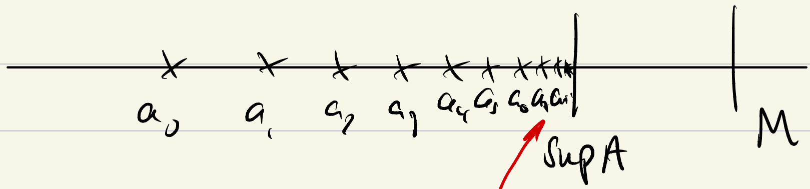
$$\lim_{n\to+\infty} b_n = \inf\{b_n ; n\in\mathbb{N}\}$$

+ (a_n) increasing, for every n $a_{n+1} \geq a_n$

+ (a_n) decreasing for every n $a_{n+1} \leq a_n$

"Proof" It follows from the least upper bound axiom but I will just illustrate the idea on a drawing:

Suppose some sequence $(a_n)_{n\in\mathbb{N}}$ is increasing and bounded, $A = \{a_n, n \in \mathbb{N}\}$



the terms of the sequence have no choice but to accumulate near sup A. (They may reach it and then the sequence will be constant)

Fancy mathematical way of saying "consequence"

Corollary: An increasing sequence either converges or diverges to infinity:

$$(\forall A > 0, \exists N \in \mathbb{N}, \forall n \geq N, \quad a_n \geq A)$$

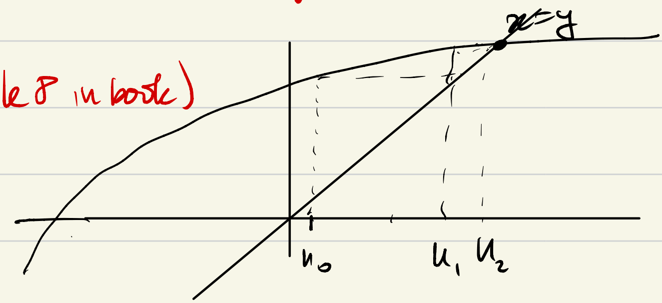
we write $\boxed{\lim_{n\to+\infty} a_n = +\infty}$.

Exercise Can you write what $\lim_{n\to+\infty} a_n = -\infty$ means?

Remark In fact it is sufficient that the monotonic behaviour be attained after a finite number of terms.

Complete example (Example 8 in book)

$$\begin{cases} u_0 = 1 \\ u_{n+1} = \sqrt{6+u_n} \end{cases}$$



If (u_n) converges then since $x \mapsto \sqrt{6+x}$ is a continuous function: $\ell = \sqrt{\ell+6} \iff \ell^2 - \ell - 6 = 0$

$\iff (\ell+2)(\ell-3) = 0$

so the limit is either -2 or 3.

Now we need to show that a limit exists.

First we note that: $x \mapsto \sqrt{6+x}$ is an increasing function. Also $u_0 = 1$, $u_1 = \sqrt{7}$ $u_0 < u_1$.

and by immediate induction (u_n) is increasing.

so the limit is either 3 or $+\infty$.

If you can show it is bounded we are ok.

But $f([0,3]) \subset [0,3]$ so: $\forall n \in \mathbb{N}$, $u_n \in [0,3]$

Some facts that we will use

+ A function $f: I \to \mathbb{R}$ is continuous at $x_0 \in I$ iff
$\lim_{n\to+\infty} f(a_n) = f(x_0)$ for every sequence $a_n \xrightarrow[n\to+\infty]{} x_0$.

+ Let $a, L \in \mathbb{R} \cup \{-\infty, \infty\}$ $\quad f: \mathbb{R} \to \mathbb{R}$

$\lim_{x\to a} f(x) = L$ iff for every sequence $(x_n)_{n\in\mathbb{N}}$
such that $\lim_{n\to+\infty} x_n = a$,
$\lim_{n\to+\infty} f(x_n) = L$

In general we will not apply the definition to calculate limits but will use these results that you know from before: (Phew!)

Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be convergent sequences,

Such that: $\lim_{n\to+\infty} a_n = a$, $\lim_{n\to+\infty} b_n = b$

$$\lim_{n\to+\infty} (a_n + b_n) = \lim_{n\to+\infty} a_n + \lim_{n\to+\infty} b_n$$

$$\lim_{n\to+\infty} (a_n b_n) = ab$$

if $b \neq 0$ $\quad \lim_{n\to+\infty} \frac{a_n}{b_n} = \frac{a}{b}$. $\quad$ etc...

Comparison

Suppose that $\lim_{n\to+\infty} a_n$ and $\lim_{n\to+\infty} b_n$ exist (they may be $+\infty$ or $-\infty$).

Then:

If there is $n_0 \in \mathbb{N}$ such that

for any $n \geq n_0$, $a_n \leq b_n$

then $\lim_{n\to+\infty} a_n \leq \lim_{n\to+\infty} b_n$

"Squeeze" theorem (Théorème des gendarmes)

If (a_n), (b_n), (c_n) are three sequences:

and ① $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n = L \in \mathbb{R}$

② there is $n_0 \in \mathbb{N}$, such that:

for all $n \geq n_0$, $a_n \leq c_n \leq b_n$

then $\lim_{n\to+\infty} c_n$ exists and

$\lim_{n\to+\infty} c_n = L$.

Example: if $\lim_{n\to+\infty} |a_n| = 0$, then $-|a_n| \leq a_n \leq |a_n|$

so $\lim_{n\to+\infty} a_n = 0$.

Algebraic operations and limits

It would be really great if to calculate the limit of an expression we could just split it up into parts that have a known limit and then just apply algebraic operations to the limits.
This works for finite limits but we sometimes run into trouble when we try to include "$+\infty$ and $-\infty$"

NOT NUMBERS

Addition table

$\lim_{n\to+\infty} b_n$ \ $\lim_{n\to+\infty} a_n$	$a\in\mathbb{R}$	$+\infty$	$-\infty$
$b\in\mathbb{R}$	$a+b$	$+\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	
$-\infty$	$-\infty$		$-\infty$

The squares /// are indeterminate forms.
We cannot extend "continuously" addition to include $+\infty, -\infty$.

Eg. $a_n = n$ $\quad \lim_{n\to+\infty} a_n = +\infty$ $\quad a_n + b_n = 0$

$b_n = -n$ $\quad \lim_{n\to+\infty} b_n = -\infty$, so $\lim_{n\to+\infty} a_n + b_n = 0$

but if you look at $c_n = -n+1$ then $\lim_{n\to+\infty} c_n = -\infty$

however, $a_n + b_n = 1$ so $\lim\limits_{n \to +\infty} (a_n + b_n) = 1$

Multiplication table

$\lim\limits_{n\to+\infty} b_n$ \ $\lim\limits_{n\to+\infty} a_n$	0	$a>0$	$a<0$	$+\infty$	$-\infty$
0	0	0	0		
$b>0$	0	ab	ab	$+\infty$	$-\infty$
$b<0$	0	ab	ab	$-\infty$	$+\infty$
$+\infty$		$+\infty$	$-\infty$	$+\infty$	$-\infty$
$-\infty$		$-\infty$	$+\infty$	$-\infty$	$+\infty$

Example

$a_n = \frac{1}{n^2}$ $\qquad \lim\limits_{n\to+\infty} a_n = 0$ $\qquad a_n b_n = \frac{1}{n} \xrightarrow[n\to+\infty]{} 0$

$b_n = n$ $\qquad \lim\limits_{n\to+\infty} b_n = +\infty$

However:

$a_n = \frac{1}{n}$ $\qquad \lim\limits_{n\to+\infty} a_n = 0$ $\qquad a_n b_n = 1 \xrightarrow[m\to+\infty]{} 1$

$b_n = n$ $\qquad \lim\limits_{n\to+\infty} b_n = +\infty$

$\boxed{\text{The indeterminate forms have to be dealt with on a case by case basis}}$

l'Hospital's rule

A rule that is sometimes invaluable is l Hospitals rule.

Prop Let f and g be differentiable functions and suppose that $f(a) = g(a) = 0$ then

$$\lim_{x \to a} \frac{f(a)}{g(a)}$$ is indeterminate of the form $0 \cdot \infty$

If $f'(a) \neq 0$ and $g'(a) \neq 0$ then,

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}}.$$

Proof

$$\frac{f(x)}{g(x)} = \frac{f(x) - \overset{=0}{f(a)}}{g(x) - g(a)} = \underbrace{\frac{f(x)-f(a)}{x-a}}_{\xrightarrow{x \to a} f'(a)} \cdot \underbrace{\frac{x-a}{g(x)-g(a)}}_{\xrightarrow{x \to a} \frac{1}{g'(a)}}.$$

Example ① $\lim_{x \to 0} \frac{\sin(x)}{x}$

$f(x) = \sin(x)$, $f(0) = 0$

$g(x) = x$ $g(0) = 0$

$f'(x) = \cos(x)$, $f'(0) = 1$

$g'(x) = 1$

L'Hospital's rule $\Rightarrow$ $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$.

Therefore: $\lim_{n \to +\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n \to +\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$

$= \lim_{x \to 0} \frac{\sin x}{x} = 1$

because

$\lim_{n \to +\infty} \frac{1}{n} = 0$.

1] Series

A series, written "$\sum_{n\in\mathbb{N}} a_n$", is the data consisting of ① a sequence of reals $(a_n)_{n\in\mathbb{N}}$ known as the general term of the series.

② The sequences of $\boxed{\text{partial sums:}}$ $S_N = \sum_{n=0}^{N} a_n$.

We say that a sequence converges if $(S_N)_{N\in\mathbb{N}}$ converges to a finite limit, and in this case we write $\sum_{n=0}^{+\infty} a_n = \lim_{N\to+\infty} S_N$.

Examples: $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $\left(\sum_{n\geq 0} (-1)^n\right)$ does not converge.

Special case. Series with non-negative terms

Assume that the general term of $\sum_{n\geq 0} a_n$

is such that: $\forall n \in \mathbb{N},\ a_n \geq 0$.

It follows that the sequence of partial sums

is non-decreasing and therefore the

$\lim_{N\to+\infty} S_N$ exists in $\mathbb{R}_+ \cup \{+\infty\}$.

To prove that such a sequence converges

it is necessary and sufficient to show that

$\exists M \in \mathbb{R}_+^*,\ \forall N \in \mathbb{N},\ 0 \leq S_N \leq M$.

Fundamental example : Geometric series.

Let $q > 0$. Consider $\sum_{n \geq \mathbb{N}} q^n$.

$q \neq 1$

Can we determine the sum ?

$$S_{N+1} = 1 + q + q^2 + \dots + q^{N+1}$$

magic. $S_{N+1} - 1 = q S_N$.

$$\begin{array}{c} S_{N+1} - 1 = q S_N \\ \| \\ S_N + q^{N+1} - 1 = q S_N \end{array}$$

$$(1-q) S_N = 1 - q^{N+1}.$$

if $q \neq 1$ $\quad \boxed{S_N = \dfrac{1-q^{N+1}}{1-q}}$

So we can study it directly:

if $q > 1 \Rightarrow S_N \longrightarrow +\infty$.

if $q<1$ $\quad S_N \xrightarrow{} \frac{1}{1-q}$.

$$\Rightarrow \sum_{n=0}^{+\infty} q^n = \frac{1}{1-q} \quad \text{if } 0<q<1$$

Ex: $\displaystyle\sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{+\infty} \frac{1}{2^n} = 2$.

if $q=1$ what happens: $\quad S_N = N+1 \xrightarrow[N\to+\infty]{} +\infty$.

Criterion for convergence of positive sequences

⚠ Common error (please don't make it!)

If $\left(\sum_{n\geq 0} a_n\right)$ converges then $\lim\limits_{N\to+\infty} (S_N - S_{N-1}) = 0$

but $S_N - S_{N-1} = a_N$ so $\lim\limits_{N\to+\infty} a_N = 0$.

The term general of a convergent series necessarily converges to 0, but this is NOT sufficient; in particular:

$$\sum_{n=1}^{+\infty} \frac{1}{n} \quad \text{diverges.}$$

Proof that $\sum_{n\geq 1} \frac{1}{n}$ diverges (using the "integral" test)

I can't calculate the partial sums ☹ but I can estimate them!

The map $x \longmapsto \frac{1}{x}$ is decreasing on $\mathbb{R}_+^*$, so

for any $x \in [n, n+1]$, $\frac{1}{n+1} \leq \frac{1}{x} \leq \frac{1}{n}$.

<u>Now</u>: $\frac{1}{n+1} = \int_n^{n+1} \frac{1}{n+1}\,dx \leq \int_n^{n+1} \frac{1}{x}\,dx \leq \int_n^{n+1} \frac{1}{n} = \frac{1}{n}$

But if I sum over this inequality:

$$\sum_{n=1}^{N} \frac{1}{n+1} \leq \underbrace{\sum_{n=1}^{N} \int_n^{n+1} \frac{1}{x}\,dx}_{\int_1^{N+1} \frac{1}{x}\,dx \,=\, \ln(N+1)} \leq \sum_{n=1}^{N} \frac{1}{n}$$

therefore for any $N \in \mathbb{N}$, $\ln(N+1) \leq \sum_{n=1}^{N} \frac{1}{n}$

Since $\lim_{N\to+\infty} \ln(N+1) = +\infty$

$$\lim_{N\to+\infty} \sum_{n=1}^{N} \frac{1}{n} = +\infty.$$

$\boxed{\text{The series diverges to } +\infty.}$

Divergence test

$\boxed{\text{If } (a_n)_{n\in\mathbb{N}} \text{ does not converge to } 0 \text{ then the series } \left(\sum a_n\right)_{n\in\mathbb{N}} \text{ does not converge}}$

The converse is false.

BASIC COMPARISON THEOREM FOR POSITIVE SERIES

Recall that to show that a positive series converges it is sufficient to show that the partial sums are bounded.

Thm: Suppose that there is a constant $M>0$ and a rank n_0 such that for every $n \geq n_0$, $0 \leq a_n \leq M b_n$

then ① $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges

② $\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges

Proof ① If $0 \leq a_n \leq M b_n$ for $n \geq n_0$ and $(\sum b_n)$ cv

then $$0 \leq \sum_{n=n_0}^{N} a_n \leq M \sum_{n=n_0}^{N} b_n \leq M \sum_{n=n_0}^{+\infty} b_n$$

Therefore the partial sums of the positive series $(\sum_{n \geq n_0} a_n)$ are bounded and so $\sum_{n=n_0}^{+\infty} a_n < +\infty$.

therefore $$\sum_{n=0}^{+\infty} a_n = \sum_{n=0}^{n_0-1} a_n + \sum_{n=n_0}^{+\infty} a_n < +\infty$$

② Similar argument.

Remark If $0 \leq a_n \leq M b_n$ for $n \geq n_0$

and $\sum_{n=0}^{+\infty} b_n = +\infty$ then this inequality does not

teach us anything about $\left(\sum_{n\geq 0} a_n\right)$. Similarly

convergence of $\left(\sum_{n\geq 0} a_n\right)$ does not tell us anything

about $\left(\sum b_n\right)$

The series $\left(\sum_{n\geq 1} \frac{1}{n^p}\right)$, $p \in \mathbb{R}$

We will now increase our population of examples through the study of the series $\left(\sum_{n\geq 1} \frac{1}{n^p}\right)$.

This will illustrate some of the techniques.

<u>Case 1</u> $p \leq 0$

If $p \leq 0$ then the sequence $\left(\frac{1}{n^p}\right)_{n\in\mathbb{N}^*}$ does not converge to zero, by the Divergence test, $\left(\sum_{n\geq 1} \frac{1}{n^p}\right)$ does not converge. Since they are positive series they <u>diverge to $+\infty$</u>.

That was easy! So we restrict to <u>$p > 0$</u>

We already know from last lecture that

$\sum_{n \geq 1} \frac{1}{n}$ diverges to $+\infty$. We shall now

try to apply our Comparison theorem to

study some cases.

Consider the function f defined by $f(p) = \frac{1}{n^p}$
$= e^{-p \ln n}$

where $n \geq 1$ is a fixed integer.

Since $f'(p) = -(\ln n) \frac{1}{n^p}$, since $n \geq 1$,

$f'(p) \leq 0$ and so the function f is decreasing.

In particular if $p \leq 1$ then

$$f(1) \leq f(p)$$

ie.
$$\frac{1}{n} \leq \frac{1}{n^p}$$

Since n was fixed but arbitrary this shows that for all $n \geq 1$, and all $0 < p \leq 1$

$$0 < \frac{1}{n} \leq \frac{1}{n^p}.$$

Since we know that $\sum_{n=0}^{+\infty} \frac{1}{n} = +\infty$, by the comparison theorem this shows that:

When $0 < p \leq 1$, $\sum_{n=0}^{+\infty} \frac{1}{n^p} = +\infty$

However, we learn nothing about the case $p > 1$.

For this we need another argument...

The case $p>1$

We shall repeat the argument we used for $\sum \frac{1}{n}$.

The idea is to compare the partial sums $\sum_{n=1}^{N} \frac{1}{n^p}$, which we don't know how to compute, to $\int_1^N \frac{1}{x^p}\,dx$, which we do. "we will try to squeeze $\frac{1}{x^p}$" between two terms of our sum". For this we consider now for $p>1$ fixed but arbitrary the function f defined by $f(x) = \frac{1}{x^p} = x^{-p}$ $(x>0)$, then $f'(x) = -p\,x^{-(p+1)}$, so f is decreasing on $(0,+\infty) = \mathbb{R}_+^*$.

In particular, for any $x \in [n, n+1]$, where $n \geq 1$

is an arbitrary integer, we have:

$$\frac{1}{(n+1)^p} \leqslant \frac{1}{x^p} \leqslant \frac{1}{n^p}.$$

Integrating from n to $n+1$ we find:

$$\underbrace{\int_n^{n+1} \frac{1}{(n+1)^p}\,dx}_{=\frac{1}{(n+1)^p}} \leqslant \int_n^{n+1} \frac{1}{x^p}\,dx \leqslant \underbrace{\int_n^{n+1} \frac{1}{n^p}\,dx}_{=\frac{1}{n^p}}$$

so:

$$\boxed{\frac{1}{(n+1)^p} \leqslant \int_n^{n+1} \frac{1}{x^p}\,dx \leqslant \frac{1}{n^p}}$$

<u>for every $n \geq 1$</u>

Now we sum these inequalities up to $N \in \mathbb{N}$

$$\sum_{n=1}^{N} \frac{1}{(n+1)^p} \leq \sum_{n=1}^{N} \int_n^{n+1} \frac{1}{x^p}\, dx \leq \sum_{n=1}^{N} \frac{1}{n^p}$$

But: $$\sum_{n=1}^{N} \int_n^{n+1} \frac{1}{x^p}\, dx = \int_1^{N+1} \frac{1}{x^p}\, dx.$$

Therefore:

$$\boxed{\sum_{n=1}^{N} \frac{1}{(n+1)^p} \leq \int_1^{N+1} \frac{1}{x^p}\, dx \leq \sum_{n=1}^{N} \frac{1}{n^p}}$$

Now:

$$\int_1^{N+1} \frac{1}{x^p}\, dx = \left[\frac{1}{(1-p)} \frac{1}{x^{p-1}}\right]_1^{N+1}$$

$$= \frac{1}{(1-p)} \frac{1}{(N+1)^{p-1}} + \frac{1}{p-1}$$

Since $p>1$, $\lim_{N \to +\infty} \frac{1}{(N+1)^{p-1}} \xrightarrow[N \to +\infty]{} 0$

we see that the middle term has a finite limit when $N \to +\infty$.

It follows now from the first inequality that

$$\sum_{n=1}^{N} \frac{1}{(n+1)^p} = \sum_{n=2}^{N+1} \frac{1}{n^p} = \sum_{n=1}^{N+1} \frac{1}{n^p} - 1$$

is bounded. Therefore as it is a positive series

$$\boxed{\sum_{n=1}^{+\infty} \frac{1}{n^p} < +\infty \quad \text{when } p > 1}$$

Classical tests for convergence of positive series

In general since we cannot calculate partial sums we will use our comparison theorem and known examples to infer convergence or divergence of arbitrary positive series. The following tests summarise the most used arguments.

Test 1 : The limit test

Consider two (eventually) positive series $(\sum a_n)$, $(\sum b_n)$. Suppose that:

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = L \in \mathbb{R}_+ \cup \{\infty\}$$

Then: if $L < +\infty$ then $\sum a_n$ cv if $\sum b_n$ cv.

if $L = +\infty$, if $\sum b_n$ diverges then so does $\sum a_n$

Remark: if $L > 0$ and finite then $\sum a_n$ cv iff $\sum b_n$ cv.

Example: $\sum_{n\geq 1} \sin\left(\frac{1}{n^2}\right)$. This is a positive series and so we will compare with $\sum_{n\geq 1} \frac{1}{n^2}$ using the limit test: $a_n = \sin\left(\frac{1}{n^2}\right)$ $b_n = \frac{1}{n^2}$

$$\frac{a_n}{b_n} = n^2 \sin\left(\frac{1}{n^2}\right)$$

$$\lim_{n\to+\infty} n^2 \sin\left(\frac{1}{n^2}\right) = \lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

therefore since $\sum_{n\geq 1} \frac{1}{n^2}$ converges $\sum_{n\geq 1} \sin\left(\frac{1}{n^2}\right)$ converges by the limit test.

The next tests follow by comparing to geometric series

Test 2: Cauchy's root test

Consider a positive series $\sum_{n \in \mathbb{N}} a_n$.

Suppose that $\lim_{n \to +\infty} a_n^{\frac{1}{n}} = \rho$

If $\rho < 1$ then the series converges.

If $\rho > 1$ then the series diverges

If $\rho = 1$ the test is inconclusive.

Remark Try to apply this test to $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n}$ to see why the case $\rho = 1$ is inconclusive.

Remark Note the resemblance between these cases and that we found when studying $\sum_{n \geq 1} \frac{1}{q^n}$ $q > 0$

Test 3 D'Alembert's ratio test

Consider a (eventually) positive series

Suppose $\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = \rho$ then:

- if $\rho < 1$ the series converges
- if $\rho > 1$ the series diverges
- if $\rho = 1$ the test is inconclusive.

Example $\sum_{n \geq 0} \frac{1}{n!}$ $a_n = \frac{1}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \xrightarrow[n \to +\infty]{} 0$$

Therefore by D'Alembert's ratio test $\sum_{n \geq 0} \frac{1}{n!}$ converges.

Examples of application of our criterion:

× $\left(\sum_{n\geq 0} n\right)$, $\left(\sum_{n\geq 0} n^2\right)$ $\left(\sum_{n\geq 1} n\sin\left(\frac{1}{n}\right)\right)$

What can be said about these series ?

THEY DIVERGE

Why: The general term does not converge to 0.

⇒ Begin your study of a series by first checking that it doesn't grossly diverge.

If the general term converges to 0, then now we have to start some work:

Ex 1: $\left(\sum_{n\geq 3} \frac{1}{n(n-1)(n-2)}\right)$ → general term converges to 0

The limit test (compare with $\sum \frac{1}{n^3}$) shows that this series converges.

It turns out we can in fact calculate this sum it is a telescopic series.

Since the general term is a rational function we can do a partial fraction decomposition.

$$\frac{1}{n(n-2)(n-1)} = \frac{1}{2n} + \frac{1}{2(n-2)} - \frac{1}{n-1}$$

$$= \frac{1}{2}\left(\frac{1}{n} + \frac{1}{(n-2)} - \frac{2}{n-1}\right)$$

<u>Therefore</u>: For $N \geq 3$

Finite so I can seperate

$$S_N = \frac{1}{2}\sum_{n=3}^{N}\left(\frac{1}{n} + \frac{1}{(n-2)} - \frac{2}{n-1}\right)$$

$$= \frac{1}{2}\left(\sum_{n=3}^{N}\frac{1}{n} + \sum_{n=3}^{N}\frac{1}{n-2} - 2\sum_{n=3}^{N}\frac{1}{n-1}\right)$$

$$= \frac{1}{2}\left(\sum_{n=3}^{N}\frac{1}{n} + \sum_{n=1}^{N-2}\frac{1}{n} - 2\sum_{n=2}^{N-1}\frac{1}{n}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{2} + \frac{1}{N} - \frac{1}{N-1} + 1\right)$$

$$= \frac{1}{4}\left(1 - \frac{1}{2N(N-1)}\right) \xrightarrow[N\to+\infty]{} \frac{1}{4}$$

therefore $\boxed{\sum_{n=3}^{+\infty} \frac{1}{n(n-1)(n-2)} = \frac{1}{4}}$

Remark : This computation also proves convergence.

Ex 2 : $\left(\sum_{n\geq 3} \frac{2^n + 3^n}{7^n}\right)$

This is a slight adaptation of a geometric series

Since $\frac{2^n+3^n}{7^n} = \underbrace{\left(\frac{2}{7}\right)}_{<1}^n + \underbrace{\left(\frac{3}{7}\right)}_{<1}^n$

So it is the sum of two convergent geometric series & therefore converges.

$$\sum_{n=3}^{+\infty} \frac{2^n+3^n}{7^n} = \sum_{n=3}^{+\infty}\left(\frac{2}{7}\right)^n + \sum_{n=3}^{+\infty}\left(\frac{3}{7}\right)^n$$

$$= \sum_{n=0}^{+\infty}\left(\frac{2}{7}\right)^{n+3} + \sum_{n=0}^{+\infty}\left(\frac{3}{7}\right)^{n+3}$$

no worries here as everything is positive

Therefore: $$\sum_{n=3}^{+\infty} \frac{2^n+3^n}{7^n} = \left(\frac{2}{7}\right)^3 \left(\frac{1}{1-\frac{2}{7}}\right) + \left(\frac{3}{7}\right)^3 \left(\frac{1}{1-\frac{3}{7}}\right)$$

$$= \left(\frac{2}{7}\right)^3 \frac{7}{5} + \left(\frac{3}{7}\right)^3 \left(\frac{7}{4}\right)$$

$$= \frac{8}{49} \times \frac{1}{5} + \frac{27}{49} \times \frac{1}{4}$$

$$= \frac{1}{49}\left(\frac{32 + 135}{20}\right)$$

$$= \frac{1}{49}\left(\frac{167}{20}\right) = \frac{167}{980}$$

These are a few rare cases where we can evaluate the sum immediately, in general we can only talk about convergence.

Example 3 $\left(\sum_{n\geq 1} \ln\left(1+\frac{1}{n^2}\right)\right)$ Converges by the limit test.

Example 4 $\left(\sum_{n\geq 1}\left(\frac{1+\sin n}{n^2}\right)\right)$ the limit test doesn't work but:

$$0 \leq \frac{1+\sin n}{n^2} \leq \frac{2}{n^2}$$

So by the comparison theorem $\sum_{n\geq 1}\left(\frac{1+\sin n}{n^2}\right)$ converges.

Example 5 $\sum_{n=0}^{+\infty} \frac{(2n)!}{(n!)^2}$

Lets try D'Alembert's test:

$$\frac{(2(n+1))!}{(n+1)!^2} \times \frac{(n!)^2}{(2n)!} = \frac{(2n+1)(2n+2)}{(n+1)^2} \xrightarrow[n\to+\infty]{} 4 > 1$$

so $\sum_{n=0}^{+\infty} \frac{(2n)!}{(n!)^2}$ diverges

General culture Stirlings formula. $\boxed{n! \underset{n\to+\infty}{\sim} \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$

the ratio converges to 1.

Example 6 $\left(\sum_{n\geq 1} \frac{2^{n+1}}{n^n}\right) \longrightarrow$ root test

$$\lim_{n\to+\infty} \left(\frac{2^{n+1}}{n^n}\right)^{\frac{1}{n}} = \lim_{n\to+\infty} e^{\frac{(n+1)\ln 2}{n} - \ln n}$$

$$= 0$$

Therefore $\sum_{n\geq 1} \frac{2^{n+1}}{n^n}$ converges by (Cauchy's) root test

Example 7 $\sum_{n=1}^{\infty} \frac{1}{\pi^n - n^\pi}$

$$\pi^n - n^\pi = e^{n\ln\pi} - e^{\pi \ln n}$$

$$= e^{n\ln\pi}\left(1 - e^{\pi\ln n - n\ln\pi}\right)$$

$$= \underbrace{e^{n\ln\pi}}_{+\infty}\left(1 - \underbrace{e^{-n\ln\pi\left(1 - \pi\frac{\ln n}{n}\right)}}_{0}\right)$$

The general term converges to 0 — Solution ① root test

$$(\pi^n - n^\pi)^{-\frac{1}{n}} = e^{-\frac{1}{n}\ln(\pi^n - n^\pi)}$$

$$= e^{-\ln\pi + \ln\left(1 - e^{-n\ln\pi\left(1 - \pi\frac{\ln n}{n}\right)}\right)}$$

so $\lim\limits_{n\to+\infty} (\pi^n - n^\pi)^{-\frac{1}{n}} = \frac{1}{\pi} < 1$

Solution ② limit test

$$\frac{\pi^n}{\pi^n - n^\pi} = \frac{1}{1 - \frac{n^\pi}{\pi^n}}$$

$$\frac{n^\pi}{\pi^n} = e^{\pi\ln n - n\ln\pi} = e^{\pi n\left(\frac{\ln n}{n} - \ln\pi\right)} \xrightarrow[n\to+\infty]{} 0$$

$\lim\limits_{n\to+\infty} \frac{\pi^n}{\pi^n - n^\pi} = 1$ ie $\frac{1}{\pi^n} \underset{n\to+\infty}{\sim} \frac{1}{\pi^n - n^\pi}$

Since $\sum\limits_{n\geq 1} \frac{1}{\pi^n}$ converges so does $\sum\limits_{n\geq 1} \frac{1}{\pi^n - n^\pi}$.

Every case is different: practice

APPROXIMATING THE SUM OF A POSITIVE SERIES

In general, we cannot determine a formula for the sum of a series. Although we cannot find an expression for the partial sums we can evaluate them numerically.

Since, by definition, $\lim_{N\to+\infty} \sum_{n=0}^{N} a_n = \sum_{n=0}^{+\infty} a_n = s$

we know that $\lim_{N\to+\infty} |s - s_N| = \left|\sum_{n=N+1}^{+\infty} a_n\right| \xrightarrow[n\to+\infty]{} 0$

So for large enough N we can use s_N as an approximation for s to a given precision $\varepsilon > 0$.

But to do this we need to determine what "N large enough means"

Therefore we would like to estimate $\left|\sum_{n=N+1}^{+\infty} a_n\right|$

for large N. In this lecture we will present two methods for this.

(A) Geometric bounds

If $\sum_{n\geq 1} \frac{1}{q^n}$ is a convergent geometric series

we can calculate its "tail" or "remainder"

$$\boxed{\sum_{n=N+1}^{+\infty} \frac{1}{q^n} = \frac{q^{N+1}}{1-q}.}$$

← you should know how to find this fast

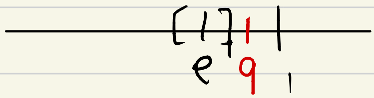
Whilst it is not particularly useful to use the partial sums to estimate the sum in this case (we know it explicitly!) It can be useful for getting bounds on sums that we can't calculate but that we can compare to geometric series.

This includes series to which we can apply the root or ratio tests. To see this let us study the proof of this test.

Proof of the ratio test in the case $\frac{a_{n+1}}{a_n} \to \rho < 1$

Suppose $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} \to \rho < 1$ $\quad a_n \geq 0$ for all $n \in \mathbb{N}$.

Choose a number $\rho < q < 1$ and note that there is $N_0 \in \mathbb{N}$ such that



for $n \geq N_0$ $\quad 0 \leq \frac{a_{n+1}}{a_n} \leq q$

then for all $n \geq N_0$ $\quad a_{n+1} \leq q a_n$

hence for all $n \in \mathbb{N}$, $0 \leq a_{N_0+n} \leq q^n a_{N_0}$ $\quad (*)$

This concludes the proof of the test but we can exploit further

For any $N \geq N_0$

$$0 \leq \sum_{n=N+1}^{+\infty} a_n = \sum_{n=N+1-N_0}^{+\infty} a_{n+N_0} \leq \sum_{n=N-N_0+1}^{+\infty} q^n a_{N_0}$$

$$\boxed{0 \leq \sum_{n=N+1}^{+\infty} a_n \leq \frac{q^{N-N_0+1}}{1-q} a_{N_0}}$$

2 choices in this formula $\rightarrow q$
$\rightarrow N_0$

<u>Example</u>: $\sum_{n\geq 1} \frac{1}{n!}$, $\frac{a_{n+1}}{a_n} = \frac{1}{n+1}$ take $q = \frac{1}{2}$

we see that for all $n \geq 1$ $\frac{a_{n+1}}{a_n} \leq \frac{1}{2}$, ie $N_0 = 1$

so we can apply the above if $N \geq 1$

$$\sum_{n=N+1}^{+\infty} \frac{1}{n!} \leq \left(\frac{1}{2}\right)^N \times \frac{1}{1-\frac{1}{2}} \times 1 = \frac{1}{2^{N-1}}$$

we could in this case get a much better error estimate if take $q = \frac{1}{N+1}$ then $N_0 = N$ and we have

$$\boxed{\sum_{n=N+1}^{+\infty} \frac{1}{n!} \leq \frac{1}{N!} \frac{1}{N+1} \frac{1}{1-\frac{1}{N+1}} = \frac{1}{N!\,N}}$$

N.B In the text book, they calculate $\sum_{n=N}^{+\infty} \frac{1}{n!}$

so using the above estimate $\boxed{\sum_{n=N}^{+\infty} \frac{1}{n!} = \frac{1}{N} + \sum_{n=N+1}^{+\infty} \frac{1}{n!} \leq \frac{N+1}{N!\,N}}$

Remark: The proof of the root test is almost identical to this and one can derive similar bounds:

Proposition

① Take $\rho < q < 1$, find N_0 such that for all $n \geq N_0$, $(a_n)^{\frac{1}{n}} < q$, then $a_n < q^n$ if $n \geq n_0$

so for $N \geq N_0$, $\boxed{0 \leq \sum_{n=N+1}^{+\infty} a_n \leq \sum_{n=N+1}^{+\infty} q^n = \frac{q^{N+1}}{1-q}}$

Example $\sum_{n \geq 0}^{+\infty} \frac{2^{n+1}}{n^n}$ $\quad 0 \leq \left(\frac{2^{n+1}}{n^n}\right)^{\frac{1}{n}} = \frac{2\, 2^{\frac{1}{n}}}{n}$ decreases to 1

$\leq \frac{4}{n}$ ← easier to estimate.

lets take $q = \frac{4}{N}$, $\frac{4}{n} \leq \frac{4}{N} = q \iff n \geq N = N_0$

Hence, $\boxed{\sum_{n=N+1}^{+\infty} a_n \leq 4\left(\frac{4}{N}\right)^N \frac{1}{N-1}}$

(B) Integral bounds.

Thm Let $f : \mathbb{R}_+^0 \longrightarrow \mathbb{R}_+$ be continuous non-negative decreasing function.

Consider the positive series $\sum_{n \geq 1} f(n)$ then:

1. for every choice $0 \leq N \leq M$ $\quad N, M \in \mathbb{N}$

$$\sum_{n=N+1}^{M+1} f(n) \leq \int_N^{M+1} f(x)\,dx \leq \sum_{n=N}^{M} f(n) \quad (IE)$$

2. $\sum_{n=1}^{+\infty} f(n) < +\infty$ if and only if $\int_1^{+\infty} f(x)\,dx < +\infty$

Proof: See the study of $\sum \frac{1}{n^p}$.

if $\sum_{n=1}^{+\infty} f(n) < +\infty$ sending $N \to +\infty$ in (IE)

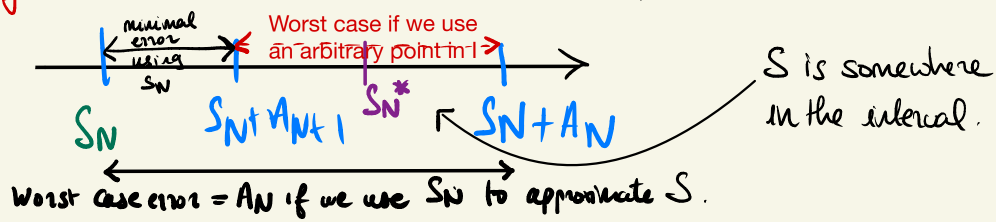
$$\underbrace{\int_{N+1}^{+\infty} f(x)\,dx}_{A_{N+1}} \leq \sum_{n=N+1}^{+\infty} f(n) \leq \underbrace{\int_N^{+\infty} f(x)\,dx}_{A_N}$$

This tells us that $S = \sum_{n=1}^{+\infty} f(n)$ is in the interval: $[S_N + A_{N+1}, \ S_N + A_N] = I$

Therefore any number in this interval approximates

$$\boxed{S \text{ with error at most } A_N - A_{N+1}}.$$

In particular, if we use $s_N^* = \frac{A_N + A_{N+1}}{2} + S_N$ (the midpoint of the interval) this gives a slightly better estimate than S_N.



$$S - S_N^* = S - S_N - \frac{A_N + A_{N+1}}{2}$$

therefore:

$$-\frac{A_N - A_{N+1}}{2} \leq S - S_N^* \leq \frac{A_N - A_{N+1}}{2}$$

ie $\boxed{|S - S_N^*| \leq \frac{A_N - A_{N+1}}{2}.}$

whereas our best upper bound on $S - S_N$ is

$$\boxed{0 \leq A_{N+1} \leq S - S_N \leq A_N}$$

<u>Example</u> $\sum_{n\geq 1} \frac{1}{n^p}$ $f(x) = \frac{1}{x^p}$ $p > 1$

$$\int_{N+1}^{+\infty} \frac{1}{x^p}dx \leq \sum_{n=N+1}^{+\infty} \frac{1}{n^p} \leq \int_N^{+\infty} \frac{1}{x^p}dx$$

$$\frac{1}{p-1}\frac{1}{(N+1)^{p-1}} \leq \sum_{n=N+1}^{+\infty} \frac{1}{n^p} \leq \frac{1}{p-1}\frac{1}{N^{p-1}}$$

so $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ is in the interval $\left[\frac{1}{p-1}\frac{1}{(N+1)^{p-1}} + S_N, \frac{1}{p-1}\frac{1}{N^{p-1}} + S_N\right]$

S_N estimates $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ with error at most $\frac{1}{p-1}\frac{1}{N^{p-1}}$

Lets take $p=5$, $N=5$,

S_5 estimates $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ with precision $\simeq 0{,}0004$

S_5^+ estimates $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ with precision $\simeq 0{,}0001$

it converges fast!

IV - Series with arbitrary general term

⚠ It is no longer sufficient to show that the partial sums are bounded.

The divergence test still applies.

* $\sum_{n\in\mathbb{N}} (-1)^n$ does not converge because the general term does not converge to 0.

N.B. Some people use quite liberally the term diverge and will say $\sum_{n\in\mathbb{N}} (-1)^n$ diverges and $\sum_{n\geq 1} \frac{1}{n}$ diverges to infinity.

I prefer to reserve the term "diverge" for series that diverge to $\pm\infty$

Example $\displaystyle\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = \underbrace{\overbrace{\underbrace{1}_{S_1} - \frac{1}{2}}^{S_2} + \frac{1}{3}}_{S_3} \underbrace{\overbrace{- \frac{1}{4}}^{>0}}_{<0} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7}$

S_4

So $S_2 < S_4$ $\qquad$ $S_1 > S_3$

what we notice is then that:

(S_{2N}) is increasing and (S_{2N+1}) is decreasing

$$\begin{cases} S_{2N+2} - S_{2N} = \dfrac{1}{2N+1} - \dfrac{1}{2N+2} > 0 \\ S_{2N+3} - S_{2N+1} = \dfrac{-1}{2N+2} + \dfrac{1}{2N+3} < 0 \end{cases}$$

Additionnally, $S_{2N+1} - S_{2N} = \dfrac{1}{2N+1} > 0$

and $\displaystyle\lim_{N\to+\infty} S_{2N+1} - S_{2N} = 0$

(S_{2N}) and (S_{2N+1}) are ADJACENT SEQUENCES.

increasing $\qquad\qquad\qquad\qquad$ (S_{2N+1}) decreasing

- $S_{2N} \leq S_{2N+1} \leq S_1$ $\qquad$ forall $N \in \mathbb{N}$

$\Rightarrow$ (S_{2N}) converges by the monotone convergence theorem

Similarly: $\underbrace{S_2 \leq S_{2N}}_{(S_{2N}) \text{ increasing}} \leq \underbrace{S_{2N+1}}_{\text{decreasing}}$ for $N \geq 1$

Therefore (S_{2N+1}) converges by the monotone convergence theorem but:

$$0 = \lim_{N \to +\infty} S_{2N+1} - S_{2N} = \lim_{N \to +\infty} S_{2N+1} - \lim_{N \to +\infty} S_{2N}$$

Therefore they converge to the same limit!

Conclusion $(S_N)_{N \in \mathbb{N}}$ converges to this limit too!

$\Rightarrow \sum_{n \geq 1} \frac{(-1)^n}{n}$ converges!

This is the prototype example of the following theorem

Theorem (Alternating series theorem)

Let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence of positive real numbers that converges to 0.

then the alternating series: $\sum (-1)^n a_n$ converges.

Proof: Exercise.

Example $\displaystyle\sum_{n\geq 2} \frac{(-1)^n}{\ln n}$ converges

Indeed $a_n = \dfrac{1}{\ln n}$, $\ln$ is an increasing function on $(0,+\infty)$, therefore if $n\geq 2$,

$$0 \leq \ln n \leq \ln(n+1)$$

$$\Rightarrow \quad a_{n+1} = \frac{1}{\ln(n+1)} \leq \frac{1}{\ln n} = a_n \quad \text{for } n\geq 2.$$

Since $\displaystyle\lim_{n\to+\infty} \frac{1}{\ln n} = 0$, the alternating series converges.

Remark Theorem still applies if the hypotheses are satisfied after a finite number of terms.

e.g. $\sum_{n\geq 0} (-1)^n \sin\left(\frac{5\pi}{n+1}\right)$

$a_0 = \sin(5\pi) = 0$

$a_1 = \sin\left(\frac{5\pi}{2}\right) = \sin\left(2\pi + \frac{\pi}{2}\right) = 1$

$a_2 = \sin\left(\frac{5\pi}{3}\right) = \sin\left(2\pi - \frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$a_3 = \sin\left(\frac{5\pi}{4}\right) = \sin\left(\pi + \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$a_4 = \sin(\pi) = 0$

$a_5 = \sin\left(\frac{5\pi}{6}\right) = \sin\left(\pi - \frac{\pi}{6}\right) = \frac{1}{2}$

$a_6 = \sin\left(\frac{5\pi}{7}\right) \simeq 0{,}78 > a_5$

(...)

but for $n \geq 9$, $\frac{5\pi}{n+1} \in \left[0, \frac{\pi}{2}\right]$, where sin is increasing, therefore $(a_n) = \left(\sin\left(\frac{5\pi}{n+1}\right)\right)$ is decreasing for $n \geq 9$.

So we write $\sum_{n\geq 0} (-1)^n \sin\left(\frac{5\pi}{n+1}\right) = \underbrace{\sum_{n=0}^{9} (-1)^n \sin\left(\frac{5\pi}{n+1}\right)}_{\text{finite sum}} + \underbrace{\sum_{n\geq 10} (-1)^n \sin\left(\frac{5\pi}{n+1}\right)}_{\text{converges by the theorem}}$

What happens if it is not alternating?

Definition A series $\left(\sum_{n\geq 0} a_n\right)$ is said to be **absolutely convergent** if the positive series $\left(\sum_{n\geq 0} |a_n|\right)$ converges.

Ex: $\sum \frac{(-1)^n}{n^2}$ converges absolutely, $\sum \frac{(-1)^n}{n}$ does not.

Theorem: Absolutely convergent series converge.

Proof: omitted, relies on the completeness of $\mathbb{R}$.

Examples: $\sum_{n\in\mathbb{N}} q^n$, $|q|<1$

converges absolutely and therefore converges.

WARNING: The converse is FALSE.

$\sum \frac{(-1)^n}{n}$ converges $\qquad \sum \frac{1}{n}$ diverges

Series that converge but that are not absolutely convergent are sometimes called semi or conditionally convergent.

THE GOOD NEWS : To show the convergence of a series $\sum a_n$, I can try to show that it converges absolutely and study the positive series $\sum |a_n|$.

I CAN APPLY MY CONVERGENCE TESTS TO THE POSITIVE SERIES $\sum |a_n|$.

Example: $\displaystyle\sum_{n\geq 1} \frac{\cos(n)}{n^4}$

We test for absolute convergence

$$0 \leq \frac{|\cos(n)|}{n^4} \leq \frac{1}{n^4}$$

therefore $\displaystyle\sum_{n\geq 1} \frac{\cos(n)}{n^4}$ converges absolutely and therefore converges.

Dirichlet's test (not in book)

Consider a series of the form $\sum_{n\geq 0} a_n b_n$

Assume that:
- (a_n) is a non-increasing sequence converging to 0.
- $\exists C \in \mathbb{R}_+^*$, $\left|\sum_{n=0}^{N} b_n\right| < C$

then $\sum_{n\geq 0} a_n b_n$ converges.

Proof Discrete "integration by parts" $B_n = \sum_{k=0}^{n} b_n$, $B_{-1} = 0$

$$\sum_{n=0}^{N} a_n b_n = \sum_{n=0}^{N} a_n (B_n - B_{n-1})$$

$$= \sum_{n=0}^{N} a_n B_n - \sum_{n=0}^{N-1} a_{n+1} B_n$$

$$= a_N B_N + \sum_{n=0}^{N-1} (a_n - a_{n+1}) B_n$$

$$= \underbrace{a_N B_N}_{\xrightarrow[N\to\infty]{} 0} + \underbrace{\sum_{n=0}^{N-1} (a_n - a_{n+1}) B_n}_{\text{Converges absolutely}}$$

$\square$

$$\sum_{n=0}^{N-1} |(a_n - a_{n+1})| |B_n| \leqslant C \sum_{n=0}^{N-1} (a_n - a_{n+1}) = C(a_0 - a_N)$$

The error estimate:

$$\sum_{n=M+1}^{N} a_n b_n = \sum_{n=M+1}^{N} a_n (B_n - B_{n-1}) = \sum_{n=M+1}^{N} a_n B_n - \sum_{n=M}^{N-1} a_{n+1} B_n$$

$$= a_N B_N - a_{M+1} B_M - \sum_{n=M+1}^{N-1} (a_n - a_{n+1}) B_n$$

$$\left| \qquad \right| = \left| a_N B_N - a_{M+1} B_M - \sum_{n=M+1}^{N-1} (a_n - a_{n+1}) B_n \right|$$

$$\left| \sum_{n=M+1}^{+\infty} a_n b_n \right| \leqslant 2C a_{M+1}$$

<u>Example</u> $\sum_{n \in \mathbb{N}^*} \frac{\sin(n)}{n}$. See tutorial.

POWER SERIES

History: The modern theory of power series began in fact with Newton, who even considered it his greatest mathematical discovery.

Other important names: Abel, Cauchy, Euler

Motivation: Define new functions by considering series that depend on a parameter $\left(\sum_{n\geq 0} a_n(t)\right)$.

Applications: "solving differential equations"

If we consider the set T composed of the values for t for which $\left(\sum_{n\geq 0} a_n(t)\right)$ is a convergent series one can define a function $f(t) = \sum_{n=0}^{+\infty} a_n(t)$, $t \in T$.

Several natural questions: what is T like?

Does f have any nice properties

<u>Example</u> Let us consider the exponential function.

Recall that it is the unique solution to:
the "Cauchy" problem $\begin{cases} y' = y \\ y(0) = 1. \end{cases}$

By the fundamental theorem of analysis:

$$e^x = 1 + \int_0^x 1 . e^t \, dt$$

integrate 1

however, we may integrate by parts to find:

$$e^x = 1 + x + \int_0^x (x-t) e^t \, dt$$

Repeating the trick:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{1}{2} \int_0^x (x-t)^2 e^t \, dt$$

and again...

$$e^x = 1 + x + \frac{x^2}{2} + \frac{1}{3!} x^3 + \frac{1}{3!} \int_0^x (x-t)^3 e^t \, dt$$

$(\ldots)$ $\quad e^x \overset{?}{=} \sum_{n=0}^{+\infty} \frac{x^n}{n!},$

Does it make sense to continue the process indefinitely ?

Def A power series is a series depending on a parameter t of the form:

$$\left(\sum_{n\geq 0} a_n t^n\right)$$

Remark: It looks like an "infinite" polynomial...

Ex: $\sum_{n\geq 0} \frac{1}{n!} t^n$, let us study its convergence.

Fix $t \in \mathbb{R}^*$, then we apply the ratio test to the positive series: $\sum_{n\geq 0} \frac{1}{n!}|t|^n$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|t|}{n+1} \xrightarrow[n\to+\infty]{} 0 \quad \text{for fixed } t.$$

So $\sum_{n\geq 0} \frac{1}{n!} t^n$ converges absolutely therefore converges

for every fixed $t \in \mathbb{R}$. (convergence for $t=0$ is obvious)

We can do better, a priori the "way it is converging" may depend on t, but in fact:

Let $R>0$, $0<|t|<R$, then:

$$\left|\frac{a_{n+1}}{a_n}\right| \leq \frac{|t|}{n+1} \leq \frac{R}{n+1} \xrightarrow[n\to+\infty]{} 0$$

But now this is uniform on the disk of radius R.

So the way it converges, in some way is similar for every t in the interval $(-R, R)$.

All power series exhibit similar behaviour.

Prop $\sum a_n t^n$, suppose that for some $t_0 \in \mathbb{R}^+$ the series converges then for all $R < |t_0|$, $\sum a_n t^n$ converges absolutely for all $|t| < R$

Proof $a_n t_0^n \xrightarrow[n \to +\infty]{} 0$, let $M > 0$ and chose $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ $|a_n t_0^n| \leq M$, then:

$$|a_n t^n| \leq |a_n t_0^n| \left|\frac{t}{t_0}\right|^n \leq M \underbrace{\left|\frac{R}{t_0}\right|^n}_{\text{general term of a convegent sequence}}$$

$\square$

Remark: These means that power series converge on intervals of the form $(-R, R)$.

The above motivates the following definition:

Definition: we define the radius of convergence of a power series to be:

$$\boxed{R = \sup \{ |z| ,\ \sum a_n z^n \text{ converges} \}}.$$

R gives the size of the largest open interval $(-R, R)$, on which we have absolute convergence.

N.B. We do not know what happens at the endpoints.

The problem of finding R is completely solved

Theorem (Cauchy-Hadamard) Let $\left(\sum_{n \geq 0} a_n t^n\right)$ be a power series, then:

$$\boxed{\frac{1}{R} = \limsup_{n \to +\infty} |a_n|^{\frac{1}{n}} = \inf_{m \in \mathbb{N}} \sup_{n \geq m} |a_n|^{\frac{1}{n}}}.$$

In particular, if $\lim\limits_{n \to \infty} |a_n|^{\frac{1}{n}}$ exists then $\boxed{\frac{1}{R} = \lim\limits_{n \to +\infty} |a_n|^{\frac{1}{n}}}$

$a_n = 1$ for all n

Example: $\sum_{n\geq 0} \frac{x^n}{n!}$, $R = +\infty$, $\sum_{n\geq 0} x^n$, $R = 1$.

we even have $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$, $|x|<1$.

We have answered the question where do power series converge? Now we answer the question about the properties of the sum:

Theorem Let $\left(\sum_{n\geq 0} a_n t^n\right)$ be a power series with radius of convergence given by $R>0$. Let

$f(t) = \sum_{n=0}^{+\infty} a_n t^n$, $t \in (-R, R)$ then:

① f is a continuous function on $(-R,R)$

② f is differentiable on $(-R,R)$ and:

$$\boxed{f'(t) = \sum_{n=0}^{+\infty} n a_n t^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} t^n}$$

We can differentiate term by term, like a polynomial.

Example: We shall show that $e^{x} = \sum_{n=0}^{+\infty} \frac{1}{n!} x^n$.

Define $f(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$, by the theorem

we have: $f'(x) = \sum_{n=0}^{+\infty} \frac{(n+1)}{(n+1)!} x^n - \sum_{n=0}^{+\infty} \frac{x^n}{n!} = f(x)$

Furthermore, $f(0) = \sum_{n=0}^{+\infty} \frac{0^n}{n!} = 1$.

so $\begin{cases} f' = f \\ f(0) = 1 \end{cases}$, by uniqueness:

$$\boxed{e^{x} = \sum_{n=0}^{+\infty} \frac{x^n}{n!}}$$

Remark: The theorem tells us what happens on the open interval $(-R, R)$.

Remark The theorem also tells us that differentiating does not cause the radius of convergence to decrease.

<u>Example</u>: cos and sin can be expanded in series. To have an easy way to remember the formulae we allow ourselves to work in $\mathbb{C}$

$$\mathbb{C} = \{a + ib,\ a, b \in \mathbb{R}\} \qquad i^2 = -1.$$

$$z \in \mathbb{C}, \qquad |z|^2 = a^2 + b^2$$

$$e^{ix} = \cos(x) + i\sin(x)$$

$$e^{ix} = \sum_{n=0}^{+\infty} \frac{(ix)^n}{n!} = \sum_{p=0}^{+\infty} \frac{(ix)^{2p}}{(2p)!} + \sum_{p=0}^{+\infty} \frac{(ix)^{2p+1}}{(2p+1)!}$$

split into odd an even parts

using $i^2 = -1$, we find.

$$e^{ix} = \underbrace{\sum_{p=0}^{+\infty} \frac{(-1)^p x^{2p}}{2p!}}_{\cos(x)} + i \underbrace{\sum_{p=0}^{+\infty} \frac{(-1)^p x^{2p+1}}{(2p+1)!}}_{\sin(x)}$$

26 sept

Algebraic operations on power series

Let $\sum_{n\geq 0} a_n t^n$, $\sum_{n\geq 0} b_n t^n$ be power series

with radii of convergence R_a and R_b.

Let $c \in \mathbb{R}$, $c \neq 0$ ($c=0$ is trivial)

① if $b_n = c a_n$ then $\sum_{n=0}^{+\infty} \underbrace{c a_n}_{b_n} t^n = c \cdot \sum_{n=0}^{+\infty} a_n t^n$

and $R_b = R_a$

② $b_n = c^n a_n$ then $R_b = \dfrac{R}{c}$.

③ The radius of convergence R_{a+b} of $\sum (a_n + b_n) t^n$

satisfies, $R_{a+b} \geq \min(R_a, R_b)$ and if

$|t| < \min(R_a, R_b)$ $\sum_{n=0}^{+\infty} (a_n + b_n) t^n = \sum_{n=0}^{+\infty} a_n t^n + \sum_{n=0}^{+\infty} b_n t^n$.

Multiplication of power series.

Theorem (Merten's) Let $\left(\sum_{n\geq 0} a_n\right)$ and $\left(\sum_{n\geq 0} b_n\right)$ be two convergent series at least one of which converges ABSOLUTELY then if $c_n = \sum_{k=0}^{n} a_k b_{n-k}$,

$\sum_{n\geq 0} c_n$ converges and:

$$\boxed{\sum_{n=0}^{+\infty} c_n = \left(\sum_{n=0}^{+\infty} a_n\right)\left(\sum_{n=0}^{+\infty} b_n\right)}$$

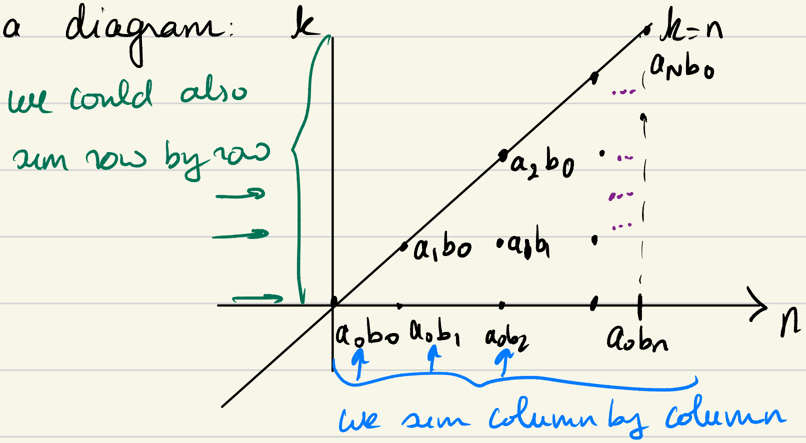
N.B. Its like a "distributivity" property.

Proof of Merten's theorem

Assume $\sum_{n=0}^{+\infty} |a_n| < +\infty$

Let us first investigate the partial sums of $\sum_{n\geq 0} c_n$

For $N \in \mathbb{N}$, $\sum_{n=0}^{N} c_n = \sum_{n=0}^{N} \sum_{k=0}^{n} a_k b_{n-k}$, to rewrite this finite sum it is informative to represent the terms on a diagram:



Therefore: $\sum_{n=0}^{N} \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{N} \sum_{n=k}^{N} a_k b_{n-k}$

$$= \sum_{k=0}^{N} a_k \sum_{n=0}^{N-k} b_n$$

reindex

Now morally we want to take $N \to +\infty$

but it is slightly more complicated.

Set : $A = \sum_{n=0}^{+\infty} a_n \qquad B = \sum_{n=0}^{+\infty} a_n$

Fix $\varepsilon \in \mathbb{R}_+^*$

$$\sum_{n=0}^{N} \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{N} a_k \sum_{n=0}^{N-k} b_n$$

$$= \sum_{k=0}^{N} a_k B + \sum_{k=0}^{N} a_k \underbrace{\left(\sum_{n=0}^{N-k} b_n - B \right)}_{\sum_{n=N-k+1}^{+\infty} b_n}$$

$$= \left(\sum_{k=0}^{N} a_k - A \right) B + AB + \sum_{k=0}^{N} a_k \sum_{n=N-k+1}^{+\infty} b_n$$

$$\left| \sum_{n=0}^{N} c_n - AB \right| \leqslant \underbrace{\left| \sum_{k=N+1}^{+\infty} a_k \right|}_{\text{can be made smaller than } \frac{\varepsilon}{3B}} B + \left| \sum_{k=0}^{N} a_k \underbrace{\sum_{n=N-k+1}^{+\infty} b_n}_{\text{problem this is not necessarily small}} \right|$$

$$\leqslant \frac{\varepsilon}{3} + \sum_{k=0}^{N} |a_k| \left| \sum_{n=N-k+1}^{+\infty} b_n \right|$$

Now the trick, we introduce $N_0 \in \mathbb{N}$ and write:

$$\underbrace{\sum_{k=N_0}^{N} |a_k| \left| \sum_{n=N-k+1}^{+\infty} b_n \right|}_{\leq \left(\sum_{k=N_0}^{+\infty} |a_k| \right) B} + \sum_{k=0}^{N_0} |a_k| \underbrace{\left| \sum_{n=N-k+1}^{+\infty} b_n \right|}$$

now the worst term in this sum is

$$N-N_0+1 \xrightarrow[N\to+\infty]{} +\infty$$

Since $\sum_{k=0}^{+\infty} |a_k| < +\infty$, there is N_1 such that if

$N_0 \geq N_1$, $\quad \sum_{k=N_0}^{+\infty} |a_k| < \frac{\varepsilon}{3B}$. So we fix $N_0 \geq N_1$.

now for any N large enough $\quad N-N_0+1 \geq N_2$

where N_2 is chosen such that $\left| \sum_{n=M}^{+\infty} b_n \right| \leq \frac{\varepsilon}{3A}$

for all $M \geq N_2$.

Now it follows that; for any $N \geq N_2$

$$\left| \sum_{n=0}^{N} c_n - AB \right| \leq \varepsilon.$$

Application Multiplication of power series

Apply the above with $\sum_{n\geq 0} a_n t^n$ and $\sum_{n\geq 0} b_n t^n$

the radius of convergence $R \geq \min(R_a, R_b)$

and:

$$\boxed{\left(\sum_{n=0}^{+\infty} a_n t^n \sum_{n=0}^{+\infty} b_n t^n\right) = \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) t^n}$$

for $|t| < \min(R_a, R_b)$.

N.B same rule as for polynomials!

Example $e^x e^y = e^{x+y}$

$$e^x e^y = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \sum_{k=0}^{+\infty} \frac{y^k}{k!} \underset{\substack{\uparrow \\ \text{Merten's} \\ \text{theorem}}}{=} \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} \frac{x^k y^{n-k}}{k!\,(n-k)!}\right)$$

But (Binomial theorem), $(x+y)^n = \sum_{n=0}^{n} \binom{n}{k} x^k y^{n-k}$

Therefore:

$$\boxed{e^x e^y = \sum_{n=0}^{+\infty} \frac{1}{n!}\left(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\right) = \sum_{k=0}^{+\infty} \frac{1}{n!}(x+y)^n = e^{x+y}}$$

Integration term by term

Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ be a convergent power series with radius of convergence $R>0$.

Let $x<R$, then

$$\boxed{\int_0^x f(t)\,dt = \sum_{n=0}^{+\infty} \frac{a_n}{n+1} t^{n+1} = \sum_{n=1}^{+\infty} \frac{a_{n-1}}{n} t^n}$$

and the RHS is a power series with radius of convergence at least R. (Actually it is R)

$\Rightarrow$ You can integrate term by term

Example Recall that:

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n \quad , \; |x|<1$$

using the theorem we can say that:

$$\boxed{-\ln(1-x) = \sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{+\infty} \frac{x^n}{n} .}$$

we can deduce that for $|x|<1$

$$\boxed{\ln(1+x) = \sum_{n=0}^{+\infty} (-1)^{n+1} \frac{x^n}{n}}$$

oh but wait for $x=1$ the RHS is the conditionally convergent alternating series $\displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{n}$.

and the LHS has a limit $x \to 1$, $\ln(2)$

Can we take the limit $x \to 1$ and conclude that

$$\ln 2 = \sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{n} \quad ???$$

IN GENERAL, WE CANNOT TAKE THE LIMIT, but it turns out this is okay.

Theorem (Abel) Suppose $\sum a_n t^n$ is a power series with radius of convergence $R>0$.

Suppose that $\sum_{n\geq 0} a_n t_0^n$ converges with $t_0 = \pm R$

$$\boxed{\lim_{x\to t_0} \sum_{n=0}^{+\infty} a_n t^n = \sum_{n=0}^{+\infty} a_n t_0^n}$$

proof (omitted)

Using Abel's theorem we have:

$$\boxed{\ln 2 = \lim_{x\to 1} \ln(1+x) = \lim_{x\to 1} \sum_{n=0}^{+\infty} \frac{(-1)^{n+1} x^n}{n} = \sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{n}.}$$

NON TRIVIAL STEP.

Power series centered at an arbitrary point c

Definition Let $c \in \mathbb{R}$, a power series centered at c, is a series of the form $\sum_{n \geq 0} a_n (t-c)^n$.

EVERYTHING IS EXACTLY THE SAME AS BEFORE BECAUSE YOU CAN RETRANSLATE TO 0

Set $T = t - c$.

Taylor-McLaurin Series

In the previous lectures we have seen that sometimes functions can be written as power series.

Furthermore, let us consider:

$$f(t) = \sum_{n=0}^{+\infty} a_n (t-c)^n \quad , \quad |t-c|<R$$

where R is the radius of convergence.
Note that $f(c) = a_0$

Using the differentiation theorem iteratively we conclude that f is C^∞ (differentiable to any order)

$$\boxed{a_n = \frac{f^{(n)}(c)}{n!}}$$

$f^{(n)}$ is the n^{th} derivative of the function f.

In other words the series is completely determined by f and its derivatives at the point c.

Definition Let $f : I \longrightarrow \mathbb{R}$ be an infinitely differentiable function defined on an open interval I.
We define the Taylor-Mclaurin series associated to f centered at $c \in I$, to be the power series:

$$\boxed{\sum_{n \geq 0} \frac{f^{n}(c)}{n!} (t-c)^n}$$

N.B. We have said NOTHING about the convergence of this power series which CAN HAVE vanishing radius of convergence.

stopped here

Def If for some $c \in I$, the series has non-vanishing radius of convergence $\boxed{\text{AND}}$

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{n}(c)}{n!} (t-c)^n$$

then we say that f is analytic near c.

The theory of analytical functions is best developed with $\mathbb{C}$ and so I shall venture no further on this terrain.

N.B. when f is not analytic its Taylor-Mclaurin series does not determine it uniquely.

Examples of analytic functions: polynomials, exp, cos sin, ...

The sum, product and composition of analytic functions are analytic.

Example of Taylor-Mclaurin series

1/ $f(x) = x^{\alpha}$ $\quad x \in \mathbb{R}_+^*$, near 1

$f(1) = 1 \quad f'(x) = \alpha x^{\alpha-1} \quad f''(x) = \alpha(\alpha-1) x^{\alpha-1}$

$$f^{(n)}(x) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)\, x^{\alpha-n}$$

Taylor-Mclaurin series: $\displaystyle\sum_{n\geq 0} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}(t-1)^n.$

Example 2, $f(x) = \cos(x)$ $\quad f'(x) = -\sin(x)$

$f''(x) = -\cos(x)$ $\quad f^{(3)}(x) = \sin(x)$

$f^{(2n)}(x) = (-1)^n \cos(x)$ $\quad f^{(2n+1)} = (-1)^{n+1} \sin(x)$

Therefore: $f^{(2n)}(x) = (-1)^n$, $f^{(2n+1)}(x) = 0$

Hence the Taylor-Mclaurin series at 0

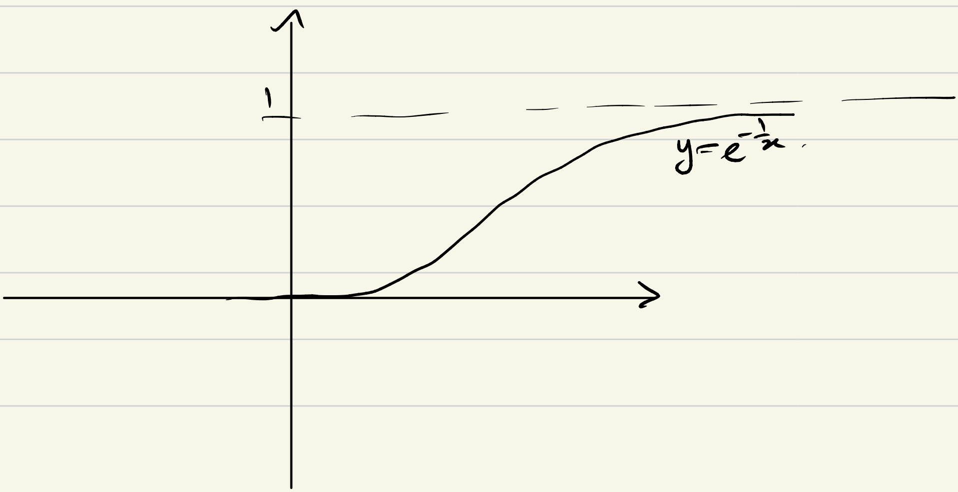
is $\displaystyle\sum_{p \geq 0} \frac{(-1)^p x^{2p}}{(2p)!}$.

If the Taylor-Mclaurin series at c has non-zero radius of convergence, does it necessarily converge to f?

$$\boxed{\text{NO}}$$

Consider $\quad f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & \text{si } x \leq 0 \end{cases}$

It is a smooth function such that $f^{(n)}(0) = 0$ for all $n \geq 0$, therefore its Taylor-Mclaurin series at 0 vanishes, but $f \neq 0$!



Conclusion: Two different C^∞ functions can have the same Taylor-Mclaurin series... So even if the power series has non-vanishing radius of convergence it might not converge to the function we started with.

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{si } x \leq 0 \end{cases}$$ is C^∞ but not analytic.

When does the Taylor-Mclaurin series converge to the function?

To answer this lets try to estimate the error.

Taylor's theorem with integral remainder.

Let f be a smooth function (differentiable to any order) on an interval open interval I and $c \in I$,

$$f(x) = f(c) + \int_c^x f'(t)\,dt ,$$

by the fundamental theorem of analysis, integrating by parts n times we arrive at:

$$\boxed{f(x) = \sum_{n=0}^{n} \frac{f'(c)}{n!}(x-c)^n + \int_c^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt}$$

Integral remainder. $R_I(f, x) = \int_c^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt.$

So if $R_I(f,x) \longrightarrow 0$ we have, convergence.

Example: Let $f(x) = x^\alpha$, work near $c=1$

We could calculate the remainder but we can in fact do better. $\Rightarrow$ $f(1+x) = (1+x)^\alpha = e^{\alpha \ln(1+x)}$

Conceptual solution.

$\ln(1+x)$ is analytic for $|x|<1$, exp is analytic. the Composition of analytic functions is analytic, so the power series converges.

Another way of doing things that does not use the notion of analytic functions is to look at the series in its own right, study its convergence and show that it satisfies a dif. eq.

$$\sum_{n\geq 0}^{+\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n$$

Observe that if $f(x) = (1+x)^\alpha$, $f'(x) = \alpha(1+x)^{\alpha-1}$
$= \alpha \dfrac{f(x)}{(1+x)}$

so f is a solution to the Cauchy problem:

$$\begin{cases} (1+x) f'(x) = \alpha f(x) \\ f(0) = 1 \end{cases}$$

Note that $\left|\dfrac{a_{n+1}}{a_n}\right| \leq \left|\dfrac{\alpha - n}{n+1}\right| \xrightarrow[n\to+\infty]{} 1$

so $R = 1$

we set: $$g(x) = \sum_{n=0}^{+\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n \quad , |x|<1$$

$$g'(x) = \sum_{n=1}^{+\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} x^{n-1}$$

$$g'(x) = \sum_{n=0}^{+\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} x^n$$

$$(1+x)g'(x) = \sum_{n=0}^{+\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} x^n + \sum_{n=0}^{+\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} x^{n+1}$$

$$= \sum_{n=1}^{+\infty} \left(\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} \right) x^n \, \alpha$$

$$= \sum_{n=1}^{+\infty} \frac{\alpha(\alpha-1)\cdots\alpha(\alpha-n+1)(\alpha\cancel{-n+n})}{n!} x^n + \alpha$$

$$= \alpha \left(\sum_{n=0}^{+\infty} \frac{\alpha(\alpha-1)\cdots\alpha(\alpha-n+1)}{n!} x^n \right) = \alpha\, g(x)$$

Therefore, $\dfrac{dg}{dx} = \dfrac{\alpha g(x)}{1+x}$ and $g(0) = 1$

$$d(\ln f) = \alpha\, d(\ln(1+x)) \Rightarrow f(x) = C(1+x)^{\alpha}$$

since $f(0)=1$, $C=1$ and $f(x) = (1+x)^{\alpha}$

This proves that:

Generalisation of the binomial theorem

$$\boxed{(1+x)^{\alpha} = \sum_{n=0}^{+\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n, \quad |x|<1}$$

or equivalently

$$\boxed{x^{\alpha} = \sum_{n=0}^{+\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} (x-1)^n \quad |x-1|<1}$$

Finding the radius of convergence using the ratio test

Prop Let $\sum a_n x^n$ be a power series and suppose that $\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = L$,

No x!

then: if $L = +\infty$, $R = 0$

if $0 < L < +\infty$, $R = \frac{1}{L}$

if $L = 0$, $R = +\infty$

Remark Less general than Cauchy-Hadamard but is sometimes easier to apply.

Proof Let $0 < |x| < r < \frac{1}{L}$ then the ratio test applies uniformly, and the series converges absolutely for all $|x| < \frac{1}{L}$, therefore $R \geq \frac{1}{L}$

If $|x| > L$ then again the ratio test applies negatively and therefore $R \leq \frac{1}{L}$. $\Rightarrow$ $\boxed{R = \frac{1}{L}}$

To be more precise, the ratio test applies negatively to $\sum |a_n||x|^n$ which cannot be absolutely convergent if $|x| > \frac{1}{L}$. If it converged conditionally at some point x_0, $|x_0| > \frac{1}{L}$ then it would converge absolutely (see Proposition at the start of the notes on Power series) for all $\frac{1}{L} < |x| < |x_0|$

but this is not possible by the above application of the ratio test.

So the series does not converge if $|x| > \frac{1}{L}$ and

hence $R \leq \frac{1}{L}$ as stated.

Example $\displaystyle\sum_{p=0}^{+\infty} \frac{(-1)^p}{(2p)!} x^{2p}$

Recall that $\cos$ is the unique solution of the equation
$$\begin{cases} y'' + y = 0 \\ y'(0) = 0 \\ y(0) = 0 \end{cases}$$

Let us show that $f(x) = \displaystyle\sum_{p=0}^{+\infty} \frac{(-1)^p}{(2p)!} x^{2p}$ satisfies this equation.

First, the radius of convergence of the power series is $R = +\infty$.

$$a_n = \begin{cases} \dfrac{(-1)^n}{n!} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

$$(|a_n|)^{\frac{1}{n}} = \begin{cases} \left(\dfrac{1}{n!}\right)^{\frac{1}{n}} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$\lim_{n\to\infty} e^{\frac{1}{n}\ln\frac{1}{n!}} = e^{-\frac{1}{n}\sum_{k=1}^{n}\ln k} \longrightarrow 0$$

compare with $\int \ln x \, dx$ using that $\ln$ is increasing.

so by the Cauchy-Hadamard theorem $\frac{1}{R} = 0 \Rightarrow R = +\infty$

Now we calculate on $\mathbb{R}$.

$$f'(x) = \sum_{p=1}^{+\infty} \frac{(-1)^p}{(2p-1)!} x^{2p-1}$$

$$f''(x) = \sum_{p=1}^{+\infty} \frac{(-1)^p}{(2p-2)!} x^{2(p-1)}$$

$$f''(x) = \sum_{p=1}^{+\infty} \frac{(-1)^p}{(2(p-1))!} x^{2(p-1)}$$

$$= \sum_{p=0}^{+\infty} \frac{(-1)^{p+1}}{(2p)!} x^{2p} = -f(x)$$

Furthermore, $f(0) = 1 = \cos(0)$, so $f(x) = \cos x$ for all $x \in \mathbb{R}$.

<u>Examples</u> Using the rules of computation of power series to find new power series expansions.

<u>Power series expansion of arctan.</u>

$\arctan(1) = 0$, $\quad \arctan'(x) = \dfrac{1}{1+x^2}$

For $|x|<1$, $|x^2|<1$ and so we can use the geometric series.

$$\frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n}$$

Now we can integrate; for $|x|<1$

$$\arctan(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

<u>Application</u>: power series expansion of π.

$\arctan(1) = \frac{\pi}{4}$, does $\arctan(1) \overset{?}{=} \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}$

$\Rightarrow$ Apply Abels theorem.

By the alternating series test $\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}$ converges,

therefore by Abel's theorem.

$$\boxed{\pi = 4 \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}}$$

Example Power series expansion of $\frac{1}{(1+x)^2}$

$$\frac{d}{dx}\left(\frac{-1}{1+x}\right) = \frac{1}{(1+x)^2}$$

if $|x|<1$, $-\frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^{n+1} x^n$

Take the derivative:

$$\boxed{\frac{1}{(1+x)^2} = \sum_{n=0}^{+\infty} (-1)^{n+1} n x^{n-1} = \sum_{n=0}^{+\infty} (-1)^n (n+1) x^n.}$$

Further applications of the Taylor-Mclaurin series

Even when the Taylor-Mclaurin series of a C^∞ function does not converge or converges but not to the function f, if we look at Taylor's integral remainder theorem we see that it does offer an approximation of f near c, which improves as we get closer and closer to c. "asymptotic"

Let $f: I \to \mathbb{R}$ be a C^∞ function I open, $c \in I$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)(x-c)^n}{k!} + \int_c^x \frac{f^{(n+1)}(t)(x-t)^n}{n!}\, dt$$

If $|x-c| < \delta$, small enough so that $x \in I$, then $f^{(n+1)}$ is bounded by some $M > 0$ and

$$\left| \int_c^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} \right| \leq M \frac{|x-c|^{n+1}}{(n+1)!}$$

"Lagrange error"

We can therefore write:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)(x-c)^n}{k!} + O\left((x-c)^{n+1}\right)$$

$O((x-c)^{n+1})$ means that it is a function of the form $|x-c|^{n+1} \times g$ where g is bounded near c.

This is known as the Taylor expansion of f near c to order n.

This can be useful for computing limits.

<u>Example</u> $\qquad \dfrac{\sin x}{x} = \dfrac{x + O(x^2)}{x}$

$$= 1 + \underbrace{\underset{x \to c}{O}(x)}$$

this is a bounded function times x

so it follows that $\boxed{\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1}$

$f(x) = \dfrac{\sin x}{x + \cos x} \qquad x \in [1, +\infty[$

$$f(x) = \frac{\sin x}{x}\left(\frac{1}{1 + \frac{\cos x}{x}}\right) = \frac{\sin x}{x}\left(1 - \frac{\cos x}{x} + O\left(\left(\frac{\cos x}{x}\right)^2\right)\right)$$

$\downarrow \to 0$

$$= \frac{\sin x}{x} - \frac{\cos x \sin x}{x^2} + O\left(\frac{1}{x^3}\right)$$

Challenge: Find $\displaystyle\lim_{x \to 0^+} \frac{(\sin x)^x - x^{\sin x}}{(\tan x)^x - x^{\tan x}}$

(its not necessarily easy but it can be done).

Example: $\tan x$ to order 5 near 0

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)}{1 - \underbrace{\left(\frac{x^2}{2!} - \frac{x^4}{4!} + O(x^6)\right)}_{\text{for } x \text{ small } < |1|}}$$

can apply $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$

$$\tan x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)\right)\left(1 + \left(\frac{x^2}{2!} - \frac{x^4}{4!}\right) + \left(\frac{x^2}{2!} - \frac{x^4}{4!}\right)^2 + O(x^6)\right)$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)\right)\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + O(x^6)\right)$$

$$= \left(x + \frac{x^3}{3} + \left(\frac{1}{5!} + \frac{5}{24} - \frac{1}{12}\right)x^5 + O(x^6)\right)$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + O(x^6)$$

so $$\frac{\tan x - x}{x^3} = \frac{1}{3} + O(x^2)$$

$$\boxed{\lim_{x \to 0} \frac{\tan x - x}{x^3} = \frac{1}{3}}$$

You can compose power series expansions but you only go to the order you want.

$$\ln(1+\tan(x)) = \ln(1 + \underbrace{x + \frac{x^3}{3} + \frac{2}{15}x^5 + O(x^6)}_{\text{something small } x \to 0})$$

$$\ln(1+x) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$$

$$\ln(1+\tan x) = x + \frac{x^3}{3} + -\frac{1}{2}\left(x + \frac{x^3}{3} + \frac{2}{15}x^5\right)^2$$

$$+ \frac{1}{3}x^3 + O(x^4)$$

$$\boxed{\ln(1+\tan x) = x + \frac{2x^3}{3} - \frac{1}{2}x^2 + O(x^4)}$$